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ELEMENTARY ALGEBRA

TEACHERS' EDITION

BY

G. A. WENTWORTH

AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS

GINN & COMPANY
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PREFACE

This book is intended for teachers, and *for teachers only.* The publishers will sell it to no persons except teachers of Wentworth's Elementary Algebra. Every teacher who purchases a copy must consider himself in honor bound not to leave it where pupils may have access to it, and not to sell it except to the publishers, Messrs. Ginn & Company.

It is expected that this book will prove of great advantage to every teacher of a class which does not have the time to solve every one of the numerous examples, by aiding him to select the examples best adapted to the needs of his particular class. It is hoped that every teacher who is pressed for time will find relief by not being obliged to work out every example for himself.

Criticisms and corrections will be thankfully received.

G. A. WENTWORTH

EXETER, NEW HAMPSHIRE
February, 1906

ELEMENTARY ALGEBRA

TEACHERS' EDITION

Exercise 1. Page 10

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $x = 5$, $y = 6$, $z = 0$, find the numerical value of:

1. $7x = 7 \times 5 = 35$.
2. $8b^2 = 8 \times 2^2 = 8 \times 4 = 32$.
3. $7c^3 = 7 \times 3^3 = 7 \times 27 = 189$.
4. $5bc = 5 \times 2 \times 3 = 30$.
5. $6c^2d = 6 \times 3^2 \times 4 = 6 \times 9 \times 4 = 216$.
6. $2c^2dx = 2 \times 3^2 \times 4 \times 5 = 2 \times 9 \times 4 \times 5 = 360$.
7. $14c^3yz = 14 \times 3^3 \times 6 \times 0 = 0$.
8. $b^3c^2y = 2^3 \times 3^2 \times 6 = 8 \times 9 \times 6 = 432$.
9. $3a^4b^2x = 3 \times 1^4 \times 2^2 \times 5 = 3 \times 1 \times 4 \times 5 = 60$.
10. $\frac{3}{8}d^2x = \frac{3}{8} \times 4^2 \times 5 = \frac{3}{8} \times 16 \times 5 = 30$.
11. $\frac{2}{9}c^3d = \frac{2}{9} \times 3^3 \times 4 = 2 \times 3 \times 4 = 24$.
12. $\frac{3}{10}b^2xy = \frac{3}{10} \times 2^2 \times 5 \times 6 = \frac{3}{10} \times 4 \times 5 \times 6 = 36$.
13. $\frac{5}{18}a^3dy^2 = \frac{5}{18} \times 1^3 \times 4 \times 6^2 = \frac{5}{18} \times 4 \times 36 = 40$.
14. $\frac{2}{3}x^2y^3z = \frac{2}{3} \times 5^2 \times 6^3 \times 0 = 0$.
15. $\frac{4}{15}c^2x^2 = \frac{4}{15} \times 3^2 \times 5^2 = 4 \times 3 \times 5 = 60$.
16. $\frac{5}{12}xy^2 = \frac{5}{12} \times 5 \times 6^2 = \frac{5}{12} \times 5 \times 36 = 75$.
17. $\frac{7}{10}d^2x^2 = \frac{7}{10} \times 4^2 \times 5^2 = \frac{7}{10} \times 16 \times 25 = 280$.
18. $\frac{13}{12}c^2d^2 = \frac{13}{12} \times 3^2 \times 4^2 = 13 \times 3 \times 4 = 156$.
19. $\dfrac{4b^4x^2}{5d^3} = \dfrac{4 \times 2^4 \times 5^2}{5 \times 4^3} = \dfrac{4 \times 16 \times 25}{5 \times 64} = 5$.

20. $\dfrac{7c^3x^2}{5by^2}=\dfrac{7\times 3^3\times 5^2}{5\times 2\times 6^2}=\dfrac{7\times 27\times 25}{5\times 2\times 36}=\dfrac{105}{8}=13\tfrac{1}{8}.$

21. $\dfrac{x^2y^2}{5c^2d^2}=\dfrac{5^2\times 6^2}{5\times 3^2\times 4^2}=\dfrac{25\times 36}{5\times 9\times 16}=\dfrac{5}{4}=1\tfrac{1}{4}.$

22. $\dfrac{9cdz}{dy}=\dfrac{9\times 3\times 4\times 0}{4\times 6}=0.$

23. $\dfrac{5a^5d^3}{4b^3x^2}=\dfrac{5\times 1^5\times 4^3}{4\times 2^3\times 5^2}=\dfrac{5\times 1\times 64}{4\times 8\times 25}=\dfrac{2}{5}.$

24. $\sqrt{b^2dx^2}=\sqrt{2^2\times 4\times 5^2}=2\times 2\times 5=20.$

25. $\sqrt{dy}=\sqrt{4}\times 6=2\times 6=12.$

26. $\sqrt[3]{27}\,d^2=\sqrt[3]{27}\times 4^2=3\times 16=48.$

27. $\sqrt{(bcy)}=\sqrt{2\times 3\times 6}=6.$

28. $\sqrt{x^2y^3z^3}=\sqrt{5^2\times 6^3\times 0^3}=0.$

29. $\sqrt[3]{9bcd}=\sqrt[3]{9\times 2\times 3\times 4}=\sqrt[3]{27\times 8}=3\times 2=6.$

30. $2\sqrt{c^2dx^2}=2\sqrt{3^2\times 4\times 5^2}=2\times 3\times 2\times 5=60.$

31. $cd\sqrt{dy^2}=3\times 4\sqrt{4\times 6^2}=3\times 4\times 2\times 6=144.$

32. $abc\sqrt{2cx^2y}=1\times 2\times 3\sqrt{2\times 3\times 5^2\times 6}=2\times 3\times 5\times 6=180.$

33. $\tfrac{3}{4}d\sqrt{dy^2}=\tfrac{3}{4}\times 4\sqrt{4\times 6^2}=\tfrac{3}{4}\times 4\times 2\times 6=36.$

34. $abcdxyz=1\times 2\times 3\times 4\times 5\times 6\times 0=0.$

35. $cx\sqrt[3]{b^2cdy^2}=3\times 5\sqrt[3]{2^2\times 3\times 4\times 6^2}=3\times 5\sqrt[3]{2^6\times 3^3}$
$=3\times 5\times 4\times 3=180.$

36. $b^2c\sqrt[3]{d^2z^2}=2^2\times 3\sqrt[3]{4^2\times 0^2}=0.$

37. $b^2dx^2y=2^2\times 4\times 5^2\times 6=4\times 4\times 25\times 6=2400.$

Exercise 2. Page 12

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$, find the numerical value of:

1. $9a+2b+3c-2f=9\times 1+2\times 2+3\times 3-2\times 0=9+4+9-0=22.$

2. $4e-3a-3b+5c=4\times 5-3\times 1-3\times 2+5\times 3=20-3-6+15=26.$

3. $8abc-bcd+9cde-def$
$=8\times 1\times 2\times 3-2\times 3\times 4+9\times 3\times 4\times 5-4\times 5\times 0$
$=48-24+540-0=564.$

4. $\dfrac{4ac}{b}+\dfrac{8bc}{d}-\dfrac{5cd}{e}=\dfrac{4\times 1\times 3}{2}+\dfrac{8\times 2\times 3}{4}-\dfrac{5\times 3\times 4}{5}$
$=6+12-12=6.$

5. $$7e + bcd - \frac{3\,bde}{2\,ac} = 7 \times 5 + 2 \times 3 \times 4 - \frac{3 \times 2 \times 4 \times 5}{2 \times 1 \times 3}$$
$$= 35 + 24 - 20 = 39.$$

6. $$abc^2 + bcd^2 - dea^2 + f^3$$
$$= 1 \times 2 \times 3^2 + 2 \times 3 \times 4^2 - 4 \times 5 \times 1^2 + 0^3$$
$$= 18 + 96 - 20 + 0 = 94.$$

7. $$e^4 + 6\,e^2b^2 + b^4 - 4\,e^3b - 4\,eb^3$$
$$= 5^4 + 6 \times 5^2 \times 2^2 + 2^4 - 4 \times 5^3 \times 2 - 4 \times 5 \times 2^3$$
$$= 625 + 600 + 16 - 1000 - 160 = 81.$$

8. $$\frac{8\,a^2 + 3\,b^2}{a^2b^2} + \frac{4\,c^2 + 6\,b^2}{c^2 - b^2} - \frac{c^2 + d^2}{e^2}$$
$$= \frac{8 \times 1^2 + 3 \times 2^2}{1^2 \times 2^2} + \frac{4 \times 3^2 + 6 \times 2^2}{3^2 - 2^2} - \frac{3^2 + 4^2}{5^2}$$
$$= \frac{8 + 12}{4} + \frac{36 + 24}{5} - \frac{9 + 16}{25} = 5 + 12 - 1 = 16.$$

9. $$\frac{d^c}{b^e} = \frac{4^3}{2^5} = \frac{64}{32} = 2.$$

10. $$\frac{e^c + b^a}{c^b - b^c} = \frac{5^3 + 2}{3^2 - 2^3} = \frac{125 + 2}{9 - 8} = 127.$$

11. $$\frac{b^c + d^c}{b^2 + d^2 - bd} = \frac{2^3 + 4^3}{2^2 + 4^2 - 2 \times 4} = \frac{8 + 64}{4 + 16 - 8} = \frac{72}{12} = 6.$$

12. $$\frac{e^c - d^c}{e^2 + ed + d^2} = \frac{5^3 - 4^3}{5^2 + 5 \times 4 + 4^2} = \frac{125 - 64}{25 + 20 + 16} = \frac{61}{61} = 1.$$

If $a = 2$, $b = 10$, $x = 3$, $y = 5$, find the numerical value of:

13. $xy + 4a \times 2 = 3 \times 5 + 4 \times 2 \times 2 = 15 + 16 = 31.$

14. $xy + 15b \div 5 = 3 \times 5 + 15 \times 10 \div 5 = 15 + 30 = 45.$

15. $$3x + 7y \div 7 + a \times y = 3 \times 3 + 7 \times 5 \div 7 + 2 \times 5$$
$$= 9 + 5 + 10 = 24.$$

16. $$6b - 8y \div 2y \times b - 2b = 6 \times 10 - \frac{8 \times 5}{2 \times 5} \times 10 - 2 \times 10$$
$$= 60 - 40 - 20 = 0.$$

17. $$(6b - 8y) \div 2y \times b + 2b = \frac{6 \times 10 - 8 \times 5}{2 \times 5} \times 10 + 2 \times 10$$
$$= 20 + 20 = 40.$$

18. $(6b - 8y) \div (2y \times b) + 2b = (6 \times 10 - 8 \times 5) \div (2 \times 5 \times 10) + 2 \times 10$
$= \frac{1}{5} + 20 = 20\frac{1}{5}.$

19. $6b - (8y \div 2y) \times b - 2b = 6 \times 10 - \dfrac{8 \times 5}{2 \times 5} \times 10 - 2 \times 10$
$= 60 - 40 - 20 = 0.$

Exercise 3. Page 14

Remove the parentheses and combine:

1. $7 + (5 + 3) = 7 + 5 + 3 = 15.$
2. $7 + (5 - 3) = 7 + 5 - 3 = 9.$
3. $8 + (6 + 2) = 8 + 6 + 2 = 16.$
4. $8 + (6 - 2) = 8 + 6 - 2 = 12.$
5. $7 - (5 - 3) = 7 - 5 + 3 = 5.$
6. $9 - (5 + 3) = 9 - 5 - 3 = 1.$
7. $8 - (6 + 2) = 8 - 6 - 2 = 0.$
8. $8 - (6 - 2) = 8 - 6 + 2 = 4.$
9. $(12 - 8) - (7 - 4) = 12 - 8 - 7 + 4 = 1.$
10. $(10 - 4) - (2 + 3) = 10 - 4 - 2 - 3 = 1.$
11. $(14 - 6) + (8 - 6) = 14 - 6 + 8 - 6 = 10.$
12. $(7 + 3) - (4 - 2) = 7 + 3 - 4 + 2 = 8.$

If $a = 8$, $b = 5$, $c = 6$, $d = 3$, find the value of:

13. $(a + b) + (c + d) = (8 + 5) + (6 + 3) = 8 + 5 + 6 + 3 = 22.$
14. $(a + b) - (c + d) = (8 + 5) - (6 + 3) = 8 + 5 - 6 - 3 = 4.$
15. $(a + b) + (c - d) = (8 + 5) + (6 - 3) = 8 + 5 + 6 - 3 = 16.$
16. $(a + b) - (c - d) = (8 + 5) - (6 - 3) = 8 + 5 - 6 + 3 = 10.$
17. $(a - b) + (c - d) = (8 - 5) + (6 - 3) = 8 - 5 + 6 - 3 = 6.$
18. $(a - b) - (c - d) = (8 - 5) - (6 - 3) = 8 - 5 - 6 + 3 = 0.$
19. $(a - b) + (c + d) = (8 - 5) + (6 + 3) = 8 - 5 + 6 + 3 = 12.$
20. $(c + d) - (a - b) = (6 + 3) - (8 - 5) = 6 + 3 - 8 + 5 = 6.$

Exercise 4. Page 16

Perform the indicated operations, and find the numerical value of each expression if $a = 4$, $b = 3$, $c = 2$:

1. $5(ab + c) = 5ab + 5c = 5 \times 4 \times 3 + 5 \times 2 = 60 + 10 = 70.$
2. $4(ac + b) = 4ac + 4b = 4 \times 4 \times 2 + 4 \times 3 = 32 + 12 = 44.$
3. $3(a + bc) = 3a + 3bc = 3 \times 4 + 3 \times 3 \times 2 = 12 + 18 = 30.$
4. $7(ab - c) = 7ab - 7c = 7 \times 4 \times 3 - 7 \times 2 = 84 - 14 = 70.$
5. $6(ac - b) = 6ac - 6b = 6 \times 4 \times 2 - 6 \times 3 = 48 - 18 = 30.$
6. $5a(b - c) = 5ab - 5ac = 5 \times 4 \times 3 - 5 \times 4 \times 2 = 60 - 40 = 20.$

7. $2ab - a(bc - a) = 2ab - abc + a^2$
$= 2 \times 4 \times 3 - 4 \times 3 \times 2 + 4 \times 4 = 24 - 24 + 16 = 16.$

8. $c + 2ab(ac - b) = c + 2a^2bc - 2ab^2$
$= 2 + 2 \times 4^2 \times 3 \times 2 - 2 \times 4 \times 3^2 = 2 + 192 - 72 = 122.$

9. $3ac - c(b + c) = 3ac - bc - c^2$
$= 3 \times 4 \times 2 - 3 \times 2 - 2^2 = 24 - 6 - 4 = 14.$

10. $5ab - (b^2 + b) = 5ab - b^2 - b$
$= 5 \times 4 \times 3 - 3^2 - 3 = 60 - 9 - 3 = 48.$

11. $6bc - 4(ab - 3c) = 6bc - 4ab + 12c$
$= 6 \times 3 \times 2 - 4 \times 4 \times 3 + 12 \times 2 = 36 - 48 + 24 = 12.$

12. $ab + b(a - c) = ab + ab - bc$
$= 2ab - bc = 2 \times 4 \times 3 - 3 \times 2 = 24 - 6 = 18.$

13. $(a - c)b - bc = ab - bc - bc$
$= ab - 2bc = 4 \times 3 - 2 \times 3 \times 2 = 12 - 12 = 0.$

14. $(a - b)c + 3ac = ac - bc + 3ac$
$= 4ac - bc = 4 \times 4 \times 2 - 3 \times 2 = 32 - 6 = 26.$

15. $(2a + 3b)b - 2ab = 2ab + 3b^2 - 2ab = 3b^2 = 3 \times 3^2 = 27.$

16. $(a^2 - b^2)c - (a^2 - c^2) = a^2c - b^2c - a^2 + c^2$
$= 4^2 \times 2 - 3^2 \times 2 - 4^2 + 2^2 = 32 - 18 - 16 + 4 = 2.$

17. $(a^2 - c^2)b - (b^2 - c^2) = a^2b - bc^2 - b^2 + c^2$
$= 4^2 \times 3 - 3 \times 2^2 - 3^2 + 2^2 = 48 - 12 - 9 + 4 = 31.$

18. $2(bc + ac) - c(b^2 + c^2) = 2bc + 2ac - b^2c - c^3$
$= 2 \times 3 \times 2 + 2 \times 4 \times 2 - 3^2 \times 2 - 2^3 = 12 + 16 - 18 - 8 = 2.$

Exercise 5. Page 17

Perform the indicated operations, and find the numerical value of each quotient if $a = 8$, $b = 4$, $c = 2$:

1. $(a + b) \div b = \frac{a}{b} + 1 = \frac{8}{4} + 1 = 2 + 1 = 3.$

2. $(a + c) \div c = \frac{a}{c} + 1 = \frac{8}{2} + 1 = 4 + 1 = 5.$

3. $(a - b) \div b = \frac{a}{b} - 1 = \frac{8}{4} - 1 = 2 - 1 = 1.$

4. $(a - b) \div c = \frac{a}{c} - \frac{b}{c} = \frac{8}{2} - \frac{4}{2} = 4 - 2 = 2.$

5. $(ac+b)\div b=\frac{ac}{b}+1=\frac{8\times 2}{4}+1=4+1=5.$

6. $(ab+c)\div c=\frac{ab}{c}+1=\frac{8\times 4}{2}+1=16+1=17.$

7. $(ac-b)\div b=\frac{ac}{b}-1=\frac{8\times 2}{4}-1=4-1=3.$

8. $(ab-c)\div c=\frac{ab}{c}-1=\frac{8\times 4}{2}-1=16-1=15.$

9. $(b^2+c^2)\div a=\frac{b^2}{a}+\frac{c^2}{a}=\frac{4^2}{8}+\frac{2^2}{8}=2+\frac{1}{2}=2\frac{1}{2}.$

10. $(b^3-ac)\div b=b^2-\frac{ac}{b}=4^2-\frac{8\times 2}{4}=16-4=12.$

11. $(a^2-bc)\div c=\frac{a^2}{c}-b=\frac{8^2}{2}-4=32-4=28.$

12. $(b^2+ac)\div bc=\frac{b}{c}+\frac{a}{b}=\frac{4}{2}+\frac{8}{4}=2+2=4.$

Exercise 6. Page 21

Find the value of x and verify the answer:

1. $3x+1=19.$

Transpose, $3x=19-1.$
Combine, $3x=18.$
Divide by 3, $x=6.$

Verification

$$\begin{aligned}3\times 6+1&=19.\\ 18+1&=19.\\ 19&=19.\end{aligned}$$

2. $2x+5=11.$

Transpose, $2x=11-5.$
Combine, $2x=6.$
Divide by 2, $x=3.$

Verification

$$\begin{aligned}2\times 3+5&=11.\\ 6+5&=11.\\ 11&=11.\end{aligned}$$

3. $17x=7x+10.$

Transpose, $17x-7x=10.$
Combine, $10x=10.$
Divide by 10, $x=1.$

Verification

$$\begin{aligned}17\times 1&=7\times 1+10.\\ 17&=7+10.\\ 17&=17.\end{aligned}$$

4. $5x+2=17.$

Transpose, $5x=17-2.$
Combine, $5x=15.$
Divide by 5, $x=3.$

Verification

$$\begin{aligned}5\times 3+2&=17.\\ 15+2&=17.\\ 17&=17.\end{aligned}$$

5. $4x + 7 = 6x + 1.$

Transpose, $4x - 6x = 1 - 7.$
Change signs, $6x - 4x = 7 - 1.$
Combine, $2x = 6.$
Divide by 2, $x = 3.$

Verification

$$4 \times 3 + 7 = 6 \times 3 + 1.$$
$$12 + 7 = 18 + 1.$$
$$19 = 19.$$

6. $3x - 7 = 2x + 2.$

Transpose, $3x - 2x = 2 + 7.$
Combine, $x = 9.$

Verification

$$3 \times 9 - 7 = 2 \times 9 + 2.$$
$$27 - 7 = 18 + 2.$$
$$20 = 20.$$

7. $7x - 12 = 3x + 4.$

Transpose, $7x - 3x = 4 + 12.$
Combine, $4x = 16.$
Divide by 4, $x = 4.$

Verification

$$7 \times 4 - 12 = 3 \times 4 + 4.$$
$$28 - 12 = 12 + 4.$$
$$16 = 16.$$

8. $21x - 4 = 14x + 17.$

Transpose, $21x - 14x = 17 + 4.$
Combine, $7x = 21.$
Divide by 7, $x = 3.$

Verification

$$21 \times 3 - 4 = 14 \times 3 + 17.$$
$$63 - 4 = 42 + 17.$$
$$59 = 59.$$

9. $3x + 13 = 4x + 6.$

Transpose, $3x - 4x = 6 - 13.$
Change signs, $4x - 3x = 13 - 6.$
Combine, $x = 7.$

Verification

$$3 \times 7 + 13 = 4 \times 7 + 6.$$
$$21 + 13 = 28 + 6.$$
$$34 = 34.$$

10. $7x - 9 = 6x + 1.$

Transpose, $7x - 6x = 1 + 9.$
Combine, $x = 10.$

Verification

$$7 \times 10 - 9 = 6 \times 10 + 1.$$
$$70 - 9 = 60 + 1.$$
$$61 = 61.$$

11. $8x - 14 = 6x + 6.$

Transpose, $8x - 6x = 6 + 14.$
Combine, $2x = 20.$
Divide by 2, $x = 10.$

Verification

$$8 \times 10 - 14 = 6 \times 10 + 6.$$
$$80 - 14 = 60 + 6.$$
$$66 = 66.$$

12. $x - 3 = 15 - 2x.$

Transpose, $x + 2x = 15 + 3.$
Combine, $3x = 18.$
Divide by 3, $x = 6.$

Verification

$$6 - 3 = 15 - 2 \times 6.$$
$$6 - 3 = 15 - 12.$$
$$3 = 3.$$

13. $6x - 6 = 8 + 4x.$

Transpose, $6x - 4x = 8 + 6.$
Combine, $2x = 14.$
Divide by 2, $x = 7.$

Verification

$$6 \times 7 - 6 = 8 + 4 \times 7.$$
$$42 - 6 = 8 + 28.$$
$$36 = 36.$$

14. $x - 2 = 2x - 7.$

Transpose, $x - 2x = 2 - 7.$
Change signs, $2x - x = 7 - 2.$
Combine, $x = 5.$

Verification

$$5 - 2 = 2 \times 5 - 7.$$
$$5 - 2 = 10 - 7.$$
$$3 = 3.$$

15. $42 - 21x + 15 - 5x = 68 - 11 - 4x.$

Transpose, $4x - 21x - 5x = 68 - 11 - 42 - 15.$
Change signs, $21x + 5x - 4x = 11 + 42 + 15 - 68.$
Combine, $22x = 0.$
Divide by 22, $x = 0.$

Verification. $42 - 21 \times 0 + 15 - 5 \times 0 = 68 - 11 - 4 \times 0.$
$42 - 0 + 15 - 0 = 68 - 11 - 0.$
$57 = 57.$

16. $60 - 4x - 3x + 6x = 77 - 7x - 11.$

Transpose, $6x - 4x - 3x + 7x = 77 - 11 - 60.$
Combine, $6x = 6.$
Divide by 6, $x = 1.$

Verification. $60 - 4 \times 1 - 3 \times 1 + 6 \times 1 = 77 - 7 \times 1 - 11.$
$60 - 4 - 3 + 6 = 77 - 7 - 11.$
$59 = 59.$

17. $28x - 21 + 42 - 28x = 22 - 4x.$

Transpose, $28x - 28x + 4x = 22 + 21 - 42.$
Combine, $4x = 1.$
Divide by 4, $x = \frac{1}{4}.$

Verification. $28 \times \frac{1}{4} - 21 + 42 - 28 \times \frac{1}{4} = 22 - 4 \times \frac{1}{4}.$
$7 - 21 + 42 - 7 = 22 - 1.$
$21 = 21.$

18. $9(13 - x) - 4x = 5(21 - 2x) + 9x.$

Remove parentheses, $117 - 9x - 4x = 105 - 10x + 9x.$
Transpose, $10x - 9x - 9x - 4x = 105 - 117.$
Change signs, $9x + 9x + 4x - 10x = 117 - 105.$
Combine, $12x = 12.$
Divide by 12, $x = 1.$

Verification. $9(13-1)-4\times 1=5(21-2\times 1)+9\times 1.$

$$108-4=95+9.$$

$$104=104.$$

19. $6(x-5)+2x=8x-2(x+10).$

Remove parentheses, $6x-30+2x=8x-2x-20.$

Transpose, $6x+2x-8x+2x=30-20.$

Combine, $2x=10.$

Divide by 2, $x=5.$

Verification. $6(5-5)+2\times 5=8\times 5-2(5+10).$

$$0+10=40-30.$$

$$10=10.$$

20. $199+15x-(2x-5)=17(x+17)-13x-22.$

Remove parentheses,

$$199+15x-2x+5=17x+289-13x-22.$$

Transpose, $15x-2x-17x+13x=289-22-199-5.$

Combine, $9x=63.$

Divide by 9, $x=7.$

Verification. $199+15\times 7-(2\times 7-5)=17(7+17)-13\times 7-22.$

$$199+105-9=408-91-22.$$

$$295=295.$$

21. $8(3x-2)-7x-5(12-3x)+28=8(3x+2)-32.$

Remove parentheses,

$$24x-16-7x-60+15x+28=24x+16-32.$$

Transpose, $24x-7x+15x-24x=16-32+16+60-28.$

Combine, $8x=32.$

Divide by 8, $x=\underline{4}.$

Verification.

$$8(3\times 4-2)-7\times 4-5(12-3\times 4)+28=8(3\times 4+2)-32.$$

$$80-28-0+28=112-32.$$

$$80=80.$$

22. $7(3x-6)+5(x-3)+4(17-x)=77.$

Remove parentheses,

$$21x-42+5x-15+68-4x=77$$

Transpose, $21x+5x-4x=42+15-68+77.$

Combine, $22x=66.$

Divide by 22, $x=\underline{3}.$

Verification. $7(3 \times 3 - 6) + 5(3 - 3) + 4(17 - 3) = 77.$
$21 + 0 + 56 = 77.$
$77 = 77.$

23. $364 - 15(15x - 7) - 3(13 - 2x) + 8 = 6(x - 2).$

Remove parentheses,
$364 - 225x + 105 - 39 + 6x + 8 = 6x - 12.$
Transpose, $6x - 225x - 6x = 39 - 364 - 105 - 8 - 12.$
Change signs, $225x + 6x - 6x = 364 + 105 + 8 + 12 - 39.$
Combine, $225x = 450.$
Divide by 225, $x = 2.$

Verification. $364 - 15(15 \times 2 - 7) - 3(13 - 2 \times 2) + 8 = 6(2 - 2).$
$364 - 345 - 27 + 8 = 0.$
$0 = 0.$

24. $111 + 39x = 7(18 - 3x) + 3(20 - 5x).$

Remove parentheses, $111 + 39x = 126 - 21x + 60 - 15x.$
Transpose, $39x + 21x + 15x = 126 + 60 - 111.$
Combine, $75x = 75.$
Divide by 75, $x = 1.$

Verification. $111 + 39 \times 1 = 7(18 - 3 \times 1) + 3(20 - 5 \times 1).$
$111 + 39 = 105 + 45.$
$150 = 150.$

25. $8(x - 10) - 9 + 3(15 - x) + 2(18 - x) - 22 = 0.$

Remove parentheses,
$8x - 80 - 9 + 45 - 3x + 36 - 2x - 22 = 0.$
Transpose, $8x - 3x - 2x = 80 + 9 - 45 - 36 + 22.$
Combine, $3x = 30.$
Divide by 3, $x = 10.$

Verification. $8(10 - 10) - 9 + 3(15 - 10) + 2(18 - 10) - 22 = 0.$
$0 - 9 + 15 + 16 - 22 = 0.$
$0 = 0.$

26. $3(x + 5) - 5(x - 4) + 22 = 2(x + 6) + 3(11 - x) + 4.$

Remove parentheses,
$3x + 15 - 5x + 20 + 22 = 2x + 12 + 33 - 3x + 4.$
Transpose, $3x - 5x - 2x + 3x = 12 + 33 + 4 - 15 - 20 - 22.$
Change signs, $5x + 2x - 3x - 3x = 15 + 20 + 22 - 12 - 33 - 4.$
Combine, $x = 8.$

Verification. $3(8+5)-5(8-4)+22=2(8+6)+3(11-8)+4.$

$$39-20+22=28+9+4.$$

$$41=41.$$

27. $2(16-x)+3(5x-4)=12(3+x)-2(12-x).$

Remove parentheses, $32-2x+15x-12=36+12x-24+2x.$

Transpose, $15x-2x-12x-2x=36-24-32+12.$

Change signs, $2x+12x+2x-15x=24+32-36-12.$

Combine, $x=8.$

Verification. $2(16-8)+3(5\times 8-4)=12(3+8)-2(12-8).$

$$16+108=132-8.$$

$$124=124.$$

Exercise 7. Page 22

1. Write in symbols: a diminished by b; a increased by b; a multiplied by b; a divided by b; the square of a; the square root of a; the cube root of a; the square of a multiplied by the fourth power of b.

$$a-b;\ a+b;\ ab;\ \frac{a}{b};$$

$$a^2;\ \sqrt{a};\ \sqrt[3]{a};\ a^2b^4.$$

2. A man sold a horse for a dollars and lost b dollars on the cost. What did the horse cost him?

$$\$(a+b).$$

3. If a is an integral number, what integral number immediately precedes it? What integral number immediately follows it?

$$a-1;\ a+1.$$

4. A man bought a horse for m dollars and sold it for n dollars. If m is greater than n, how much did he lose?

$$\$(m-n).$$

5. A man sold a cow for c dollars and gained a dollars on the cost. What did he pay for the cow?

$$\$(c-a).$$

6. If the length of a day is t hours, what is the length of the night?

$(24-t)$ hours.

7. A rectangular field is a rods long and b rods wide. What is the area of the field?

ab square rods.

8. A farmer receives c cents for one bushel of corn. What does he receive for a bushels of corn?

ac cents.

9. A man pays b dollars for one cord of wood. What must he pay for a cords of wood?

ab dollars.

10. A boy runs a yards in one second. How many yards will he run in t seconds?

ta.

11. The area of a rectangular field is a square rods and the length of the field is b rods. What is the breadth?

$\frac{a}{b}$ rods.

12. A train travels c hours at the rate of a miles an hour. Find the distance traveled.

ca miles.

13. A horse goes c miles in b hours. How many miles does he go in one hour?

$$\frac{c}{b}.$$

14. The product is p and the multiplier is m. Find the multiplicand.

$$\frac{p}{m}.$$

15. The divisor is a, the quotient b, and the remainder c. Find the dividend.

$ab + c$.

16. The divisor is a, the dividend b, and the remainder c. Find the quotient.

$$\frac{b-c}{a}.$$

17. The difference between two numbers is a and the larger number is b. Find the smaller number.

$b - a$.

18. The difference between two numbers is a and the smaller number is b. Find the larger number.

$b + a$.

19. Write the excess of a over b.

$a - b$.

20. What is the excess of $5x + 10$ over $3x - 7$?

$$\begin{aligned} 5x + 10 - (3x - 7) &= 5x + 10 - 3x + 7 \\ &= 2x + 17. \end{aligned}$$

21. What is the excess of $3x - 16$ over $40 - 7x$?

$$\begin{aligned} 3x - 16 - (40 - 7x) &= 3x - 16 - 40 + 7x \\ &= 10x - 56. \end{aligned}$$

22. Express in cents the value of a quarters and b dimes.

$25a + 10b$.

23. A man has a dollars, b half dollars, and c quarters. He spends d half dollars and e quarters. How many cents has he left?

$$100a + 50b + 25c - (50d + 25e)$$
$$= 100a + 50b + 25c - 50d - 25e.$$

24. George is a years old to-day. How old was he b years ago? How old will he be c years hence?

$(a - b)$ years; $(a + c)$ years.

25. Frank is a years old to-day. In how many years will he be b times as old?

$ba - a$.

26. A boy can run a yards in b seconds. How many yards can he run in one second?

$$\frac{a}{b}.$$

27. A man has oats enough for n horses m days. How many days would the oats last a horses?

$$\frac{mn}{a}.$$

28. A cubical box is a inches long. Find the contents of the box.

a^3 cubic inches.

29. A gallon contains 231 cubic inches. How many gallons will a rectangular tank hold, a inches long, b inches wide, and c inches high?

$$\frac{abc}{231}.$$

30. How many bricks a inches long, b inches wide, and c inches high can be placed in a room l inches long, m inches wide, and n inches high?

$$\frac{lmn}{abc}.$$

Exercise 8. Page 27

1. The sum of two numbers is 60 and the greater is four times the less. Find the numbers.

Let $x =$ the larger number.

Then $60 - x =$ the smaller number,

and $4(60 - x) =$ four times the smaller number.

$\therefore x = 4(60 - x)$.

Remove parenthesis, $x = 240 - 4x$.

Transpose, $x + 4x = 240$.

Combine, $5x = 240$.

Divide by 5, $x = 48$.

$60 - x = 12$.

Therefore, the required numbers are 48 and 12.

2. The difference between two numbers is 12 and their sum is 76. Find the numbers.

Let $x =$ the smaller number.
Then $x + 12 =$ the larger number,
and $x + x + 12 =$ the sum of the numbers.
But $76 =$ the sum of the numbers.
$\therefore x + x + 12 = 76.$
Transpose, $x + x = 76 - 12.$
Combine, $2x = 64.$
Divide by 2, $x = 32.$
$x + 12 = 44.$

Therefore, the required numbers are 44 and 32.

3. The difference between two numbers is 11 and their sum is 97. Find the numbers.

Let $x =$ the smaller number.
Then $x + 11 =$ the larger number,
and $x + x + 11 =$ the sum of the numbers.
But $97 =$ the sum of the numbers.
$\therefore x + x + 11 = 97.$
Transpose, $x + x = 97 - 11.$
Combine, $2x = 86.$
Divide by 2, $x = 43.$
$x + 11 = 54.$

Therefore, the required numbers are 54 and 43.

4. Three times a certain number is equal to the number increased by 24. Find the number.

Let $x =$ the number.
Then $3x =$ three times the number,
and $x + 24 =$ the number increased by 24.
$\therefore 3x = x + 24.$
Transpose, $3x - x = 24.$
Combine, $2x = 24.$
Divide by 2, $x = 12.$

Therefore, the required number is 12.

5. The sum of two numbers is 39 and the larger exceeds the smaller by 9. Find the numbers.

Let $x =$ the smaller number.
Then $x + 9 =$ the larger number,
and $x + x + 9 =$ the sum of the numbers.
But $39 =$ the sum of the numbers.
$\therefore x + x + 9 = 39.$
Transpose, $x + x = 39 - 9.$
Combine, $2x = 30.$
Divide by 2, $x = 15.$
$x + 9 = 24.$

Therefore, the required numbers are 24 and 15.

6. The sum of two numbers is 63 and the larger exceeds twice the smaller by 3. Find the numbers.

Let $x =$ the smaller number.
Then $63 - x =$ the larger number,
and $2x + 3 =$ twice the smaller number plus 3.
But $2x + 3 =$ the larger number.
$\therefore 2x + 3 = 63 - x.$
Transpose, $2x + x = 63 - 3.$
Combine, $3x = 60.$
Divide by 3, $x = 20.$
$63 - x = 43.$

Therefore, the required numbers are 43 and 20.

7. The sum of two numbers is 49 and twice the smaller exceeds the larger by 14. Find the numbers.

Let $x =$ the smaller number.
Then $49 - x =$ the larger number,
and $2x =$ twice the smaller number.
$\therefore 2x - (49 - x) = 14.$
Remove the parenthesis,
$2x - 49 + x = 14.$
Transpose, $2x + x = 14 + 49.$
Combine, $3x = 63.$
Divide by 3, $x = 21.$
$49 - x = 28.$

Therefore, the required numbers are 28 and 21.

✓ **8.** Find three consecutive numbers whose sum is 39.

Let $x =$ the middle number.
Then $x - 1 =$ the smallest number,
$x + 1 =$ the largest number,
and $x + x - 1 + x + 1 =$ the sum of the numbers.
But $39 =$ the sum of the numbers.
$\therefore x + x - 1 + x + 1 = 39.$
Transpose, $x + x + x = 39 + 1 - 1.$
Combine, $3x = 39.$
Divide by 3, $x = 13.$
$x - 1 = 12.$
$x + 1 = 14.$

Therefore, the required numbers are 12, 13, and 14.

✓ **9.** Find five consecutive numbers whose sum is 70.

Let $x =$ the middle number.
Then $x - 1 =$ the next smaller number,
$x - 2 =$ the smallest number,
$x + 1 =$ the next larger number,
$x + 2 =$ the largest number,
and $x + x - 1 + x - 2 + x + 1 + x + 2 =$ the sum of the numbers.
But $70 =$ the sum of the numbers.
$\therefore x + x - 1 + x - 2 + x + 1 + x + 2 = 70.$
Transpose, $x + x + x + x + x = 70 + 1 + 2 - 1 - 2.$
Combine, $5x = 70.$
Divide by 5, $x = 14.$
$x - 2 = 12.$
$x - 1 = 13.$
$x + 1 = 15.$
$x + 2 = 16.$

Therefore, the required numbers are 12, 13, 14, 15, and 16.

✓ **10.** If a certain number is multiplied by 12, the product is 168. Find the number.

Let $x =$ the number.
Then $12x =$ the number multiplied by 12.
But $168 =$ the number multiplied by 12.
$\therefore 12x = 168.$
Divide by 12, $x = 14.$

Therefore, the required number is 14.

11. The difference of two numbers is 22 and three times the smaller exceeds twice the larger by 6. Find the numbers.

Let $x =$ the larger number.
Then $x - 22 =$ the smaller number,
$3(x - 22) =$ three times the smaller number,
$2x =$ twice the larger number,
and $3(x - 22) - 2x =$ the excess of three times the smaller over twice the larger number.
But $6 =$ this excess.
$\therefore 3(x - 22) - 2x = 6.$
Remove the parenthesis,
$3x - 66 - 2x = 6.$
Transpose, $3x - 2x = 6 + 66.$
Combine, $x = 72.$
$x - 22 = 50.$

Therefore, the required numbers are 72 and 50.

12. Find a number such that when 9 is added to three times the number the sum is 42.

Let $x =$ the number.
Then $3x =$ three times the number,
and $3x + 9 =$ three times the number plus 9.
But $42 =$ this sum.
$\therefore 3x + 9 = 42.$
Transpose, $3x = 42 - 9.$
Combine, $3x = 33.$
Divide by 3, $x = 11.$

Therefore, the required number is 11.

13. A man sold a quantity of wood for \$49, half of it at \$3 a cord and the other half at \$4 a cord. How many cords of wood did he sell?

Let $2x =$ the number of cords of wood.
Then $x =$ half the number of cords of wood,
$3x =$ the number of dollars received for the wood sold at \$3 a cord,
$4x =$ the number of dollars received for the wood sold at \$4 a cord,
and $3x + 4x =$ the number of dollars received for the wood.
But $49 =$ the number of dollars received for the wood.

$\therefore 3x + 4x = 49.$

Combine, $7x = 49.$

Divide by 7, $x = 7.$

$2x = 14.$

Therefore, the man sold 14 cords of wood.

14. The sum of the ages of a father and son is 42 years, and the father is five times as old as the son. What is the age of each?

Let $x =$ the number of years in the age of the son.
Then $5x =$ the number of years in the age of the father,
and $5x + x =$ the number of years in the sum of the ages.
But $42 =$ the number of years in the sum of the ages.

$\therefore 5x + x = 42.$

Combine, $6x = 42.$

Divide by 6, $x = 7.$

$5x = 35.$

Therefore, the father is 35 years old and the son 7 years old.

15. A tree 120 feet high was broken so that the length of the part broken off was four times the length of the part left standing. Find the length of each part.

Let $x =$ the number of feet in the part left standing.
Then $4x =$ the number of feet in the part broken off,
and $x + 4x =$ the number of feet in the whole tree.
But $120 =$ the number of feet in the whole tree.

$\therefore x + 4x = 120.$

Combine, $5x = 120.$

Divide by 5, $x = 24.$

$4x = 96.$

Therefore, the length of the part broken off was 96 feet; of the part left standing 24 feet.

16. Two men start from the same place and travel in *opposite* directions, one 35 miles a day and the other 25 miles a day. In how many days will they be 360 miles apart?

Let $x =$ the required number of days.
Then $35x =$ the number of miles the first travels,
$25x =$ the number of miles the second travels,
and $35x + 25x =$ the number of miles they together travel.
But $360 =$ the number of miles they together travel.

$\therefore 35x + 25x = 360.$

Combine, $60x = 360.$

Divide by 60, $x = 6.$

Therefore, the required number of days is 6.

17. Two men start from the same place and travel in the *same* direction, one 35 miles a day and the other 25 miles a day. In how many days will they be 360 miles apart?

Let $x =$ the required number of days.

Then $35x =$ the number of miles the first travels,

$25x =$ the number of miles the second travels,

and $35x - 25x =$ the number of miles between them in x days.

But $360 =$ the number of miles between them in x days.

$\therefore 35x - 25x = 360.$

Combine, $10x = 360.$

Divide by 10, $x = 36.$

Therefore, the required number of days is 36.

18. Divide \$21 among A, B, and C so that A and B shall each receive three times as much as C.

Let $x =$ the number of dollars C receives.

Then $3x =$ the number of dollars A receives,

$3x =$ the number of dollars B receives,

and $x + 3x + 3x =$ the number of dollars A, B, and C together receive.

But $21 =$ the number of dollars A, B, and C together receive.

$\therefore x + 3x + 3x = 21.$

Combine, $7x = 21.$

Divide by 7, $x = 3.$

$3x = 9.$

Therefore, A receives \$9; B, \$9; and C, \$3.

19. A, B, and C buy a summer cottage for \$3000. B pays twice as much as A, and C pays as much as A and B together. How much does each pay?

Let $x =$ the number of dollars A pays.

Then $2x =$ the number of dollars B pays,

$x + 2x =$ the number of dollars C pays,

and $x + 2x + x + 2x =$ the number of dollars A, B, and C together pay.

But $3000 =$ the number of dollars A, B, and C together pay.

$$\therefore x + 2x + x + 2x = 3000.$$

Combine, $6x = 3000.$

Divide by 6, $x = 500.$

$$2x = 1000.$$

$$x + 2x = 1500.$$

Therefore, A pays $500; B, $1000; and C, $1500.

20. Divide 48 into two parts such that one part shall exceed the other by 6.

Let $x =$ the greater part.

Then $x - 6 =$ the smaller part,

and $x + x - 6 =$ the whole number.

But $48 =$ the whole number.

$$\therefore x + x - 6 = 48.$$

Transpose, $x + x = 48 + 6.$

Combine, $2x = 54.$

Divide by 2, $x = 27.$

$$x - 6 = 21.$$

Therefore, the required parts are 27 and 21.

21. A father is twice as old as his son; 11 years ago he was three times as old as his son. Find the age of each.

Let $x =$ the number of years in the age of the son.

Then $2x =$ the number of years in the age of the father,

$x - 11 =$ the number of years in the age of the son 11 years ago,

and $2x - 11 =$ the number of years in the age of the father 11 years ago.

But 11 years ago the father was three times as old as the son.

$$\therefore 2x - 11 = 3(x - 11).$$

Remove the parenthesis, $2x - 11 = 3x - 33.$

Transpose, $2x - 3x = 11 - 33.$

Change the signs, $3x - 2x = 33 - 11.$

Combine, $x = 22.$

$$2x = 44.$$

Therefore, the father is 44 years old and the son 22 years old.

22. A man is four times as old as his son; in 18 years he will be only twice as old. Find the age of each.

Let $x =$ the number of years in the age of the son.

Then $4x =$ the number of years in the age of the father,

$x + 18 =$ the number of years in the age of the son 18 years hence,

and $4x + 18 =$ the number of years in the age of the father 18 years hence.

But in 18 years the father will be twice as old as the son.

$$\therefore 4x + 18 = 2(x + 18).$$

Remove the parenthesis, $4x + 18 = 2x + 36.$

Transpose, $4x - 2x = 36 - 18.$

Combine, $2x = 18.$

Divide by 2, $x = 9.$

$$4x = 36.$$

Therefore, the father is 36 years old and the son 9 years old.

23. A dealer bought 25 dozen oranges for $9.25. For a part he paid 40 cents a dozen and for the remainder 35 cents a dozen. How many dozen of each kind did he buy?

Let $x =$ the number of dozen he bought at 40 cents.

Then $25 - x =$ the number of dozen he bought at 35 cents,

$40x =$ the number of cents he paid for the first kind,

$35(25 - x) =$ the number of cents he paid for the second kind,

and $40x + 35(25 - x) =$ the number of cents he paid in all.

But $925 =$ the number of cents he paid in all.

$$\therefore 40x + 35(25 - x) = 925.$$

Remove the parenthesis,

$$40x + 875 - 35x = 925.$$

Transpose, $40x - 35x = 925 - 875.$

Combine, $5x = 50.$

Divide by 5, $x = 10.$

$$25 - x = 15.$$

Therefore, the dealer bought 10 dozen at 40 cents and 15 dozen at 35 cents.

24. I have $3.85 in quarters and ten-cent pieces and I have three times as many ten-cent pieces as quarters. How many coins of each kind have I?

Let $x =$ the number of quarters.

Then $3x =$ the number of ten-cent pieces,

$25x =$ the number of cents in the value of the quarters,

$30x =$ the number of cents in the value of the ten-cent pieces,

and $25x + 30x =$ the number of cents in the value of all the coins.

But $385 =$ the number of cents in the value of all the coins.

$$\therefore 25x + 30x = 385.$$

Combine, $55x = 385.$

Divide by 55, $x = 7.$

$3x = 21.$

Therefore, I have 7 quarters and 21 ten-cent pieces.

25. Divide 75 into two parts such that one part shall exceed twice the other by 9.

Let $x =$ the smaller part.

Then $75 - x =$ the larger part,

$2x =$ twice the smaller part,

and $75 - x - 2x =$ the excess of the larger over twice the smaller part.

But $9 =$ the excess of the larger over twice the smaller part.

$$\therefore 75 - x - 2x = 9.$$

Transpose, $75 - 9 = x + 2x.$

Transpose, $x + 2x = 75 - 9.$

Combine, $3x = 66.$

Divide by 3, $x = 22.$

$75 - x = 53.$

Therefore, the required parts are 53 and 22.

26. Two trains start at the same time from two cities 450 miles apart and travel towards each other, one at the rate of 50 miles an hour and the other at the rate of 40 miles an hour. In how many hours will the trains meet?

Let $x =$ the number of hours in which the trains will meet.

Then $50x =$ the number of miles the first travels,

$40x =$ the number of miles the second travels,

and $50x + 40x =$ the number of miles the two together travel.

But $450 =$ the number of miles the two together travel.

$$\therefore 50x + 40x = 450.$$

Combine, $90x = 450.$

Divide by 90, $x = 5.$

Therefore, the trains will meet in 5 hours.

27. A dealer buys 30 tons of coal and 40 cords of wood for $400. The coal costs twice as much per ton as the wood costs per cord. How much does he pay for a ton of coal and how much for a cord of wood?

Let x = the number of dollars in the cost of 1 cord of wood.
Then $2x$ = the number of dollars in the cost of 1 ton of coal,
$60x$ = the number of dollars paid for the coal,
$40x$ = the number of dollars paid for the wood,
and $60x + 40x$ = the number of dollars paid in all.
But 400 = the number of dollars paid in all.

$$\therefore 60x + 40x = 400.$$

Combine, $100x = 400.$
Divide by 100, $x = 4.$
$2x = 8.$

Therefore, the dealer pays $8 for a ton of coal and $4 for a cord of wood.

28. A man was hired for 50 days. Each day he worked he was to receive $2, and each day he was idle he was to pay 50 cents. At the end of the 50 days he received $80. How many days did he work?

Let x = the number of days he worked.
Then $50 - x$ = the number of days he was idle,
$200x$ = the number of cents he received for the days he worked,
$50(50 - x)$ = the number of cents he paid for the days he was idle,
and $200x - 50(50 - x)$ = the net number of cents he received.
But 8000 = the net number of cents he received.

$$\therefore 200x - 50(50 - x) = 8000.$$

Remove the parenthesis,

$$200x - 2500 + 50x = 8000.$$

Transpose, $200x + 50x = 8000 + 2500.$
Combine, $250x = 10500.$
Divide by 250, $x = 42.$

Therefore, the man worked 42 days.

29. Ten men agreed to buy a camp together, but two declined to take their share and each of the others had to pay $5 more for his share. What was the cost of the camp?

Let x = the number of dollars each agreed to pay.
Then $10x$ = the number of dollars in the cost of the camp,
$x + 5$ = the number of dollars each of eight did pay,
and $8(x + 5)$ = the number of dollars in the cost of the camp.

$$\therefore 10x = 8(x + 5).$$

Remove the parenthesis, $10x = 8x + 40.$

Transpose, $10x - 8x = 40.$

Combine, $2x = 40.$

Divide by 2, $x = 20.$

$$10x = 200.$$

Therefore, the camp cost $200.

30. Divide 37 into two parts such that the sum of twice the greater part and three times the smaller shall be 87.

Let $x =$ the greater part.
Then $37 - x =$ the smaller part,
$2x =$ twice the greater part,
$3(37 - x) =$ three times the smaller part,
and $2x + 3(37 - x) =$ the sum of twice the greater part and three times the smaller part.
But $87 =$ this sum.

$$\therefore 2x + 3(37 - x) = 87.$$

Remove the parenthesis,

$$2x + 111 - 3x = 87.$$

Transpose, $2x - 3x = 87 - 111.$

Change the signs, $3x - 2x = 111 - 87.$

Combine, $x = 24.$

$$37 - x = 13.$$

Therefore, the required parts are 24 and 13.

31. The sum of two numbers is 35 and three times the larger number is equal to four times the smaller. Find the numbers.

Let $x =$ the larger number.
Then $35 - x =$ the smaller number,
$3x =$ three times the larger number,
and $4(35 - x) =$ four times the smaller number.

$$\therefore 3x = 4(35 - x).$$

Remove the parenthesis, $3x = 140 - 4x.$

Transpose, $3x + 4x = 140.$

Combine, $7x = 140.$

Divide by 7, $x = 20.$

$$35 - x = 15.$$

Therefore, the required numbers are 20 and 15.

32. The sum of two numbers is 72 and twice the larger number exceeds four times the smaller by 12. Find the numbers.

Let $x =$ the larger number.
Then $72 - x =$ the smaller number,
$2x =$ twice the larger number,
and $4(72 - x) =$ four times the smaller number.

$$\therefore 2x = 4(72 - x) + 12.$$

Remove the parenthesis, $2x = 288 - 4x + 12.$
Transpose, $2x + 4x = 288 + 12.$
Combine, $6x = 300.$
Divide by 6, $x = 50.$
$72 - x = 22.$

Therefore, the required numbers are 50 and 22.

33. Four years ago a man was seven times as old as his son and 16 years hence he will be only twice as old as his son. Find the age of each.

Let $x =$ the number of years in the age of the son 4 years ago.
Then $7x =$ the number of years in the age of the father 4 years ago,
$x + 4 =$ the number of years in the age of the son now,
$7x + 4 =$ the number of years in the age of the father now,
$x + 4 + 16 =$ the number of years in the age of the son 16 years hence,
and $7x + 4 + 16 =$ the number of years in the age of the father 16 years hence.

But 16 years hence the father will be twice as old as the son.

$$\therefore 7x + 4 + 16 = 2(x + 4 + 16).$$

Remove the parenthesis,

$$7x + 4 + 16 = 2x + 8 + 32.$$

Transpose, $7x - 2x = 8 + 32 - 4 - 16.$
Combine, $5x = 20.$
Divide by 5, $x = 4.$
$x + 4 = 8.$
$7x + 4 = 32.$

Therefore, the father is 32 years old and the son is 8 years old.

34. Ten years ago A was three times as old as B and ten years hence he will be 18 years less than twice as old as B. Find the age of each.

Let $x =$ the number of years in B's age 10 years ago.
Then $3x =$ the number of years in A's age 10 years ago,
$x + 10 =$ the number of years in B's age now,
$3x + 10 =$ the number of years in A's age now,
$x + 10 + 10 =$ the number of years in B's age 10 years hence,
and $3x + 10 + 10 =$ the number of years in A's age 10 years hence.

But 10 years hence A's age will be 18 years less than twice B's age.

$$\therefore 3x + 10 + 10 = 2(x + 10 + 10) - 18.$$

Remove the parenthesis,

$$3x + 10 + 10 = 2x + 20 + 20 - 18.$$

Transpose, $3x - 2x = 20 + 20 - 18 - 10 - 10.$
Combine, $x = 2.$
$x + 10 = 12.$
$3x + 10 = 16.$

Therefore, A is 16 years old and B is 12 years old.

35. Four times the excess of a certain number over 9 is equal to 6 less than twice the number. Find the number.

Let $x =$ the number.
Then $4(x - 9) =$ four times the excess of the number over 9,
and $2x - 6 =$ 6 less than twice the number.

But these two numbers are equal.

$$\therefore 4(x - 9) = 2x - 6.$$

Remove the parenthesis, $4x - 36 = 2x - 6.$
Transpose, $4x - 2x = 36 - 6.$
Combine, $2x = 30.$
Divide by 2, $x = 15.$

Therefore, the required number is 15.

36. A can hoe 4 rows in an hour; B, 3 rows; and C, 2 rows. How many hours will it take the three together to hoe 126 rows?

Let $x =$ the number of hours required.
Then $4x =$ the number of rows A will hoe,
$3x =$ the number of rows B will hoe,
$2x =$ the number of rows C will hoe,
and $4x + 3x + 2x =$ the number of rows the three together will hoe.
But $126 =$ the number of rows the three together will hoe.

$\therefore 4x + 3x + 2x = 126.$

Combine, $9x = 126.$

Divide by 9, $x = 14.$

Therefore, it will take the three together 14 hours to hoe 126 rows.

37. A cistern that will hold 900 gallons has three pipes. The first lets in 7 gallons a minute, the second 15 gallons, and the third 23 gallons. In how many minutes will the three pipes fill the cistern?

Let $x =$ the number of minutes it takes the three pipes to fill the cistern.

Then $7x =$ the number of gallons let in by the first pipe,

$15x =$ the number of gallons let in by the second pipe,

$23x =$ the number of gallons let in by the third pipe,

and $7x + 15x + 23x =$ the number of gallons let in by the three pipes together.

But $900 =$ the number of gallons let in by the three pipes together.

$\therefore 7x + 15x + 23x = 900.$

Combine, $45x = 900.$

Divide by 45, $x = 20.$

Therefore, the three pipes together will fill the cistern in 20 minutes.

38. A tree 90 feet high was broken so that the length of the part broken off was five times the length of the part left standing. Find the length of each part.

Let $x =$ the number of feet in the length of the part left standing.

Then $5x =$ the number of feet in the length of the part broken off,

and $x + 5x =$ the number of feet in the length of the whole tree.

But $90 =$ the number of feet in the length of the whole tree.

$\therefore x + 5x = 90.$

Combine, $6x = 90.$

Divide by 6, $x = 15.$

$5x = 75.$

Therefore, the length of the part broken off is 75 feet and the length of the part left standing is 15 feet.

39. A man has \$1.10 in dimes and nickels and he has 15 coins in all. How many coins of each kind has he?

Let $x =$ the number of dimes.
Then $15 - x =$ the number of nickels,
$10x =$ the number of cents in the dimes,
$5(15 - x) =$ the number of cents in the nickels,
and $10x + 5(15 - x) =$ the number of cents in the whole.
But $110 =$ the number of cents in the whole.

$$\therefore 10x + 5(15 - x) = 110.$$

Remove the parenthesis,

$$10x + 75 - 5x = 110.$$

Transpose, $10x - 5x = 110 - 75.$
Combine, $5x = 35.$
Divide by 5, $x = 7.$
$15 - x = 8.$

Therefore, the man has 7 dimes and 8 nickels.

40. A man has 9 dollars in half dollars and quarters and he has four times as many quarters as half dollars. How many coins of each kind has he?

Let $x =$ the number of half dollars.
Then $4x =$ the number of quarters,
$50x =$ the number of cents in the half dollars,
$100x =$ the number of cents in the quarters,
and $50x + 100x =$ the number of cents in the whole.
But $900 =$ the number of cents in the whole.

$$\therefore 50x + 100x = 900.$$

Combine, $150x = 900.$
Divide by 150, $x = 6.$
$4x = 24.$

Therefore, the man has 6 half dollars and 24 quarters.

41. A man has 12 hours for an excursion. How far into the country can he ride a bicycle at the rate of 15 miles an hour so as to return in time, driving a horse at the rate of 5 miles an hour?

Let $x =$ the number of hours he can ride the bicycle.
Then $12 - x =$ the number of hours he can drive the horse,
$15x =$ the number of miles he will ride the bicycle,
and $5(12 - x) =$ the number of miles he will drive the horse.

But these two distances are equal.

$$\therefore 15x = 5(12 - x).$$

Remove the parenthesis, $15x = 60 - 5x.$

Transpose, $15x + 5x = 60.$

Combine, $20x = 60.$

Divide by 20, $x = 3.$

$$15x = 45.$$

Therefore, the man can ride a bicycle 45 miles into the country.

42. A, whose horse travels at the rate of 10 miles an hour, starts 2 hours after B, whose horse travels at the rate of 8 miles an hour. How many miles must A drive to overtake B?

Let x = the number of hours A must drive.

Then $x + 2$ = the number of hours B drives,

$10x$ = the number of miles A drives,

and $8(x + 2)$ = the number of miles B drives.

But A and B drive the same distance.

$$\therefore 10x = 8(x + 2).$$

Remove the parenthesis, $10x = 8x + 16.$

Transpose, $10x - 8x = 16.$

Combine, $2x = 16.$

Divide by 2, $x = 8.$

$$10x = 80.$$

Therefore, A must drive 80 miles.

43. If a certain number is diminished by 9 and the remainder multiplied by 9, the result is the same as if the number were diminished by 6 and the remainder multiplied by 6. Find the number.

Let x = the number.

Then $9(x - 9)$ = the number diminished by 9 and the remainder multiplied by 9,

and $6(x - 6)$ = the number diminished by 6 and the remainder multiplied by 6.

But these two numbers are equal.

$$\therefore 9(x - 9) = 6(x - 6).$$

Remove the parentheses,

$$9x - 81 = 6x - 36.$$

Transpose, $9x - 6x = 81 - 36.$

Combine, $3x = 45.$

Divide by 3, $x = 15.$

Therefore, the required number is 15.

44. If 5 times a certain number is diminished by 3 and the remainder multiplied by 23, the result is the same as if 7 times the number were increased by 9 and the sum multiplied by 7. Find the number.

Let $x =$ the number.

Then $23(5x - 3) =$ 23 times the remainder obtained by diminishing 5 times the number by 3,

and $7(7x + 9) =$ 7 times the sum obtained by increasing 7 times the number by 9.

But these two numbers are equal.

$$\therefore 23(5x - 3) = 7(7x + 9).$$

Remove the parentheses,

$$115x - 69 = 49x + 63.$$

Transpose, $115x - 49x = 63 + 69.$

Combine, $66x = 132.$

Divide by 66, $x = 2.$

Therefore, the required number is 2.

45. A and B have together \$20; A and C, \$22; B and C, \$26. How much has each?

Let $x =$ the number of dollars A has.

Then $20 - x =$ the number of dollars B has,

$22 - x =$ the number of dollars C has,

and $20 - x + 22 - x =$ the number of dollars B and C have together.

But $26 =$ the number of dollars B and C have together.

$$\therefore 26 = 20 - x + 22 - x.$$

Transpose, $x + x = 20 + 22 - 26.$

Combine, $2x = 16.$

Divide by 2, $x = 8.$

$$20 - x = 12.$$

$$22 - x = 14.$$

Therefore, A has \$8; B, \$12; and C, \$14.

46. The sum of the ages of a father and his son is 60 years, and the father's age is 3 years more than twice the age of his son. Find the age of each.

Let $x =$ the number of years in the age of the son.

Then $2x + 3 =$ the number of years in the age of the father,

and $x + 2x + 3 =$ the number of years in the sum of their ages.

But $60 =$ the number of years in the sum of their ages.

$\therefore x + 2x + 3 = 60.$

Transpose, $x + 2x = 60 - 3.$

Combine, $3x = 57.$

Divide by 3, $x = 19.$

$2x + 3 = 41.$

Therefore, the father is 41 years old and the son is 19 years old.

47. A farmer bought 16 sheep. If he had bought 4 sheep more for the same money, each sheep would have cost him one dollar less. How much did he pay for a sheep?

Let $x =$ the number of dollars he paid for 1 sheep.
Then $16x =$ the number of dollars he paid for the sheep,
and $20(x - 1) =$ the number of dollars he would have paid for the sheep if he had bought 4 more and paid for each sheep 1 dollar less.

$\therefore 20(x - 1) = 16x.$

Remove the parenthesis,

$20x - 20 = 16x.$

Transpose, $20x - 16x = 20.$

Combine, $4x = 20.$

Divide by 4, $x = 5.$

Therefore, the farmer paid $5 for each sheep.

48. A man has $64 in five-dollar bills and one-dollar bills. He has three times as many five-dollar bills as one-dollar bills. How many bills of each kind has he?

Let $x =$ the number of one-dollar bills.
Then $3x =$ the number of five-dollar bills.
$x =$ the number of dollars in the one-dollar bills,
$15x =$ the number of dollars in the five-dollar bills,
and $x + 15x =$ the number of dollars in all.
But $64 =$ the number of dollars in all.

$\therefore x + 15x = 64.$

Combine, $16x = 64.$

Divide by 16, $x = 4.$

$3x = 12.$

Therefore, the man has 12 five-dollar bills and 4 one-dollar bills.

49. The current of a river runs 2 miles an hour. A man can paddle a canoe a certain distance up the river in 7 hours and the same distance down in 3 hours. How many miles an hour can he paddle in still water?

Let $x =$ the number of miles per hour he can paddle in still water.

Then $x - 2 =$ the number of miles per hour he can paddle upstream,

$x + 2 =$ the number of miles per hour he can paddle downstream,

$7(x - 2) =$ the number of miles he paddles upstream,

and $3(x + 2) =$ the number of miles he paddles downstream.

But these two distances are equal.

$$\therefore 7(x - 2) = 3(x + 2).$$

Remove the parentheses, $7x - 14 = 3x + 6.$

Transpose, $7x - 3x = 6 + 14.$

Combine, $4x = 20.$

Divide by 5, $x = 5.$

Therefore, the man can paddle 5 miles per hour in still water.

50. An express train, which travels 45 miles an hour, starts 2 hours after an accommodation train, which it overtakes in 4 hours. What is the rate of the accommodation train?

Let $x =$ the number of miles per hour the accommodation train goes.

Then $6x =$ the number of miles the accommodation train goes,

and $4 \times 45 = 180 =$ the number of miles the express train goes.

But the trains travel the same distance.

$$\therefore 6x = 180.$$

Divide by 6, $x = 30.$

Therefore, the accommodation train travels 30 miles per hour.

51. A sum of money was divided among A, B, and C in such a way that A received four times as much as B, and B twice as much as C. If A received $35 more than C, how great a sum of money was divided?

Let $x =$ the number of dollars C received.

Then $2x =$ the number of dollars B received,

$8x =$ the number of dollars A received,

$11x =$ the number of dollars that was divided,

and $8x - x =$ the number of dollars A received more than C.

But $35 =$ the number of dollars A received more than C.

$$\therefore 8x - x = 35.$$

Combine, $7x = 35.$

Divide by 7, $x = 5.$

$$11x = 55.$$

Therefore, the sum of $55 was divided.

52. At an election there were 2 candidates and 1280 votes were cast. The successful candidate had a majority of 40. How many votes were cast for each candidate?

Let $x =$ the number of votes the successful candidate received.

Then $x - 40 =$ the number of votes the unsuccessful candidate received,

and $x + x - 40 =$ the number of votes cast.

But $1280 =$ the number of votes cast.

$$\therefore x + x - 40 = 1280.$$

Transpose, $x + x = 1280 + 40.$

Combine, $2x = 1320.$

Divide by 2, $x = 660.$

$$x - 40 = 620.$$

Therefore, the candidates received 660 votes and 620 votes respectively.

53. Three times the excess of a certain number over 8 is equal to twice the number plus 1. Find the number.

Let $x =$ the number.

Then $3(x - 8) =$ three times the excess of the number over 8,

and $2x + 1 =$ twice the number plus 1.

But these two numbers are equal.

$$\therefore 3(x - 8) = 2x + 1.$$

Remove the parenthesis, $3x - 24 = 2x + 1.$

Transpose, $3x - 2x = 1 + 24.$

Combine, $x = 25.$

Therefore, the required number is 25.

54. A man has $56 in ten-dollar bills and one-dollar bills. He has four times as many one-dollar bills as ten-dollar bills. How many bills of each kind has he?

Let $x =$ the number of ten-dollar bills.

Then $4x =$ the number of one-dollar bills,

$10x =$ the number of dollars in the ten-dollar bills,

$4x =$ the number of dollars in the one-dollar bills,

and $10x + 4x =$ the number of dollars in all.

But $56 =$ the number of dollars in all.

$$\therefore 10x + 4x = 56.$$

Combine, $14x = 56.$

Divide by 14, $x = 4.$

$$4x = 16.$$

Therefore, the man has 4 ten-dollar bills and 16 one-dollar bills.

55. Nine times a certain number exceeds 56 by as much as five times the number is less than 56. Find the number.

Let $x =$ the number.

Then $9x - 56 =$ the excess of nine times the number over 56,

and $56 - 5x =$ the excess of 56 over five times the number.

But these two numbers are equal.

$$\therefore 9x - 56 = 56 - 5x.$$

Transpose, $9x + 5x = 56 + 56.$

Combine, $14x = 112.$

Divide by 14, $x = 8.$

Therefore, the required number is 8.

56. Divide 39 into two parts such that the greater part exceeds twice the smaller by 1 less than twice the smaller part.

Let $x =$ the greater part.

Then $39 - x =$ the smaller part,

$x - 2(39 - x) =$ the excess of the greater part over twice the smaller part,

and $2(39 - x) - 1 = 1$ less than twice the smaller part.

But these last two numbers are equal.

$$\therefore x - 2(39 - x) = 2(39 - x) - 1.$$

Remove the parentheses,

$$x - 78 + 2x = 78 - 2x - 1.$$

Transpose, $x + 2x + 2x = 78 + 78 - 1.$

Combine, $5x = 155.$

Divide by 5, $x = 31.$

$$39 - x = 8.$$

Therefore, the required parts are 31 and 8.

57. Ten years ago A was three times as old as B, and 20 years ago A was five times as old as B. Find the age of each.

Let x = the number of years in B's age 10 years ago.
Then $3x$ = the number of years in A's age 10 years ago,
$x + 10$ = the number of years in B's age now,
$3x + 10$ = the number of years in A's age now,
$x + 10 - 20$ = the number of years in B's age 20 years ago,
and $3x + 10 - 20$ = the number of years in A's age 20 years ago.

But 20 years ago A was five times as old as B.

$$\therefore 3x + 10 - 20 = 5(x + 10 - 20).$$

Remove the parenthesis,

$$3x + 10 - 20 = 5x + 50 - 100.$$

Transpose, $3x - 5x = 50 - 100 - 10 + 20.$
Change the signs, $5x - 3x = 100 - 50 + 10 - 20.$
Combine, $2x = 40.$
Divide by 2, $x = 20.$

$$x + 10 = 30.$$
$$3x + 10 = 70.$$

Therefore, A is 70 years old and B is 30 years old.

58. A flag pole 105 feet high was broken so that the length of the part broken off was six times the length of the part left standing. Find the length of each part.

Let x = the number of feet in the part left standing.
Then $6x$ = the number of feet in the part broken off,
and $x + 6x$ = the number of feet in the whole pole.
But 105 = the number of feet in the whole pole.

$$\therefore x + 6x = 105.$$

Combine, $7x = 105.$
Divide by 7, $x = 15.$

$$6x = 90.$$

Therefore, the part broken off was 90 feet long and the part left standing was 15 feet long.

59. A river flows at the rate of 3 miles an hour. A man rows a certain distance up the river in 12 hours and rows back to the starting point in 3 hours. How many miles an hour can he row in still water?

Let $x =$ the number of miles he can row in 1 hour in still water.

Then $x - 3 =$ the number of miles he can row in 1 hour upstream,

$x + 3 =$ the number of miles he can row in 1 hour downstream,

$12(x - 3) =$ the number of miles he rows upstream,

and $3(x + 3) =$ the number of miles he rows downstream.

But these two distances are equal.

$$\therefore 12(x - 3) = 3(x + 3).$$

Remove the parentheses,

$$12x - 36 = 3x + 9.$$

Transpose, $12x - 3x = 9 + 36.$

Combine, $9x = 45.$

Divide by 9, $x = 5.$

Therefore, the man can row 5 miles an hour in still water.

60. A man finds that it takes his naphtha launch 2 hours to go 24 miles with the tide, and 4 hours to go 8 miles against the tide. Find the rate of the tide in miles an hour.

Let $x =$ the number of miles the tide runs per hour.

Now the launch moves against the tide 8 miles in 4 hours, or 2 miles an hour, and with the tide it moves 24 miles in 2 hours, or 12 miles an hour.

Then $12 - x =$ the actual number of miles the launch moves through the water in 1 hour,

and $x + 2 =$ the actual number of miles the launch moves through the water in 1 hour.

But these two rates are equal.

$$\therefore x + 2 = 12 - x.$$

Transpose, $x + x = 12 - 2.$

Combine, $2x = 10.$

Divide by 2, $x = 5.$

Therefore, the tide runs 5 miles an hour.

61. A merchant has two kinds of coffee, one kind costing 35 cents and the other 40 cents a pound. He makes a mixture of 100 pounds. If a pound of the mixture costs him 39 cents, how many pounds of each kind does he take?

Let $x =$ the number of pounds of the 35-cent coffee.
Then $100 - x =$ the number of pounds of the 40-cent coffee,
$35x =$ the number of cents in the value of the 35-cent coffee,
$40(100 - x) =$ the number of cents in the value of the 40-cent coffee,
and $35x + 40(100 - x) =$ the number of cents in the value of the mixture.
But $100 \times 39 =$ the number of cents in the value of the mixture.

$$\therefore 35x + 40(100 - x) = 100 \times 39.$$

Remove the parenthesis,

$$35x + 4000 - 40x = 3900.$$

Transpose, $35x - 40x = 3900 - 4000.$
Change the signs, $40x - 35x = 4000 - 3900.$
Combine, $5x = 100.$
Divide by 5, $x = 20.$
$100 - x = 80.$

Therefore, the merchant takes 20 pounds costing 35 cents and 80 pounds costing 40 cents.

62. A had $27 and B had $21. A paid B a certain sum and then B had twice as much as A had left. How many dollars did A pay B?

Let $x =$ the number of dollars A paid B.
Then $27 - x =$ the number of dollars A then had,
and $21 + x =$ the number of dollars B then had.
But B then had twice as many dollars as A then had.

$$\therefore 21 + x = 2(27 - x).$$

Remove the parenthesis, $21 + x = 54 - 2x.$
Transpose, $x + 2x = 54 - 21.$
Combine, $3x = 33.$
Divide by 3, $x = 11.$

Therefore, A paid B $11.

63. A man has a certain number of dollars, half dollars, and quarters. The number of quarters is twice the number of half dollars, and the number of half dollars is twice the number of dollars. If he has $45 in all, how many coins of each kind has he?

Let $x =$ the number of dollars.
Then $2x =$ the number of half dollars,
$4x =$ the number of quarters,

x = the number of dollars in the value of the dollars,
x = the number of dollars in the value of the half dollars,
x = the number of dollars in the value of the quarters,
and $x + x + x$ = the number of dollars in the value of the whole.
But 45 = the number of dollars in the value of the whole.

$$\therefore x + x + x = 45.$$

Combine, $3x = 45.$

Divide by 3, $x = 15.$

$$2x = 30.$$

$$4x = 60.$$

Therefore, the man has 15 dollars, 30 half dollars, and 60 quarters.

Exercise 9. Page 37

Perform mentally the indicated addition:

$$\begin{array}{rrrrrr}
\textbf{1.} & \textbf{2.} & \textbf{3.} & \textbf{4.} & \textbf{5.} & \textbf{6.} \\
+19 & -22 & -34 & -17 & +17 & +33 \\
\underline{-12} & \underline{-14} & \underline{+19} & \underline{+24} & \underline{+24} & \underline{-48} \\
+7 & -36 & -15 & +7 & +41 & -15
\end{array}$$

$$\begin{array}{rrrrrr}
\textbf{7.} & \textbf{8.} & \textbf{9.} & \textbf{10.} & \textbf{11.} & \textbf{12.} \\
+27 & -16 & -25 & -17 & +26 & -42 \\
-32 & -24 & +48 & -23 & -11 & +86 \\
\underline{+45} & \underline{+14} & \underline{-11} & \underline{-10} & \underline{-15} & \underline{-29} \\
+40 & -26 & +12 & -50 & 0 & +15
\end{array}$$

$$\begin{array}{rrrrrr}
\textbf{13.} & \textbf{14.} & \textbf{15.} & \textbf{16.} & \textbf{17.} & \textbf{18.} \\
-19 & +22 & -36 & +29 & +22 & -17 \\
-29 & -16 & +32 & -17 & +27 & -34 \\
\underline{+39} & \underline{+8} & \underline{+12} & \underline{-12} & \underline{+36} & \underline{+50} \\
-9 & +14 & +8 & 0 & +85 & -1
\end{array}$$

$$\begin{array}{rrrrrr}
\textbf{19.} & \textbf{20.} & \textbf{21.} & \textbf{22.} & \textbf{23.} & \textbf{24.} \\
28 & -46 & 28 & -14 & -12 & 80 \\
-19 & -19 & 19 & 22 & -17 & -60 \\
-17 & -24 & -12 & -38 & -22 & -40 \\
\underline{12} & \underline{16} & \underline{-7} & \underline{15} & \underline{-29} & \underline{24} \\
4 & -73 & 28 & -15 & -80 & 4
\end{array}$$

Exercise 10. Page 38

Perform mentally the indicated addition:

$$\begin{array}{rrrrr}
\textbf{1.} & \textbf{2.} & \textbf{3.} & \textbf{4.} & \textbf{5.} \\
7a & -4xy & 11a^2b & 3ac & 12x^2 \\
5a & -5xy & -5a^2b & -2ac & -18x^2 \\
-4a & 6xy & 8a^2b & 5ac & 6x^2 \\
-6a & 2xy & -6a^2b & -4ac & -3x^2 \\
\hline
2a & -xy & 8a^2b & 2ac & -3x^2
\end{array}$$

$$\begin{array}{rrrrr}
\textbf{6.} & \textbf{7.} & \textbf{8.} & \textbf{9.} & \textbf{10.} \\
-17a^2 & 12xyz & -7ab & 4y^2 & -2abc \\
-15a^2 & -17xyz & -4ab & 6y^2 & -3abc \\
-16a^2 & -8xyz & -2ab & 8y^2 & -4abc \\
32a^2 & 10xyz & 8ab & -12y^2 & -5abc \\
\hline
-16a^2 & -3xyz & -5ab & 6y^2 & -14abc
\end{array}$$

Exercise 11. Page 39

1. Find the algebraic sum of $4a$, $-7a$, a, $8a$, $-12a$.

$$4a - 7a + a + 8a - 12a = 13a - 19a = -6a.$$

2. Find the algebraic sum of $7c$, $5c$, $8c$, $-14c$, $-2c$.

$$7c + 5c + 8c - 14c - 2c = 20c - 16c = 4c.$$

3. Find the algebraic sum of $8ab$, $-7ab$, $-4ab$, $-ab$.

$$8ab - 7ab - 4ab - ab = 8ab - 12ab = -4ab.$$

4. Find the algebraic sum of $7xy$, $4xy$, $-8xy$, $-5xy$.

$$7xy + 4xy - 8xy - 5xy = 11xy - 13xy = -2xy.$$

5. Find the algebraic sum of $4abc$, $6abc$, $-8abc$, $10abc$.

$$4abc + 6abc - 8abc + 10abc = 20abc - 8abc = 12abc.$$

6. Find the algebraic sum of $14b$, $-16b$, $-2b$, $8b$.

$$14b - 16b - 2b + 8b = 22b - 18b = 4b.$$

7. Find the algebraic sum of $8yz$, $-7yz$, $-3yz$, $-2yz$.

$$8yz - 7yz - 3yz - 2yz = 8yz - 12yz = -4yz.$$

8. Find the algebraic sum of $-6x^2$, $8x^2$, $-4x^2$, $2x^2$.

$$-6x^2 + 8x^2 - 4x^2 + 2x^2 = 10x^2 - 10x^2 = 0.$$

9. Find the algebraic sum of $-4a^2b$, $-7a^2b$, $16a^2b$, $-8a^2b$.

$$-4a^2b-7a^2b+16a^2b-8a^2b=16a^2b-19a^2b=-3a^2b.$$

10. Find the algebraic sum of $-b^2m^3$, $7b^2m^3$, $-4b^2m^3$, $3b^2m^3$.

$$-b^2m^3+7b^2m^3-4b^2m^3+3b^2m^3=10b^2m^3-5b^2m^3=5b^2m^3.$$

11. Find the algebraic sum of $5a^2b$, $-3a^2b$, $4a^2b$, $-7a^2b$.

$$5a^2b-3a^2b+4a^2b-7a^2b=9a^2b-10a^2b=-a^2b.$$

12. Find the algebraic sum of $4a^2c$, $-10a^2c$, $6a^2c$, $-9a^2c$.

$$4a^2c-10a^2c+6a^2c-9a^2c=10a^2c-19a^2c=-9a^2c.$$

13. Find the algebraic sum of $2mn$, $5mn$, $-mn$, $-4mn$.

$$2mn+5mn-mn-4mn=7mn-5mn=2mn.$$

14. Find the algebraic sum of $3yz$, $2yz$, $-10yz$, $3yz$.

$$3yz+2yz-10yz+3yz=8yz-10yz=-2yz.$$

15. Find the algebraic sum of $20xy$, $-7xy$, $-8xy$, $5xy$.

$$20xy-7xy-8xy+5xy=25xy-15xy=10xy.$$

16. Find the algebraic sum of $-8z^4$, $18z^4$, $-15z^4$, $11z^4$.

$$-8z^4+18z^4-15z^4+11z^4=29z^4-23z^4=6z^4.$$

17. Express in one term $-7x^2-18x^2+7x^2+7x^2+20x^2+3x^2$.

$$-7x^2-18x^2+7x^2+7x^2+20x^2+3x^2=37x^2-25x^2=12x^2.$$

18. Express in one term $4a^2+a^2-9a^2+11a^2-6a^2-7a^2$.

$$4a^2+a^2-9a^2+11a^2-6a^2-7a^2=16a^2-22a^2=-6a^2.$$

19. Express in one term $-3ab^2-5ab^2-29ab^2+4ab^2+5ab^2$.

$$-3ab^2-5ab^2-29ab^2+4ab^2+5ab^2=9ab^2-37ab^2=-28ab^2.$$

20. Express in one term $3a^2x+6a^2x+4a^2x-9a^2x-13a^2x$.

$$3a^2x+6a^2x+4a^2x-9a^2x-13a^2x=13a^2x-22a^2x=-9a^2x.$$

21. Express in one term $-5a^2b^2-10a^2b^2+9a^2b^2+3a^2b^2-5a^2b^2$.

$$-5a^2b^2-10a^2b^2+9a^2b^2+3a^2b^2-5a^2b^2=12a^2b^2-20a^2b^2=-8a^2b^2.$$

22. Express in one term $5a^3b-21a^3b-11a^3b+18a^3b+17a^3b$.

$$5a^3b-21a^3b-11a^3b+18a^3b+17a^3b=40a^3b-32a^3b=8a^3b.$$

23. Express in one term $41y^2z^2-43y^2z^2+61y^2z^2-2y^2z^2-27y^2z^2$.

$$41y^2z^2-43y^2z^2+61y^2z^2-2y^2z^2-27y^2z^2=102y^2z^2-72y^2z^2=30y^2z^2.$$

24. Express in one term $-3bc-4bc-31bc+28bc+7bc-bc$.

$$-3bc-4bc-31bc+28bc+7bc-bc=35bc-39bc=-4bc.$$

25. Express in one term $b^2d - 8b^2d + 7b^2d - 19b^2d + 7b^2d + 4b^2d$.

$b^2d - 8b^2d + 7b^2d - 19b^2d + 7b^2d + 4b^2d = 19b^2d - 27b^2d = -8b^2d$.

26. Express in one term $14ac - 7ac + 6ac - 9ac + ac + 4ac$.

$14ac - 7ac + 6ac - 9ac + ac + 4ac = 25ac - 16ac = 9ac$.

27. Express in one term $4xy + 3xy - 10xy - xy + 5xy - 7xy$.

$4xy + 3xy - 10xy - xy + 5xy - 7xy = 12xy - 18xy = -6xy$.

28. Express in one term $14ac^2 + 17ac^2 - 7ac^2 - 3ac^2 - 4ac^2 - 15ac^2$.

$14ac^2 + 17ac^2 - 7ac^2 - 3ac^2 - 4ac^2 - 15ac^2 = 31ac^2 - 29ac^2 = 2ac^2$.

29. Express in one term $4x^2 - 11x^2 - 7x^2 + 5x^2 - 9x^2 + 3x^2 + 5x^2$.

$4x^2 - 11x^2 - 7x^2 + 5x^2 - 9x^2 + 3x^2 + 5x^2 = 17x^2 - 27x^2 = -10x^2$.

30. Express in one term $16abcx - 11abcx - 2abcx + 3abcx + 4abcx$.

$16abcx - 11abcx - 2abcx + 3abcx + 4abcx = 23abcx - 13abcx = 10abcx$.

31. Express in one term $13cy + 12cy - 24cy + 2cy - 3cy + 8cy$.

$13cy + 12cy - 24cy + 2cy - 3cy + 8cy = 35cy - 27cy = 8cy$.

32. Express in one term $2ax - 10ax - 7ax + 3ax - 4ax + 6ax$.

$2ax - 10ax - 7ax + 3ax - 4ax + 6ax = 11ax - 21ax = -10ax$.

Exercise 12. Page 41

Perform mentally the indicated subtraction:

1.	**2.**	**3.**	**4.**	**5.**	**6.**
14	− 14	− 14	− 16	− 4	− 8
− 8	8	− 8	− 4	16	18
22	− 22	− 6	− 12	− 20	− 26

7.	**8.**	**9.**	**10.**	**11.**	**12.**
16	15	− 15	− 15	− 12	− 18
20	− 20	20	− 5	− 16	− 24
− 4	35	− 35	− 10	4	6

13.	**14.**	**15.**
18	− 18	18
24	− 18	− 18
− 6	0	36

Exercise 13. Page 41

Perform mentally the indicated subtraction:

1.	2.	3.	4.	5.
$4ab$	$-8a^2c$	$15xyz$	$-15x^2y^2$	$14abc$
$-6ab$	$8a^2c$	$-8xyz$	$-7x^2y^2$	$8abc$
$10ab$	$-16a^2c$	$23xyz$	$-8x^2y^2$	$6abc$

6.	7.	8.	9.	10.
$-18cd$	$14a^2b$	$-14a^2b$	$8ac^2$	$-4ab$
$-7cd$	$-9a^2b$	$9a^2b$	$-10ac^2$	$-8ab$
$-11cd$	$23a^2b$	$-23a^2b$	$18ac^2$	$4ab$

11. If $a = 5$, $b = -3$, $c = -2$, find the value of $a + (-b) + c$.

$$a + (-b) + c = 5 + [-(-3)] + (-2) = 5 + 3 - 2 = 6.$$

12. If $a = 5$, $b = -3$, $c = -2$, find the value of $-a - (-b) + c$.

$$-a - (-b) + c = -5 - [-(-3)] + (-2) = -5 - 3 - 2 = -10.$$

13. If $a = 5$, $b = -3$, $c = -2$, find the value of $a - b + (-c)$.

$$a - b + (-c) = 5 - (-3) + [-(-2)] = 5 + 3 + 2 = 10.$$

14. If $a=5$, $b=-3$, $c=-2$, find the value of $-(-a)-(-b)-(-c)$.

$$\begin{aligned} -(-a) - (-b) - (-c) &= -(-5) - [-(-3)] - [-(-2)] \\ &= 5 - 3 - 2 = 0. \end{aligned}$$

15. If $a = 5$, $b = -3$, $c = -2$, find the value of $a - (-b) + (-c)$.

$$a - (-b) + (-c) = 5 - [-(-3)] + [-(-2)] = 5 - 3 + 2 = 4.$$

16. If $a = 5$, $b = -3$, $c = -2$, find the value of $-a + (-b) + (-c)$.

$$-a + (-b) + (-c) = -5 + [-(-3)] + [-(-2)] = -5 + 3 + 2 = 0.$$

17. If $a = 5$, $b = -3$, $c = -2$, find the value of $-a + (-b) - (-c)$.

$$-a + (-b) - (-c) = -5 + [-(-3)] - [-(-2)] = -5 + 3 - 2 = -4.$$

18. If $a = 5$, $b = -3$, $c = -2$, find the value of $-(-a) + b + c$.

$$-(-a) + b + c = -(-5) + (-3) + (-2) = 5 - 3 - 2 = 0$$

Exercise 14. Page 45

Find mentally the product of:

1.	2.	3.	4.	5.	6.
$4c$	$2c$	$5ab$	$-2xy$	$4ac$	$8x^2$
$5c$	c	$-2ab$	$-2xy$	$-2ac$	$-2x$
$20c^2$	$2c^2$	$-10a^2b^2$	$4x^2y^2$	$-8a^2c^2$	$-16x^3$

7.	8.	9.	10.	11.	12.
$-5x^2$	$5ab^2$	$8ac$	$-8x^2y$	$-a^3b^2$	$9ab^2$
$-2x^2$	$-2a^2b$	$2a^2c^2$	$-3xy^2$	$-ab^5$	$-2ab^2$
$10x^4$	$-10a^3b^3$	$16a^3c^3$	$24x^3y^3$	a^4b^7	$-18a^2b^4$

13.	14.	15.	16.	17.	18.
$-6ax^2$	$5xy^3$	$9x^6$	$-16a^3b$	$-x^2y^2$	$7a^2b$
$-2a^2x^2$	$-4x^2y^2$	$-2x^4$	$-4ab^3$	$-3xy^5$	$2a^5b^3$
$12a^3x^4$	$-20x^3y^5$	$-18x^{10}$	$64a^4b^4$	$3x^3y^7$	$14a^7b^4$

19.	20.	21.	22.	23.	24.
$11a^2y$	$-x^{m+1}$	a^2b^m	$-x^nb^n$	a^{m+3}	a^{m+n-1}
$-3bx^4$	$-x^{2m-1}$	$-a^nb^2$	$-x^ny^n$	a^{m-4}	$-a^{m-n+1}$
$-33a^2bx^4y$	x^{3m}	$-a^{n+2}b^{m+2}$	$x^{2n}b^ny^n$	a^{2m-1}	$-a^{2m}$

25. Find mentally the product of $9xy$, $-7x^2y^3$, and $-10xy^3$.

$$(9xy)(-7x^2y^3)(-10xy^3) = 630x^4y^7.$$

26. Find mentally the product of $-7a^2b$, $6ab^2$, and $-4a^2b^2$.

$$(-7a^2b)(6ab^2)(-4a^2b^2) = 168a^5b^5.$$

27. Find mentally the product of $4a^3b$, $-6a^2b$, and $5a^3b^3$.

$$(4a^3b)(-6a^2b)(5a^3b^3) = -120a^8b^5.$$

28. Find mentally the product of $2a^4$, a^2b^2, $-6a^3b$, and $2a^5b^7$.

$$(2a^4)(a^2b^2)(-6a^3b)(2a^5b^7) = -24a^{14}b^{10}.$$

29. Find mentally the product of x^2, $-x^2y$, y^2z^2, and x^2y^3.

$$(x^2)(-x^2y)(y^2z^2)(x^2y^3) = -x^6y^6z^2.$$

30. Find mentally the product of x^{2n}, $-x^{n+m}$, $-x^{n-3}$, and x^{m-n}.

$$(x^{2n})(-x^{n+m})(-x^{n-3})(x^{m-n}) = x^{2m+3n-3}.$$

31. Find mentally the product of a^{2m+1}, a^{2n-1}, a^{m+n}, a^n, and a^{m-3n}.

$$(a^{2m+1})(a^{2n-1})(a^{m+n})(a^n)(a^{m-3n}) = a^{4m+n}.$$

32. Find mentally the product of x^{c+d}, x^{d-c}, x^{c-d}, x^4, and x^{2c-3d}.

$$(x^{c+d})(x^{d-c})(x^{c-d})(x^4)(x^{2c-3d}) = x^{3c-2d+4}.$$

Exercise 15. Page 48

Perform mentally the indicated division:

1. $\dfrac{54\,a^2y}{6\,y} = 9\,a^2.$

2. $\dfrac{108\,m^2c}{9\,m^2} = 12\,c.$

3. $\dfrac{17\,x^8y}{17\,x^4} = x^4y.$

4. $\dfrac{72\,x^5y^3}{12\,x^4y} = 6\,xy^2.$

5. $\dfrac{6\,x^7y^8}{3\,x^2y^5} = 2\,x^5y^3.$

6. $\dfrac{-21\,a^4x^2y^5}{-7\,a^3x^2y^2} = 3\,ay^3.$

7. $\dfrac{8\,a^5bc^4}{-2\,a^2c^3} = -4\,a^3bc.$

8. $\dfrac{121\,a^7x^8}{-11\,a^5x^2} = -11\,a^2x^6.$

9. $\dfrac{-96\,a^2y^7}{8\,ay^6} = -12\,ay.$

10. $\dfrac{-15\,a^7b^4}{-3\,a^3b^3} = 5\,a^4b.$

11. $\dfrac{-12\,m^4n^5}{3\,m^4n} = -4\,n^4.$

12. $\dfrac{3\,ab^7y^8}{ab^4y^3} = 3\,b^3y^5.$

13. $\dfrac{-21\,a^2b^5c^5}{-7\,ab^4c^2} = 3\,abc^3.$

14. $\dfrac{3\,m^8y^6z^7}{m^3y^5z^4} = 3\,m^5yz^3.$

15. $\dfrac{-56\,a^2cd^x}{-8\,acd^x} = 7\,a.$

16. $\dfrac{92\,a^5b^6c^7}{46\,a^3b^2c} = 2\,a^2b^4c^6.$

17. $\dfrac{14\,x^5y^4z^8}{-7\,x^5y^4z^4} = -2\,z^4.$

18. $\dfrac{12\,x^ny^m}{6\,x^2y^3} = 2\,x^{n-2}y^{m-3}.$

19. $\dfrac{-14\,b^3c^8d^9}{-2\,c^5d^4} = 7\,b^3c^3d^5.$

20. $\dfrac{36\,l^2m^5}{12\,l^2m^2} = 3\,m^3.$

21. $\dfrac{40\,a^2x^3y^4}{-8\,x^3y^2} = -5\,a^2y^2.$

22. $\dfrac{-28\,c^3d^4}{-7\,c^2d^2} = 4\,cd^2.$

23. $\dfrac{-36\,x^{n-2}y^{n-1}}{9\,x^2y^2} = -4\,x^{n-4}y^{n-3}.$

24. $\dfrac{x^3y^3}{x^n} = x^{3-n}y^3.$

25. $\dfrac{-y^7z^{12}}{-y^5z^4} = y^2z^8.$

26. $\dfrac{9\,a^3b^2c^n}{3\,a^2b^2c^{n-1}} = 3\,ac.$

27. $\dfrac{3a^{n+3}}{-a^{n-3}} = -3a^6.$

30. $\dfrac{4a^m}{2a^{m-4}} = 2a^4.$

28. $\dfrac{10a^{12}x^7}{5a^8x^5} = 2a^4x^2.$

31. $\dfrac{-18x^5y^6z}{9x^3y^3z} = -2x^2y^3.$

29. $\dfrac{-3x^{m+1}y^{m+2}}{-x^{m-1}y^{m-2}} = 3x^2y^4.$

32. $\dfrac{-x^{22}y^{18}}{-x^{14}y^{14}} = x^8y^4.$

Exercise 16. Page 51

1. Add $2a+3b-5c$; $3a-2b+3c$; $a-2b+3c$.

$$\begin{array}{rrr}
2a & +3b & -5c \\
3a & -2b & +3c \\
a & -2b & +3c \\
\hline
6a & -b & +c
\end{array}$$

2. Add $5x^3-3x^2y+3xy^2-y^3$; $-3x^3+x^2y+7y^3$; $2x^3-5x^2y-8xy^2+y^3$; $-3x^3+4x^2y+7xy^2-y^3$.

$$\begin{array}{rrrr}
5x^3 & -3x^2y & +3xy^2 & -y^3 \\
-3x^3 & +x^2y & & +7y^3 \\
2x^3 & -5x^2y & -8xy^2 & +y^3 \\
-3x^3 & +4x^2y & +7xy^2 & -y^3 \\
\hline
x^3 & -3x^2y & +2xy^2 & +6y^3
\end{array}$$

3. Add $7a-b+c-d$; $-5a+4b-8c+4d$; $-2a+5b+3c-7d$; $a-8b+4c-4d$.

$$\begin{array}{rrrr}
7a & -b & +c & -d \\
-5a & +4b & -8c & +4d \\
-2a & +5b & +3c & -7d \\
a & -8b & +4c & -4d \\
\hline
a & & & -8d
\end{array}$$

4. Add $7a-3b+2c-3d$; $5a-4b-5c+7d$; $9a+3b-4c+8$; $-7a-3b-2c-17$.

$$\begin{array}{rrrrr}
7a & -3b & +2c & -3d & \\
5a & -4b & -5c & +7d & \\
9a & +3b & -4c & & +8 \\
-7a & -3b & -2c & & -17 \\
\hline
14a & -7b & -9c & +4d & -9
\end{array}$$

5. Add $8a^3 - a^2b + 7ab^2 + 3b^3$; $-9a^3 + 4a^2b - 7ab^2 - 5b^3$; $a^3 + a^2b - ab^2 + b^3$; $a^3 - a^2b - ab^2 + 3b^3$.

$$\begin{array}{rrrr}
8a^3 & -\ a^2b & +\ 7ab^2 & +\ 3b^3\\
-9a^3 & +\ 4a^2b & -\ 7ab^2 & -\ 5b^3\\
a^3 & +\ a^2b & -\ ab^2 & +\ b^3\\
a^3 & -\ a^2b & -\ ab^2 & +\ 3b^3\\
\hline
a^3 & +\ 3a^2b & -\ 2ab^2 & +\ 2b^3
\end{array}$$

6. Add $7a - 3b + 5c - 10d$; $2b - 3c + d - 4e$; $5c - 6a - 4e + 2d$; $-3b - 8c + 7a - e$; $21e - 16c + a - 5d$.

$$\begin{array}{rrrrr}
7a & -\ 3b & +\ 5c & -\ 10d & \\
 & +\ 2b & -\ 3c & +\ d & -\ 4e\\
-6a & & +\ 5c & +\ 2d & -\ 4e\\
7a & -\ 3b & -\ 8c & & -\ e\\
a & & -\ 16c & -\ 5d & +\ 21e\\
\hline
9a & -\ 4b & -\ 17c & -\ 12d & +\ 12e
\end{array}$$

7. Add $3ab^2 - 4a^2b + a^3$; $-4ac^2 + 5ab^2 - c^3$; $-7b^3 + 2a^2b - 6ac^2$; $5a^3 - 11ab^2 - 12ac^2$.

$$\begin{array}{rrrrrr}
a^3 & -\ 4a^2b & +\ 3ab^2 & & & \\
 & & +\ 5ab^2 & -\ 4ac^2 & -\ c^3 & \\
 & +\ 2a^2b & & -\ 6ac^2 & & -\ 7b^3\\
5a^3 & & -\ 11ab^2 & -\ 12ac^2 & & \\
\hline
6a^3 & -\ 2a^2b & -\ 3ab^2 & -\ 22ac^2 & -\ c^3 & -\ 7b^3
\end{array}$$

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8. Add $a^3 + 3a^2b + 3ab^2 + b^3$; $-5ab^2 + 3a^2b - b^3 + 3a^3$; $3ab^2 - 5a^2b + 3b^3 - 3a^3$; $-5b^3 + 2a^2b - 4a^3 + 3ab^2$; $7a^3 + 6b^3 - 5a^2b + 5ab^2$.

$$\begin{array}{rrrr}
a^3 & +\ 3a^2b & +\ 3ab^2 & +\ b^3\\
3a^3 & +\ 3a^2b & -\ 5ab^2 & -\ b^3\\
-3a^3 & -\ 5a^2b & +\ 3ab^2 & +\ 3b^3\\
-4a^3 & +\ 2a^2b & +\ 3ab^2 & -\ 5b^3\\
7a^3 & -\ 5a^2b & +\ 5ab^2 & +\ 6b^3\\
\hline
4a^3 & -\ 2a^2b & +\ 9ab^2 & +\ 4b^3
\end{array}$$

9. Add $4c^2d^2 - 3c^3d + 17c^4 - 8c^5$; $14c^2d^2 + 4c^3d + 5c^4 - 3c^5$; $-c^2d^2 - 2c^3d + 4c^4 + 19c^5$; $2c^2d^2 + 5c^3d - 7c^4 + 9c^5$; $3c^2d^2 + 41c^3d - c^4 + 7c^5$.

$$\begin{array}{rrrr}
4c^2d^2 & -\ 3c^3d & +\ 17c^4 & -\ 8c^5\\
14c^2d^2 & +\ 4c^3d & +\ 5c^4 & -\ 3c^5\\
-\ c^2d^2 & -\ 2c^3d & +\ 4c^4 & +\ 19c^5\\
2c^2d^2 & +\ 5c^3d & -\ 7c^4 & +\ 9c^5\\
3c^2d^2 & +\ 41c^3d & -\ c^4 & +\ 7c^5\\
\hline
22c^2d^2 & +\ 45c^3d & +\ 18c^4 & +\ 24c^5
\end{array}$$

10. Add $a^4 + 6a^3b + 10a^2b^2 + 6ab^3 + b^4$; $-7b^4 - 3a^2b^2 + 7ab^3 + 8a^4 + 4a^3b$; $8a^2b^2 + 3ab^3 - 5a^4 + 3b^4 + 6a^3b$; $-3a^3b + 5a^2b^2 + 3a^4 - 5b^4 - 3ab^3$.

$$\begin{array}{rrrrr}
a^4 & +\ 6a^3b & +\ 10a^2b^2 & +\ 6ab^3 & +\ b^4 \\
8a^4 & +\ 4a^3b & -\ 3a^2b^2 & +\ 7ab^3 & -\ 7b^4 \\
-5a^4 & +\ 6a^3b & +\ 8a^2b^2 & +\ 3ab^3 & +\ 3b^4 \\
3a^4 & -\ 3a^3b & +\ 5a^2b^2 & -\ 3ab^3 & -\ 5b^4 \\
\hline
7a^4 & +\ 13a^3b & +\ 20a^2b^2 & +\ 13ab^3 & -\ 8b^4
\end{array}$$

11. Add $4x^4y^5z^6 - 3x^3y^4z^5 + 17x^2y^3z^4 - 8xy^2z^3$; $14x^2y^3z^4 + 4xy^2z^3 + 5x^3y^4z^5 - 3x^4y^5z^6$; $-x^4y^5z^6 - 2x^3y^4z^5 + 4xy^2z^3 + 19x^2y^3z^4$; $2x^3y^4z^5 + 5xy^2z^3 - 7x^4y^5z^6 + 9x^2y^3z^4$; $-12xy^2z^3 + 4x^4y^5z^6 - 15x^2y^3z^4 - x^3y^4z^5$; $3x^4y^5z^6 + 41x^2y^3z^4 - x^3y^4z^5 + 7xy^2z^3$.

$$\begin{array}{rrrr}
4x^4y^5z^6 & -\ 3x^3y^4z^5 & +\ 17x^2y^3z^4 & -\ 8xy^2z^3 \\
-3x^4y^5z^6 & +\ 5x^3y^4z^5 & +\ 14x^2y^3z^4 & +\ 4xy^2z^3 \\
-\ x^4y^5z^6 & -\ 2x^3y^4z^5 & +\ 19x^2y^3z^4 & +\ 4xy^2z^3 \\
-7x^4y^5z^6 & +\ 2x^3y^4z^5 & +\ 9x^2y^3z^4 & +\ 5xy^2z^3 \\
4x^4y^5z^6 & -\ x^3y^4z^5 & -\ 15x^2y^3z^4 & -\ 12xy^2z^3 \\
3x^4y^5z^6 & -\ x^3y^4z^5 & +\ 41x^2y^3z^4 & +\ 7xy^2z^3 \\
\hline
 & & 85x^2y^3z^4 &
\end{array}$$

Exercise 17. Page 53

1. From $4a - 6b + 2c$
take $3a - 3b - 3c$.

$$\begin{array}{r}
4a - 6b + 2c \\
3a - 3b - 3c \\
\hline
a - 3b + 5c
\end{array}$$

2. From $2a - 2b - 4c$
take $3a - 3b - 5c$.

$$\begin{array}{r}
2a - 2b - 4c \\
3a - 3b - 5c \\
\hline
-\ a + \ b + \ c
\end{array}$$

3. From $4x^2 + 2xy + 3y^2$
take $x^2 - xy + 2y^2$.

$$\begin{array}{r}
4x^2 + 2xy + 3y^2 \\
x^2 - \ xy + 2y^2 \\
\hline
3x^2 + 3xy + \ y^2
\end{array}$$

4. From $a^3 + 3a^2b + 3ab^2 + b^3$
take $a^3 - 3a^2b + 3ab^2 - b^3$.

$$\begin{array}{rrrr}
a^3 & +\ 3a^2b & +\ 3ab^2 & +\ b^3 \\
a^3 & -\ 3a^2b & +\ 3ab^2 & -\ b^3 \\
\hline
 & 6a^2b & & +\ 2b^3
\end{array}$$

5. From $x^4 + 3ax^3 - 2bx^2 + 3cx - 4d$ take $3x^4 + ax^3 - 4bx^2 - 3cx + d$.

$$\begin{array}{rrrrr}
x^4 & +\ 3ax^3 & -\ 2bx^2 & +\ 3cx & -\ 4d \\
3x^4 & +\ ax^3 & -\ 4bx^2 & -\ 3cx & +\ d \\
\hline
-2x^4 & +\ 2ax^3 & +\ 2bx^2 & +\ 6cx & -\ 5d
\end{array}$$

6. From $72x^4 - 78x^3y - 10x^2y^2 + 17xy^3 + 3y^4$
take $-x^4 + 36x^3y - 17xy^3 - 34y^4 + 10x^2y^2$.

$$\begin{array}{r}
72x^4 - 78x^3y - 10x^2y^2 + 17xy^3 + 3y^4 \\
\underline{-\ x^4 + 36x^3y + 10x^2y^2 - 17xy^3 - 34y^4} \\
73x^4 - 114x^3y - 20x^2y^2 + 34xy^3 + 37y^4
\end{array}$$

7. From $6x^2 + 7xy - 5y^2 - 12xyz - 8yz - 5z^2$
take $-8xy - 7yz + 3x^2 - 4y^2 + 5xyz$.

$$\begin{array}{r}
6x^2 + 7xy - 5y^2 - 12xyz - 8yz - 5z^2 \\
\underline{3x^2 - 8xy - 4y^2 + 5xyz - 7yz} \\
3x^2 + 15xy - y^2 - 17xyz - yz - 5z^2
\end{array}$$

8. From $4x^4 + 12x^3 - x^2 - 27x - 18$ take $4x^4 + 4x^3 - 17x^2 - 9x + 18$.

$$\begin{array}{r}
4x^4 + 12x^3 - x^2 - 27x - 18 \\
\underline{4x^4 \mp 4x^3 - 17x^2 - 9x + 18} \\
8x^3 + 16x^2 - 18x - 36
\end{array}$$

9. From $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$ take $3ax^3 - 7a^2x^2 + 3a^3x - 2a^4$.

$$\begin{array}{r}
x^4 - ax^3 - a^2x^2 - a^3x - 2a^4 \\
\underline{3ax^3 - 7a^2x^2 + 3a^3x - 2a^4} \\
x^4 - 4ax^3 + 6a^2x^2 - 4a^3x
\end{array}$$

10. From $6x^2 - 13xy + 6y^2 - 12xz - 13yz + 6z^2$
take $2x^2 - 5xy + 2y^2 - xz - yz - z^2$.

$$\begin{array}{r}
6x^2 - 13xy + 6y^2 - 12xz - 13yz + 6z^2 \\
\underline{2x^2 - 5xy + 2y^2 - xz - yz - z^2} \\
4x^2 - 8xy + 4y^2 - 11xz - 12yz + 7z^2
\end{array}$$

11. From $2x^4 - 6x^3 - x^2 + 15x - 10$ take $4x^4 + 6x^3 - 4x^2 - 15x - 15$

$$\begin{array}{r}
2x^4 - 6x^3 - x^2 + 15x - 10 \\
\underline{4x^4 + 6x^3 - 4x^2 - 15x - 15} \\
-2x^4 - 12x^3 + 3x^2 + 30x + 5
\end{array}$$

Exercise 18. Page 55

Simplify by removing the parentheses and combining like terms:

1. $(7a - 2b) - [(3a - c) - (2b - 3c)]$
$= 7a - 2b - [3a - c - 2b + 3c]$
$= 7a - 2b - 3a + c + 2b - 3c$
$= 4a - 2c.$

2. $(9a-4c)-\{[5a-(3b-4c)]-3b\}$
$=9a-4c-\{[5a-3b+4c]-3b\}$
$=9a-4c-\{5a-3b+4c-3b\}$
$=9a-4c-5a+3b-4c+3b$
$=4a+6b-8c.$

3. $2a-(3b+2c)-\{5b-3a-(a+b)+5c-[2a-(c+2b)]\}$
$=2a-3b-2c-\{5b-3a-a-b+5c-[2a-c-2b]\}$
$=2a-3b-2c-\{5b-3a-a-b+5c-2a+c+2b\}$
$=2a-3b-2c-5b+3a+a+b-5c+2a-c-2b$
$=8a-9b-8c.$

4. $a-\{2b+[3x-3a-(a+b)]+[2a-(b+c)]\}$
$=a-\{2b+[3x-3a-a-b]+[2a-b-c]\}$
$=a-\{2b+3x-3a-a-b+2a-b-c\}$
$=a-2b-3x+3a+a+b-2a+b+c$
$=3a+c-3x.$

5. $16-x-\{7x-[x-9x-(3-6x)]\}$
$=16-x-\{7x-[x-9x-3+6x]\}$
$=16-x-\{7x-x+9x+3-6x\}$
$=16-x-7x+x-9x-3+6x$
$=13-10x.$

6. $2a-\{3b+(2b-c)-4c+[2a-(3b-\overline{c+2b})]\}$
$=2a-\{3b+2b-c-4c+[2a-(3b-c+2b)]\}$
$=2a-\{3b+2b-c-4c+[2a-3b+c-2b]\}$
$=2a-\{3b+2b-c-4c+2a-3b+c-2b\}$
$=2a-3b-2b+c+4c-2a+3b-c+2b$
$=4c.$

7. $x^4-\{4x^3-[6x^2-(4x-1)]\}-(x^4+4x^3+6x^2+4x+1)$
$=x^4-\{4x^3-[6x^2-4x+1]\}-x^4-4x^3-6x^2-4x-1$
$=x^4-\{4x^3-6x^2+4x-1\}-x^4-4x^3-6x^2-4x-1$
$=x^4-4x^3+6x^2-4x+1-x^4-4x^3-6x^2-4x-1$
$=-8x^3-8x.$

8. $ab-[(3bce-2ab)-(5bce-bef)+(3ab-3bef)]$
$=ab-[3bce-2ab-5bce+bef+3ab-3bef]$
$=ab-3bce+2ab+5bce-bef-3ab+3bef$
$=2bce+2bef.$

9.
$$\begin{aligned}
&1 - [1 - (2 - x)] + [4x - (3 - 6x)] + [4 - (6x - 5)]\\
&= 1 - [1 - 2 + x] + [4x - 3 + 6x] + [4 - 6x + 5]\\
&= 1 - 1 + 2 - x + 4x - 3 + 6x + 4 - 6x + 5\\
&= 8 + 3x.
\end{aligned}$$

10.
$$\begin{aligned}
&4x - 3y - \{(2x + 4y) + 3x + [y - 9x - (2y - x) + (x - y)]\}\\
&= 4x - 3y - \{2x + 4y + 3x + [y - 9x - 2y + x + x - y]\}\\
&= 4x - 3y - \{2x + 4y + 3x + y - 9x - 2y + x + x - y\}\\
&= 4x - 3y - 2x - 4y - 3x - y + 9x + 2y - x - x + y\\
&= 6x - 5y.
\end{aligned}$$

11.
$$\begin{aligned}
&11x - \{7x - [8x - (9x - \overline{12a + 6x})]\}\\
&= 11x - \{7x - [8x - (9x - 12a + 6x)]\}\\
&= 11x - \{7x - [8x - 9x + 12a - 6x]\}\\
&= 11x - \{7x - 8x + 9x - 12a + 6x\}\\
&= 11x - 7x + 8x - 9x + 12a - 6x\\
&= 12a - 3x.
\end{aligned}$$

12.
$$\begin{aligned}
&2a - \{7b + [4a - 7b - (2a - \overline{6b + 4a})] - 3a\}\\
&= 2a - \{7b + [4a - 7b - (2a - 6b - 4a)] - 3a\}\\
&= 2a - \{7b + [4a - 7b - 2a + 6b + 4a] - 3a\}\\
&= 2a - \{7b + 4a - 7b - 2a + 6b + 4a - 3a\}\\
&= 2a - 7b - 4a + 7b + 2a - 6b - 4a + 3a\\
&= -a - 6b.
\end{aligned}$$

13.
$$\begin{aligned}
&a + 2b + 1 - (3a - 2b) - [a - \{2b + 1 + (3a + 2b) - 2b\}]\\
&= a + 2b + 1 - 3a + 2b - [a - \{2b + 1 + 3a + 2b - 2b\}]\\
&= a + 2b + 1 - 3a + 2b - [a - 2b - 1 - 3a - 2b + 2b]\\
&= a + 2b + 1 - 3a + 2b - a + 2b + 1 + 3a + 2b - 2b\\
&= 6b + 2.
\end{aligned}$$

14.
$$\begin{aligned}
&x^3 + (3xy + y^3 - 1) - [x^3 - (y^3 - 1)]\\
&= x^3 + 3xy + y^3 - 1 - [x^3 - y^3 + 1]\\
&= x^3 + 3xy + y^3 - 1 - x^3 + y^3 - 1\\
&= 3xy + 2y^3 - 2.
\end{aligned}$$

15.
$$\begin{aligned}
&a + 1 - [a^2 - (2a - 3) + a^2 - (5a^2 - \overline{8a - 7})]\\
&= a + 1 - [a^2 - (2a - 3) + a^2 - (5a^2 - 8a + 7)]\\
&= a + 1 - [a^2 - 2a + 3 + a^2 - 5a^2 + 8a - 7]\\
&= a + 1 - a^2 + 2a - 3 - a^2 + 5a^2 - 8a + 7\\
&= 3a^2 - 5a + 5.
\end{aligned}$$

16. $3x - \{2y + [3x - (4y - 5x + \overline{2x - 3y}) + 4y] - 2x\}$
$= 3x - \{2y + [3x - (4y - 5x + 2x - 3y) + 4y] - 2x\}$
$= 3x - \{2y + [3x - 4y + 5x - 2x + 3y + 4y] - 2x\}$
$= 3x - \{2y + 3x - 4y + 5x - 2x + 3y + 4y - 2x\}$
$= 3x - 2y - 3x + 4y - 5x + 2x - 3y - 4y + 2x$
$= -x - 5y.$

17. $3a^2 - ab + [4b^2 - \{3ab - 2a^2 + (2b^2 - \overline{3a^2 - 2ab}) - b^2\} + a^2 + ab]$
$= 3a^2 - ab + [4b^2 - \{3ab - 2a^2 + (2b^2 - 3a^2 + 2ab) - b^2\} + a^2 + ab]$
$= 3a^2 - ab + [4b^2 - \{3ab - 2a^2 + 2b^2 - 3a^2 + 2ab - b^2\} + a^2 + ab]$
$= 3a^2 - ab + [4b^2 - 3ab + 2a^2 - 2b^2 + 3a^2 - 2ab + b^2 + a^2 + ab]$
$= 3a^2 - ab + 4b^2 - 3ab + 2a^2 - 2b^2 + 3a^2 - 2ab + b^2 + a^2 + ab$
$= 9a^2 - 5ab + 3b^2.$

18. $3x^2 - 2xy + 3y^2 - \{2x^2 - [4x^2 - \overline{5xy + 2y^2} + (3x^2 + 2y^2)]\}$
$+ \{3x^2 - [2xy + 3y^2 - (2x^2 + 3xy + y^2)]\}$
$= 3x^2 - 2xy + 3y^2 - \{2x^2 - [4x^2 - 5xy - 2y^2 + 3x^2 + 2y^2]\}$
$+ \{3x^2 - [2xy + 3y^2 - 2x^2 - 3xy - y^2]\}$
$= 3x^2 - 2xy + 3y^2 - \{2x^2 - 4x^2 + 5xy + 2y^2 - 3x^2 - 2y^2\}$
$+ \{3x^2 - 2xy - 3y^2 + 2x^2 + 3xy + y^2\}$
$= 3x^2 - 2xy + 3y^2 - 2x^2 + 4x^2 - 5xy - 2y^2 + 3x^2 + 2y^2$
$+ 3x^2 - 2xy - 3y^2 + 2x^2 + 3xy + y^2$
$= 13x^2 - 6xy + y^2.$

19. $44x + [48y - (6z + \overline{3y - 7x}) + 4z] - [48y - \overline{8x + 2z} - (4x + y)]$
$= 44x + [48y - 6z - 3y + 7x + 4z] - [48y - 8x - 2z - 4x - y]$
$= 44x + 48y - 6z - 3y + 7x + 4z - 48y + 8x + 2z + 4x + y$
$= 63x - 2y.$

Exercise 19. Page 56

Inclose the last three terms in a parenthesis preceded by − :

1. $a^{12} - b^{12} + 6a^9b^3 - 6b^9a^3 + 8b^6a^6 = a^{12} - b^{12} - (6b^9a^3 - 6a^9b^3 - 8b^6a^6).$

2. $x^3 - y^3 - 5xy - 2y^2 - 2x^2 = x^3 - y^3 - (5xy + 2y^2 + 2x^2).$

3. $25z^2 + 10y^2 + 25y^2z^2 - 15x^2 + 10x^2z^2$
$= 25z^2 + 10y^2 - (-25y^2z^2 + 15x^2 - 10x^2z^2).$

4. $10x^4 + 5x^3 + 18x^2 + 30x - 360 = 10x^4 + 5x^3 - (360 - 30x - 18x^2).$

Inclose the last three terms in a parenthesis preceded by + :

5. $x^4 + 5x^3 - 16x^2 + 20x + 16 = x^4 + 5x^3 + (-16x^2 + 20x + 16).$

6. $x^4 + 4x^3y + x^2y^2 + 12xy^3 + 9y^4 = x^4 + 4x^3y + (x^2y^2 + 12xy^3 + 9y^4).$

7. $4x^4 + 4x^3y - 65x^2y^2 - 10xy^3 + 25y^4$
$= 4x^4 + 4x^3y + (-65x^2y^2 - 10xy^3 + 25y^4).$

8. $9a^4 + 18a^3b - 52a^2b^2 - 12ab^3 - 4b^4$
$= 9a^4 + 18a^3b + (-52a^2b^2 - 12ab^3 - 4b^4).$

Collect in parentheses the coefficients of x, y, and z:

9. $a^2x + 2a^2y + a^2z + abx - abz + b^2x + b^2y + b^2z$
$= a^2x + abx + b^2x + 2a^2y + b^2y + a^2z - abz + b^2z$
$= (a^2 + ab + b^2)x + (2a^2 + b^2)y + (a^2 - ab + b^2)z.$

10. $2ax - by + 3cz - dx + 3ey - 5fz + x - y - z$
$= 2ax - dx + x - by + 3ey - y + 3cz - 5fz - z$
$= (2a - d + 1)x - (b - 3e + 1)y + (3c - 5f - 1)z.$

11. $ax + bx - cz + 4ay - bz + cy + cx - by - az$
$= ax + bx + cx + 4ay - by + cy - az - bz - cz$
$= (a + b + c)x + (4a - b + c)y - (a + b + c)z.$

12. $3x - 4y + 5z + 3ax + 2by + 2cz - 5dx - 3ey - 4fz$
$= 3x + 3ax - 5dx - 4y + 2by - 3ey + 5z + 2cz - 4fz$
$= (3 + 3a - 5d)x - (4 - 2b + 3e)y + (5 + 2c - 4f)z.$

13. $mx + ny + pz + 3x - 4z + 5y - nx - my$
$= mx - nx + 3x - my + ny + 5y + pz - 4z$
$= (m - n + 3)x - (m - n - 5)y + (p - 4)z.$

14. $7ax - 3ay - 4az + 4bx - 2by + 3bz - 2cx + 2cy - 5cz$
$= 7ax + 4bx - 2cx - 3ay - 2by + 2cy - 4az + 3bz - 5cz$
$= (7a + 4b - 2c)x - (3a + 2b - 2c)y - (4a - 3b + 5c)z.$

15. $-5a^2x + 6a^3y + 3abz + 3ax + 5bx - 4b^2y + 6ay + 12bz$
$= -5a^2x + 3ax + 5bx + 6a^3y + 6ay - 4b^2y + 3abz + 12bz$
$= (-5a^2 + 3a + 5b)x + (6a^3 + 6a - 4b^2)y + (3ab + 12b)z.$

16. $a^2x - (b^2 - c^2)y + (c^2 - a^2)z + b^2x - c^2z - a^2y - b^2z$
$= a^2x + b^2x - a^2y - (b^2 - c^2)y - b^2z - c^2z + (c^2 - a^2)z$
$= (a^2 + b^2)x - (a^2 + b^2 - c^2)y - (b^2 + c^2 - c^2 + a^2)z$
$= (a^2 + b^2)x - (a^2 + b^2 - c^2)y - (a^2 + b^2)z.$

17. $12a^3x - 4b^3y + 2c^3z - 5a^3z + 7b^3x - 3c^3y + x - y$
$= 12a^3x + 7b^3x + x - 4b^3y - 3c^3y - y - 5a^3z + 2c^3z$
$= (12a^3 + 7b^3 + 1)x - (4b^3 + 3c^3 + 1)y - (5a^3 - 2c^3)z.$

18. $3ax - 2by + 5cz - 3ay - 2bx - 3bz + 2cy - 11cx - az$
$= 3ax - 2bx - 11cx - 3ay - 2by + 2cy - az - 3bz + 5cz$
$= (3a - 2b - 11c)x - (3a + 2b - 2c)y - (a + 3b - 5c)z.$

19. $az + 3y - 4bx + 2by - 3ax - z + 6bz + 2ay - 3x$
$= -3ax - 4bx - 3x + 2ay + 2by + 3y + az + 6bz - z$
$= -(3a + 4b + 3)x + (2a + 2b + 3)y + (a + 6b - 1)z.$

Exercise 20. Page 57

1. Add $2 + 3x + 5x^2 + 9x^3 + 17x^4 + 33x^5$; $1 + x - 7x^3 + 33x^4 - 130x^5$; $1 - 4x + 9x^2 - 16x^3 + 25x^4 - 36x^5$; $2 + 5x + 10x^2 + 17x^3 + 26x^4 + 37x^5$.

$$\begin{array}{r}
2 + 3x + 5x^2 + 9x^3 + 17x^4 + 33x^5 \\
1 + x - 7x^3 + 33x^4 - 130x^5 \\
1 - 4x + 9x^2 - 16x^3 + 25x^4 - 36x^5 \\
2 + 5x + 10x^2 + 17x^3 + 26x^4 + 37x^5 \\
\hline
6 + 5x + 24x^2 + 3x^3 + 101x^4 - 96x^5
\end{array}$$

2. Add $3a^2x^2 + 7ax^3 + 2x^4 - 7a^3x + 2a^4$; $5a^4 - 2a^3x - 3a^2x^2 - 5ax^3 + 2x^4$; $a^4 + 2a^3x - 4a^2x^2 + 2ax^3 + x^4$; $5a^2x^2 - 3x^4 - 4a^4 + 7a^3x + 5ax^3$.

$$\begin{array}{r}
2x^4 + 7ax^3 + 3a^2x^2 - 7a^3x + 2a^4 \\
2x^4 - 5ax^3 - 3a^2x^2 - 2a^3x + 5a^4 \\
x^4 + 2ax^3 - 4a^2x^2 + 2a^3x + a^4 \\
-3x^4 + 5ax^3 + 5a^2x^2 + 7a^3x - 4a^4 \\
\hline
2x^4 + 9ax^3 + a^2x^2 + 4a^4
\end{array}$$

3. Add $7(b - c)$; $4(b - c)$; $2a(b - c)$; $-3a(b - c)$; $-3(b - c)$; $-5(b - c)$.

$7(b - c) + 4(b - c) + 2a(b - c) - 3a(b - c) - 3(b - c) - 5(b - c)$
$= 3(b - c) - a(b - c) = (3 - a)(b - c).$

4. Add $(a + b)x + (b + c)y + (a + c)z$; $(b - c)x + (c - a)y + (a - b)z$; $(a + c)x + (a + b)y + (b + c)z$.

$$\begin{array}{r}
(a + b)x + (b + c)y + (a + c)z \\
(b - c)x + (c - a)y + (a - b)z \\
(a + c)x + (a + b)y + (b + c)z \\
\hline
(2a + 2b)x + (2b + 2c)y + (2a + 2c)z
\end{array}$$

5. Add $6m(x^2+1)-8m(x-1)+2m(x+1)$; $-3m(x^2+1)-2m(x-1)+3m(x+1)$; $2m(x^2+1)+9m(x-1)-4m(x+1)$.

$$\begin{array}{r}
6m(x^2+1)-8m(x-1)+2m(x+1)\\
-3m(x^2+1)-2m(x-1)+3m(x+1)\\
\underline{2m(x^2+1)+9m(x-1)-4m(x+1)}\\
5m(x^2+1)-\quad m(x-1)+\quad m(x+1)
\end{array}$$

$$= 5mx^2+5m-mx+m+mx+m=5mx^2+7m.$$

6. From a^4-b^4 take $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

$$\begin{array}{rrrrr}
a^4 & & & & -\ b^4\\
\underline{a^4} & \underline{-\ 4a^3b} & \underline{+\ 6a^2b^2} & \underline{-\ 4ab^3} & \underline{+\ \ b^4}\\
 & 4a^3b & -\ 6a^2b^2 & +\ 4ab^3 & -\ 2b^4
\end{array}$$

7. From $3y^3-6xy^2-4x^2y+x^3$ take $x^3+2x^2y-5xy^2+2y^3$.

$$\begin{array}{rrrr}
x^3 & -\ 4x^2y & -\ 6xy^2 & +\ 3y^3\\
\underline{x^3} & \underline{+\ 2x^2y} & \underline{-\ 5xy^2} & \underline{+\ 2y^3}\\
 & -\ 6x^2y & -\ \ xy^2 & +\ \ y^3
\end{array}$$

8. From $a^2b^2+3b^2c^2+4c^2a^2$ take $2a^2b^2-4bc+2b^2c^2$.

$$\begin{array}{rrrr}
a^2b^2 & +\ 3b^2c^2 & +\ 4c^2a^2 & \\
\underline{2a^2b^2} & \underline{+\ 2b^2c^2} & \underline{\qquad\quad} & \underline{-\ 4bc}\\
-\ \ a^2b^2 & +\ \ b^2c^2 & +\ 4c^2a^2 & +\ 4bc
\end{array}$$

9. From $(b+c)x^2+(a+c)xy-(a+b)y^2$ take $(a+c)x^2-(b-a)xy+(a-b)y^2$.

$$\begin{array}{l}
(b+c)x^2+(a+c)xy-(a+b)y^2\\
\underline{(a+c)x^2-(b-a)xy+(a-b)y^2}\\
(b-a)x^2+(b+c)xy-2ay^2
\end{array}$$

10. From $(a+b)^2-(a+b+c)+(b+c)^2$ take $(b+c)^2-2(a+b+c)-(a+b)^2$.

$$\begin{array}{rrr}
(a+b)^2 & -\quad(a+b+c) & +\ (b+c)^2\\
\underline{-\quad(a+b)^2} & \underline{-\ 2(a+b+c)} & \underline{+\ (b+c)^2}\\
2(a+b)^2 & +\quad(a+b+c) &
\end{array}$$

11. Simplify $4a-[3a-b-\{2a+b-(a-\overline{b-a})\}-2b]$.

$$\begin{aligned}
&4a-[3a-b-\{2a+b-(a-\overline{b-a})\}-2b]\\
&\quad=4a-[3a-b-\{2a+b-(a-b+a)\}-2b]\\
&\quad=4a-[3a-b-\{2a+b-a+b-a\}-2b]\\
&\quad=4a-[3a-b-2a-b+a-b+a-2b]\\
&\quad=4a-3a+b+2a+b-a+b-a+2b\\
&\quad=a+5b.
\end{aligned}$$

12. Simplify $x^2 - y^2 - [2x^2 + \{3x^2 - (2x^2 - y^2) - 2y^2\} + 2x^2]$.

$$\begin{aligned}
&x^2 - y^2 - [2x^2 + \{3x^2 - (2x^2 - y^2) - 2y^2\} + 2x^2]\\
&\quad = x^2 - y^2 - [2x^2 + \{3x^2 - 2x^2 + y^2 - 2y^2\} + 2x^2]\\
&\quad = x^2 - y^2 - [2x^2 + 3x^2 - 2x^2 + y^2 - 2y^2 + 2x^2]\\
&\quad = x^2 - y^2 - 2x^2 - 3x^2 + 2x^2 - y^2 + 2y^2 - 2x^2\\
&\quad = -4x^2.
\end{aligned}$$

13. Arrange $ax - x^2 - bx^2 + x^3 - ax^3 + bx - x + bx^3 - 2cx + ax^2$ according to the descending powers of x, and inclose the coefficients of like powers of x in parentheses.

$$\begin{aligned}
&ax - x^2 - bx^2 + x^3 - ax^3 + bx - x + bx^3 - 2cx + ax^2\\
&\quad = x^3 - ax^3 + bx^3 - x^2 + ax^2 - bx^2 - x + ax + bx - 2cx\\
&\quad = (1 - a + b)x^3 - (1 - a + b)x^2 - (1 - a - b + 2c)x.
\end{aligned}$$

14. Arrange $3ax^2 - 3bx + 2cx^3 - bx^3 + 2bx^2 - 2ax + 2cx^2 - 3cx + 3ax^3$ according to the descending powers of x, and inclose the coefficients of like powers of x in parentheses.

$$\begin{aligned}
&3ax^2 - 3bx + 2cx^3 - bx^3 + 2bx^2 - 2ax + 2cx^2 - 3cx + 3ax^3\\
&\quad = 3ax^3 - bx^3 + 2cx^3 + 3ax^2 + 2bx^2 + 2cx^2 - 2ax - 3bx - 3cx\\
&\quad = (3a - b + 2c)x^3 + (3a + 2b + 2c)x^2 - (2a + 3b + 3c)x.
\end{aligned}$$

15. From $3c + 4d - 5a - 2b$ take $2c - 3a + b - 2d$ and check the result when $a = 3$, $b = 4$, $c = 5$, $d = 6$.

Put $a = 3$, $b = 4$, $c = 5$, $d = 6$.

$$\begin{array}{rcrcr}
3c + 4d - 5a - 2b &=& 15 + 24 - 15 - 8 &=& 16\\
\underline{2c - 2d - 3a + b} &=& \underline{10 - 12 - 9 + 4} &=& \underline{-7}\\
c + 6d - 2a - 3b &=& 5 + 36 - 6 - 12 &=& 23
\end{array}$$

16. If $a = -2$, $b = -3$, $c = 4$, $d = 0$, find the value of

$$a^3 - 2b^3 + c^3 - 3abc + 4bcd + 2ac^2.$$

Put $a = -2$, $b = -3$, $c = 4$, $d = 0$.

$$\begin{aligned}
&a^3 - 2b^3 + c^3 - 3abc + 4bcd + 2ac^2\\
&\quad = (-2)^3 - 2(-3)^3 + (4)^3 - 3(-2)(-3)(4)\\
&\qquad + 4(-3)(4)(0) + 2(-2)(4)^2\\
&\quad = -8 + 54 + 64 - 72 + 0 - 64\\
&\quad = -26.
\end{aligned}$$

If $a = 5$, $b = 6$, $c = 7$, and $2s = a + b + c$, find the value of:

17. $s^2 - (s - a)(s - b) - (s - b)(s - c) - (s - a)(s - c)$.

$$2s = a + b + c = 5 + 6 + 7 = 18.$$
$$\therefore\ s = \tfrac{1}{2} \text{ of } 18 = 9;\ s - a = 9 - 5 = 4;\ s - b = 9 - 6 = 3;$$
$$s - c = 9 - 7 = 2.$$

$$s^2 - (s - a)(s - b) - (s - b)(s - c) - (s - a)(s - c)$$
$$= 9^2 - 4 \times 3 - 3 \times 2 - 4 \times 2 = 81 - 12 - 6 - 8 = 81 - 26 = 55.$$

18. $s^2 - (s - a)(s - b)(s - c) + (s - a)^2(s - b) + (s - b)^2(s - c)$
$$= 9^2 - 4 \times 3 \times 2 + 4^2 \times 3 + 3^2 \times 2 = 81 - 24 + 48 + 18 = 123.$$

19. $s^2 + (s - a)^2(s - b) + (s - b)^2(s - c) + (s - c)^2(s - a)$
$$= 9^2 + 4^2 \times 3 + 3^2 \times 2 + 2^2 \times 4 = 81 + 48 + 18 + 16 = 163.$$

20. What number must be added to $x^2 + 2xy + y^2$ that the sum may be $2x^2 - y^2$?

$$2x^2 - y^2 - (x^2 + 2xy + y^2) = 2x^2 - y^2 - x^2 - 2xy - y^2 = x^2 - 2xy - 2y^2.$$

21. What number must be added to $a^3 - 3a^2b + 3ab^2 - b^3$ that the sum may be $3a^3 - a^2b + 2b^3$?

$$3a^3 - a^2b + 2b^3 - (a^3 - 3a^2b + 3ab^2 - b^3)$$
$$= 3a^3 - a^2b + 2b^3 - a^3 + 3a^2b - 3ab^2 + b^3$$
$$= 2a^3 + 2a^2b - 3ab^2 + 3b^3.$$

22. What number must be subtracted from $2a^3 - 5b^3 + 3a^2b$ that the remainder may be $a^3 - 3a^2b + 3ab^2 - b^3$?

$$2a^3 - 5b^3 + 3a^2b - (a^3 - 3a^2b + 3ab^2 - b^3)$$
$$= 2a^3 - 5b^3 + 3a^2b - a^3 + 3a^2b - 3ab^2 + b^3$$
$$= a^3 + 6a^2b - 3ab^2 - 4b^3.$$

If $A = a^2 + b^2 + c^2$, $B = a^2 + b^2 - c^2$, $C = a^2 - b^2 + c^2$, $D = b^2 + c^2 - a^2$, find the value of:

23. $A + B + C + D$
$$= a^2 + b^2 + c^2 + a^2 + b^2 - c^2 + a^2 - b^2 + c^2 + b^2 + c^2 - a^2$$
$$= 2a^2 + 2b^2 + 2c^2.$$

24. $A - B - C + D$
$$= (a^2 + b^2 + c^2) - (a^2 + b^2 - c^2) - (a^2 - b^2 + c^2) + (b^2 + c^2 - a^2)$$
$$= a^2 + b^2 + c^2 - a^2 - b^2 + c^2 - a^2 + b^2 - c^2 + b^2 + c^2 - a^2$$
$$= -2a^2 + 2b^2 + 2c^2.$$

25. $A - B + C - D$
$= (a^2 + b^2 + c^2) - (a^2 + b^2 - c^2) + (a^2 - b^2 + c^2) - (b^2 + c^2 - a^2)$
$= a^2 + b^2 + c^2 - a^2 - b^2 + c^2 + a^2 - b^2 + c^2 - b^2 - c^2 + a^2$
$= 2a^2 - 2b^2 + 2c^2.$

26. $A - B - C - D$
$= (a^2 + b^2 + c^2) - (a^2 + b^2 - c^2) - (a^2 - b^2 + c^2) - (b^2 + c^2 - a^2)$
$= a^2 + b^2 + c^2 - a^2 - b^2 + c^2 - a^2 + b^2 - c^2 - b^2 - c^2 + a^2$
$= 0.$

If $A = 3a^2 - 2ab + 5b^2$, $B = 7a^2 - 8ab + 5b^2$, $C = 9a^2 - 5ab + 3b^2$, $D = 11a^2 - 3ab - 4b^2$, find the value of:

27. $A + B + C + D$
$= 3a^2 - 2ab + 5b^2 + 7a^2 - 8ab + 5b^2 + 9a^2 - 5ab + 3b^2$
$+ 11a^2 - 3ab - 4b^2$
$= 30a^2 - 18ab + 9b^2.$

28. $A - B - C + D$
$= (3a^2 - 2ab + 5b^2) - (7a^2 - 8ab + 5b^2) - (9a^2 - 5ab + 3b^2)$
$+ (11a^2 - 3ab - 4b^2)$
$= 3a^2 - 2ab + 5b^2 - 7a^2 + 8ab - 5b^2 - 9a^2 + 5ab - 3b^2$
$+ 11a^2 - 3ab - 4b^2$
$= -2a^2 + 8ab - 7b^2.$

29. $B - (A + C - D)$
$= B - A - C + D$
$= (7a^2 - 8ab + 5b^2) - (3a^2 - 2ab + 5b^2) - (9a^2 - 5ab + 3b^2)$
$+ (11a^2 - 3ab - 4b^2)$
$= 7a^2 - 8ab + 5b^2 - 3a^2 + 2ab - 5b^2 - 9a^2 + 5ab - 3b^2$
$+ 11a^2 - 3ab - 4b^2$
$= 6a^2 - 4ab - 7b^2.$

30. $A + B - (C + D)$
$= A + B - C - D$
$= (3a^2 - 2ab + 5b^2) + (7a^2 - 8ab + 5b^2) - (9a^2 - 5ab + 3b^2)$
$- (11a^2 - 3ab - 4b^2)$
$= 3a^2 - 2ab + 5b^2 + 7a^2 - 8ab + 5b^2 - 9a^2 + 5ab - 3b^2$
$- 11a^2 + 3ab + 4b^2$
$= -10a^2 - 2ab + 11b^2.$

31. $$\begin{aligned} & A - (B - C - D) \\ &= A - B + C + D \\ &= (3a^2 - 2ab + 5b^2) - (7a^2 - 8ab + 5b^2) + (9a^2 - 5ab + 3b^2) \\ &\quad + (11a^2 - 3ab - 4b^2) \\ &= 3a^2 - 2ab + 5b^2 - 7a^2 + 8ab - 5b^2 + 9a^2 - 5ab + 3b^2 \\ &\quad + 11a^2 - 3ab - 4b^2 \\ &= 16a^2 - 2ab - b^2. \end{aligned}$$

32. $$\begin{aligned} & B - \{C - [A - (B + D)]\} \\ &= B - \{C - [A - B - D]\} \\ &= B - \{C - A + B + D\} \\ &= B - C + A - B - D \\ &= A - C - D \\ &= (3a^2 - 2ab + 5b^2) - (9a^2 - 5ab + 3b^2) - (11a^2 - 3ab - 4b^2) \\ &= 3a^2 - 2ab + 5b^2 - 9a^2 + 5ab - 3b^2 - 11a^2 + 3ab + 4b^2 \\ &= -17a^2 + 6ab + 6b^2. \end{aligned}$$

Exercise 21. Page 60

1. Multiply $2x^2 - 3xy + y^2$ by $2x$.

$$\begin{array}{l} 2x^2 - 3xy + y^2 \\ 2x \\ \hline 4x^3 - 6x^2y + 2xy^2 \end{array}$$

2. Multiply $a^2 - 2ab + 3b^2$ by ab.

$$\begin{array}{l} a^2 - 2ab + 3b^2 \\ ab \\ \hline a^3b - 2a^2b^2 + 3ab^3 \end{array}$$

3. Multiply $1 + 2a - 3a^2$ by a^2.

$$\begin{array}{l} 1 + 2a - 3a^2 \\ a^2 \\ \hline a^2 + 2a^3 - 3a^4 \end{array}$$

4. Multiply $2a^2 - 3ab + b^2$ by $3ab$.

$$\begin{array}{l} 2a^2 - 3ab + b^2 \\ 3ab \\ \hline 6a^3b - 9a^2b^2 + 3ab^3 \end{array}$$

5. Multiply $x^7 - x^4 - 2x$ by $2x^2$.

$$\begin{array}{l} x^7 - x^4 - 2x \\ 2x^2 \\ \hline 2x^9 - 2x^6 - 4x^3 \end{array}$$

6. Multiply $x^3 - 15x - 10$ by $3ax$.

$$\begin{array}{l} x^3 - 15x - 10 \\ 3ax \\ \hline 3ax^4 - 45ax^2 - 30ax \end{array}$$

7. Multiply $2a^3 - 16a + 6$ by $3a$.

$$\begin{array}{l} 2a^3 - 16a + 6 \\ 3a \\ \hline 6a^4 - 48a^2 + 18a \end{array}$$

8. Multiply $4x^4 - x^2y^2 - 9y^4$ by $-xy$.

$$\begin{array}{l} 4x^4 - x^2y^2 - 9y^4 \\ -\ xy \\ \hline -4x^5y + x^3y^3 + 9xy^5 \end{array}$$

9. Multiply $6x^2 - 7x + 2$ by $-3x^2$.

$$\begin{array}{l} 6x^2 - 7x + 2 \\ -3x^2 \\ \hline -18x^4 + 21x^3 - 6x^2 \end{array}$$

10. Multiply $-5x^5y - 2x^4y^2 - x^2y^4$ by $-3y^2$.

$$\begin{array}{l} -5x^5y - 2x^4y^2 - x^2y^4 \\ -3y^2 \\ \hline 15x^5y^3 + 6x^4y^4 + 3x^2y^6 \end{array}$$

11. Multiply $a^2 - 3a + 10$ by $4a^2b$.

$$\begin{array}{l} a^2 - 3a + 10 \\ 4a^2b \\ \hline 4a^4b - 12a^3b + 40a^2b \end{array}$$

12. Multiply $9x^4 + 3x^2y^2 + 4y^4$ by $2xy$.

$$\begin{array}{l} 9x^4 + 3x^2y^2 + 4y^4 \\ 2xy \\ \hline 18x^5y + 6x^3y^3 + 8xy^5 \end{array}$$

13. Multiply $3x^3 - x^2 + 13x - 30$ by $2x^2$.

$$\begin{array}{l} 3x^3 - x^2 + 13x - 30 \\ 2x^2 \\ \hline 6x^5 - 2x^4 + 26x^3 - 60x^2 \end{array}$$

14. Multiply $x^2 + 2ax + a^2 - b^2$ by $-3ab$.

$$\begin{array}{l} x^2 + 2ax + a^2 - b^2 \\ -3ab \\ \hline -3abx^2 - 6a^2bx - 3a^3b + 3ab^3 \end{array}$$

15. Multiply $6ac + 9bc - 2ad - 3bd$ by $2ad$.

$$\begin{array}{l} 6ac + 9bc - 2ad - 3bd \\ 2ad \\ \hline 12a^2cd + 18abcd - 4a^2d^2 - 6abd^2 \end{array}$$

16. Multiply $3a^3 - 3a^2b + ab^2 - b^3$ by $-5ab$.

$$\begin{array}{l} 3a^3 - 3a^2b + ab^2 - b^3 \\ -5ab \\ \hline -15a^4b + 15a^3b^2 - 5a^2b^3 + 5ab^4 \end{array}$$

17. Multiply $x^4 - 6x^3 + 13x^2 - 12x + 4$ by $4x^2$.

$$\begin{array}{l} x^4 - 6x^3 + 13x^2 - 12x + 4 \\ 4x^2 \\ \hline 4x^6 - 24x^5 + 52x^4 - 48x^3 + 16x^2 \end{array}$$

18. Multiply $2a^5 - 4a^4 + 8a^3 - 12a^2 + 6a$ by $-3a$.

$$\begin{array}{l} 2a^5 - 4a^4 + 8a^3 - 12a^2 + 6a \\ -3a \\ \hline -6a^6 + 12a^5 - 24a^4 + 36a^3 - 18a^2 \end{array}$$

19. Multiply $a^6 - a^2b^4 - a^4b + b^5$ by $-2ab$.

$$\begin{array}{l} a^6 - a^2b^4 - a^4b + b^5 \\ -2ab \\ \hline -2a^7b + 2a^3b^5 + 2a^5b^2 - 2ab^6 \end{array}$$

Exercise 22. Page 62

1. Multiply $3a - 4b$ by $a - b$.

$$\begin{array}{lll} 3a & -4b & \\ a & -b & \\ \hline 3a^2 & -4ab & \\ & -3ab & +4b^2 \\ \hline 3a^2 & -7ab & +4b^2 \end{array}$$

2. Multiply $4b-5c$ by $3b+4c$.

$$\begin{array}{lll} 4b & -5c & \\ 3b & +4c & \\ \hline 12b^2 & -15bc & \\ & +16bc & -20c^2 \\ \hline 12b^2 & +bc & -20c^2 \end{array}$$

3. Multiply $4x - 9y$ by $x - 5y$.

$$\begin{array}{rrr}
4x & -9y & \\
x & -5y & \\
\hline
4x^2 & -9xy & \\
 & -20xy & +45y^2 \\
\hline
4x^2 & -29xy & +45y^2
\end{array}$$

4. Multiply $3a - 2b$ by $2a - b$.

$$\begin{array}{rrr}
3a & -2b & \\
2a & -b & \\
\hline
6a^2 & -4ab & \\
 & -3ab & +2b^2 \\
\hline
6a^2 & -7ab & +2b^2
\end{array}$$

5. Multiply $a + b$ by $x + y$.

$$\begin{array}{rrrr}
a & +b & & \\
x & +y & & \\
\hline
ax & +bx & & \\
 & & +ay & +by \\
\hline
ax & +bx & +ay & +by
\end{array}$$

6. Multiply $2a - 3b$ by $5a + 7b$.

$$\begin{array}{rrr}
2a & -3b & \\
5a & +7b & \\
\hline
10a^2 & -15ab & \\
 & +14ab & -21b^2 \\
\hline
10a^2 & -ab & -21b^2
\end{array}$$

7. Multiply $a - 5b$ by $a + 3b$.

$$\begin{array}{rrr}
a & -5b & \\
a & +3b & \\
\hline
a^2 & -5ab & \\
 & +3ab & -15b^2 \\
\hline
a^2 & -2ab & -15b^2
\end{array}$$

8. Multiply $3x - 5$ by $3x + 5$.

$$\begin{array}{rrr}
3x & -5 & \\
3x & +5 & \\
\hline
9x^2 & -15x & \\
 & +15x & -25 \\
\hline
9x^2 & & -25
\end{array}$$

9. Multiply $7x - 3$ by $5x - 4$.

$$\begin{array}{rrr}
7x & -3 & \\
5x & -4 & \\
\hline
35x^2 & -15x & \\
 & -28x & +12 \\
\hline
35x^2 & -43x & +12
\end{array}$$

10. Multiply $7a - 5b$ by $6a + 5b$.

$$\begin{array}{rrr}
7a & -5b & \\
6a & +5b & \\
\hline
42a^2 & -30ab & \\
 & +35ab & -25b^2 \\
\hline
42a^2 & +5ab & -25b^2
\end{array}$$

11. Multiply $a - b$ by $x + y$.

$$\begin{array}{rrrr}
a & -b & & \\
x & +y & & \\
\hline
ax & -bx & & \\
 & & +ay & -by \\
\hline
ax & -bx & +ay & -by
\end{array}$$

12. Multiply $3a + 5b$ by $7b - 4a$.

$$\begin{array}{rrr}
3a & +5b & \\
-4a & +7b & \\
\hline
-12a^2 & -20ab & \\
 & +21ab & +35b^2 \\
\hline
-12a^2 & +ab & +35b^2
\end{array}$$

13. Multiply $5x + 1$ by $7y - 2$.

$$\begin{array}{rrrr}
5x & +1 & & \\
7y & -2 & & \\
\hline
35xy & +7y & & \\
 & & -10x & -2 \\
\hline
35xy & +7y & -10x & -2
\end{array}$$

14. Multiply $2a + 3b$ by $2a - 3b$.

$$\begin{array}{rrr}
2a & +3b & \\
2a & -3b & \\
\hline
4a^2 & +6ab & \\
 & -6ab & -9b^2 \\
\hline
4a^2 & & -9b^2
\end{array}$$

15. Multiply $8x-7y$ by $7x+6y$.

$$\begin{array}{rrr}
8x & -7y & \\
7x & +6y & \\ \hline
56x^2 & -49xy & \\
 & +48xy & -42y^2 \\ \hline
56x^2 & -xy & -42y^2
\end{array}$$

16. Multiply $x+y$ by $x+y$.

$$\begin{array}{rrr}
x & +y & \\
x & +y & \\ \hline
x^2 & +xy & \\
 & +xy & +y^2 \\ \hline
x^2 & +2xy & +y^2
\end{array}$$

17. Multiply $x-y$ by $x-y$.

$$\begin{array}{rrr}
x & -y & \\
x & -y & \\ \hline
x^2 & -xy & \\
 & -xy & +y^2 \\ \hline
x^2 & -2xy & +y^2
\end{array}$$

18. Multiply $x-1$ by $x-2$.

$$\begin{array}{rrr}
x & -1 & \\
x & -2 & \\ \hline
x^2 & -x & \\
 & -2x & +2 \\ \hline
x^2 & -3x & +2
\end{array}$$

19. Multiply $x^2-2ax+a^2$ by $x-a$.

$$\begin{array}{rrrr}
x^2 & -2ax & +a^2 & \\
x & -a & & \\ \hline
x^3 & -2ax^2 & +a^2x & \\
 & -ax^2 & +2a^2x & -a^3 \\ \hline
x^3 & -3ax^2 & +3a^2x & -a^3
\end{array}$$

20. Multiply $a^2+2ab+b^2$ by $a+b$.

$$\begin{array}{rrrr}
a^2 & +2ab & +b^2 & \\
a & +b & & \\ \hline
a^3 & +2a^2b & +ab^2 & \\
 & +a^2b & +2ab^2 & +b^3 \\ \hline
a^3 & +3a^2b & +3ab^2 & +b^3
\end{array}$$

21. Multiply $3x^2-4x+7$ by $5x^2-x-4$.

$$\begin{array}{rrrrr}
3x^2 & -4x & +7 & & \\
5x^2 & -x & -4 & & \\ \hline
15x^4 & -20x^3 & +35x^2 & & \\
 & -3x^3 & +4x^2 & -7x & \\
 & & -12x^2 & +16x & -28 \\ \hline
15x^4 & -23x^3 & +27x^2 & +9x & -28
\end{array}$$

22. Multiply $5a^3-2a^2x+ax^2$ by $2a^2-ax+x^2$.

$$\begin{array}{rrrrr}
5a^3 & -2a^2x & +ax^2 & & \\
2a^2 & -ax & +x^2 & & \\ \hline
10a^5 & -4a^4x & +2a^3x^2 & & \\
 & -5a^4x & +2a^3x^2 & -a^2x^3 & \\
 & & +5a^3x^2 & -2a^2x^3 & +ax^4 \\ \hline
10a^5 & -9a^4x & +9a^3x^2 & -3a^2x^3 & +ax^4
\end{array}$$

23. Multiply $5a^2 - 3ab - 2b^2$ by $a^2 + 2ab$.

$$\begin{array}{rrrr}
5a^2 & -3ab & -2b^2 & \\
a^2 & +2ab & & \\ \hline
5a^4 & -3a^3b & -2a^2b^2 & \\
 & +10a^3b & -6a^2b^2 & -4ab^3 \\ \hline
5a^4 & +7a^3b & -8a^2b^2 & -4ab^3
\end{array}$$

24. Multiply $x^2 + 7x - 5$ by $x^2 - 3x + 7$.

$$\begin{array}{rrrrr}
x^2 & +7x & -5 & & \\
x^2 & -3x & +7 & & \\ \hline
x^4 & +7x^3 & -5x^2 & & \\
 & -3x^3 & -21x^2 & +15x & \\
 & & +7x^2 & +49x & -35 \\ \hline
x^4 & +4x^3 & -19x^2 & +64x & -35
\end{array}$$

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25. Multiply $3a^3b - 2a^2b^2 + ab^3$ by $2a^2 - ab - 5b^2$.

$$\begin{array}{rrrrr}
3a^3b & -2a^2b^2 & +ab^3 & & \\
2a^2 & -ab & -5b^2 & & \\ \hline
6a^5b & -4a^4b^2 & +2a^3b^3 & & \\
 & -3a^4b^2 & +2a^3b^3 & -a^2b^4 & \\
 & & -15a^3b^3 & +10a^2b^4 & -5ab^5 \\ \hline
6a^5b & -7a^4b^2 & -11a^3b^3 & +9a^2b^4 & -5ab^5
\end{array}$$

26. Multiply $x^2 - 2x + 1$ by $x^3 - 3x + 2$.

$$\begin{array}{rrrrrr}
x^2 & -2x & +1 & & & \\
x^3 & -3x & +2 & & & \\ \hline
x^5 & -2x^4 & +x^3 & & & \\
 & & -3x^3 & +6x^2 & -3x & \\
 & & & +2x^2 & -4x & +2 \\ \hline
x^5 & -2x^4 & -2x^3 & +8x^2 & -7x & +2
\end{array}$$

27. Multiply $x^2 - 5ax - 2a^2$ by $x^2 + 2ax + 3a^2$.

$$\begin{array}{rrrrr}
x^2 & -5ax & -2a^2 & & \\
x^2 & +2ax & +3a^2 & & \\ \hline
x^4 & -5ax^3 & -2a^2x^2 & & \\
 & +2ax^3 & -10a^2x^2 & -4a^3x & \\
 & & +3a^2x^2 & -15a^3x & -6a^4 \\ \hline
x^4 & -3ax^3 & -9a^2x^2 & -19a^3x & -6a^4
\end{array}$$

28. Multiply $7x^2 + y^2 - 3xy$ by $2x^3 + y - x$.

$$\begin{array}{lllllll}
7x^2 & -3xy & +y^2 & & & & \\
2x^3 & -x & +y & & & & \\
\hline
14x^5 & -6x^4y & +2x^3y^2 & & & & \\
 & & & -7x^3 & +3x^2y & -xy^2 & \\
 & & & & +7x^2y & -3xy^2 & +y^3 \\
\hline
14x^5 & -6x^4y & +2x^3y^2 & -7x^3 & +10x^2y & -4xy^2 & +y^3
\end{array}$$

29. Multiply $1 + 2x + x^2$ by $1 - 2y + y^2$.

$$\begin{array}{lllllllll}
1 & +2x & +x^2 & & & & & & \\
1 & -2y & +y^2 & & & & & & \\
\hline
1 & +2x & +x^2 & & & & & & \\
 & & & -2y & -4xy & -2x^2y & & & \\
 & & & & & & +y^2 & +2xy^2 & +x^2y^2 \\
\hline
1 & +2x & +x^2 & -2y & -4xy & -2x^2y & +y^2 & +2xy^2 & +x^2y^2
\end{array}$$

30. Multiply $x^2 + bx - c$ by $x^2 - ax + b$.

$$\begin{array}{llll}
\multicolumn{4}{l}{x^2 + bx - c} \\
\multicolumn{4}{l}{x^2 - ax + b} \\
\hline
x^4 + & bx^3 - & cx^2 & \\
- & ax^3 - & abx^2 + & acx \\
 & + & bx^2 + & b^2x - bc \\
\hline
\multicolumn{4}{l}{x^4 - (a - b)x^3 - (ab - b + c)x^2 + (ac + b^2)x - bc}
\end{array}$$

31. Multiply $x^2 - ax - b^2$ by $px + x^2 - q^2$.

$$\begin{array}{llll}
\multicolumn{4}{l}{x^2 - ax - b^2} \\
\multicolumn{4}{l}{x^2 + px - q^2} \\
\hline
x^4 - & ax^3 - & b^2x^2 & \\
+ & px^3 - & apx^2 - & b^2px \\
 & - & q^2x^2 + & aq^2x + b^2q^2 \\
\hline
\multicolumn{4}{l}{x^4 - (a - p)x^3 - (ap + b^2 + q^2)x^2 + (aq^2 - b^2p)x + b^2q^2}
\end{array}$$

32. Multiply $3x^3 - 2x^2 + x - 1$ by $5x^2 - 4x - 1$.

$$\begin{array}{llllll}
 & & 3x^3 & -2x^2 & +x & -1 \\
 & & & 5x^2 & -4x & -1 \\
\hline
15x^5 & -10x^4 & +5x^3 & -5x^2 & & \\
 & -12x^4 & +8x^3 & -4x^2 & +4x & \\
 & & -3x^3 & +2x^2 & -x & +1 \\
\hline
15x^5 & -22x^4 & +10x^3 & -7x^2 & +3x & +1
\end{array}$$

33. Multiply $3a + 4b - 5c$ by $4a - 3b + c$.

$$\begin{array}{rrrrrr}
3a & +4b & -5c & & & \\
4a & -3b & +c & & & \\ \hline
12a^2 & +16ab & -20ac & & & \\
 & -9ab & & -12b^2 & +15bc & \\
 & & +3ac & & +4bc & -5c^2 \\ \hline
12a^2 & +7ab & -17ac & -12b^2 & +19bc & -5c^2
\end{array}$$

34. Multiply $a^2 - 2ab + b^2$ by $5a^2 - 5ab + 4b^2$.

$$\begin{array}{rrrrr}
a^2 & -2ab & +b^2 & & \\
5a^2 & -5ab & +4b^2 & & \\ \hline
5a^4 & -10a^3b & +5a^2b^2 & & \\
 & -5a^3b & +10a^2b^2 & -5ab^3 & \\
 & & +4a^2b^2 & -8ab^3 & +4b^4 \\ \hline
5a^4 & -15a^3b & +19a^2b^2 & -13ab^3 & +4b^4
\end{array}$$

35. Multiply $2x^3 - 4x^2y - 5xy^2$ by $4x^3 + 8x^2y + 5xy^2$.

$$\begin{array}{rrrrr}
2x^3 & -4x^2y & -5xy^2 & & \\
4x^3 & +8x^2y & +5xy^2 & & \\ \hline
8x^6 & -16x^5y & -20x^4y^2 & & \\
 & +16x^5y & -32x^4y^2 & -40x^3y^3 & \\
 & & +10x^4y^2 & -20x^3y^3 & -25x^2y^4 \\ \hline
8x^6 & & -42x^4y^2 & -60x^3y^3 & -25x^2y^4
\end{array}$$

36. Multiply $a^2 - 3a + 1$ by $2a^2 + 4a - 3$.

$$\begin{array}{rrrrr}
a^2 & -3a & +1 & & \\
2a^2 & +4a & -3 & & \\ \hline
2a^4 & -6a^3 & +2a^2 & & \\
 & +4a^3 & -12a^2 & +4a & \\
 & & -3a^2 & +9a & -3 \\ \hline
2a^4 & -2a^3 & -13a^2 & +13a & -3
\end{array}$$

37. Multiply $a^m + 2a^{m+1} - 3a^{m+2} - 1$ by $a + 1$.

$$\begin{array}{rrrrrr}
-3a^{m+2} & +2a^{m+1} & +a^m & -1 & & \\
a & +1 & & & & \\ \hline
-3a^{m+3} & +2a^{m+2} & +a^{m+1} & & -a & \\
 & -3a^{m+2} & +2a^{m+1} & +a^m & & -1 \\ \hline
-3a^{m+3} & -a^{m+2} & +3a^{m+1} & +a^m & -a & -1
\end{array}$$

38. Multiply $a^n - 4a^{n-1} - 5a^{n-2} + a^{n-3}$ by $a - 1$.

$$\begin{array}{rrrrr}
 & a^n - 4a^{n-1} & - 5a^{n-2} & + a^{n-3} & \\
 & a - 1 & & & \\
\hline
a^{n+1} & - 4a^n & - 5a^{n-1} & + a^{n-2} & \\
 & - a^n & + 4a^{n-1} & + 5a^{n-2} & - a^{n-3} \\
\hline
a^{n+1} & - 5a^n & - a^{n-1} & + 6a^{n-2} & - a^{n-3}
\end{array}$$

39. Multiply $a^n - a^{n-1}b + a^{n-2}b^2$ by $a - b$.

$$\begin{array}{rrrr}
a^n & - a^{n-1}b & + a^{n-2}b^2 & \\
a & - b & & \\
\hline
a^{n+1} & - a^nb & + a^{n-1}b^2 & \\
 & - a^nb & + a^{n-1}b^2 & - a^{n-2}b^3 \\
\hline
a^{n+1} & - 2a^nb & + 2a^{n-1}b^2 & - a^{n-2}b^3
\end{array}$$

40. Multiply $1 - ax + bx^2 - cx^3$ by $1 - x^2 + x$.

$$\begin{array}{llllll}
1 & - ax & + bx^2 & - cx^3 & & \\
1 & + x & - x^2 & & & \\
\hline
1 & - ax & + bx^2 & - cx^3 & & \\
 & + x & - ax^2 & + bx^3 & - cx^4 & \\
 & & - x^2 & + ax^3 & - bx^4 & + cx^5 \\
\hline
1 & + (1 - a)x & + (b - a - 1)x^2 & + (a + b - c)x^3 & - (b + c)x^4 & + cx^5
\end{array}$$

41. Multiply $4b^3 - 2b + 1 - 3b^2$ by $4b^2 + 5b + 2b^3$.

$$\begin{array}{rrrrrr}
4b^3 & - 3b^2 & - 2b & + 1 & & \\
2b^3 & + 4b^2 & + 5b & & & \\
\hline
8b^6 & - 6b^5 & - 4b^4 & + 2b^3 & & \\
 & + 16b^5 & - 12b^4 & - 8b^3 & + 4b^2 & \\
 & & + 20b^4 & - 15b^3 & - 10b^2 & + 5b \\
\hline
8b^6 & + 10b^5 & + 4b^4 & - 21b^3 & - 6b^2 & + 5b
\end{array}$$

42. Multiply $a^3 + 3a - 2a^2 + 1$ by $4a^2 + 3 - 2a$.

$$\begin{array}{rrrrrr}
 & & a^3 & - 2a^2 & + 3a & + 1 \\
 & & 4a^2 & - 2a & + 3 & \\
\hline
4a^5 & - 8a^4 & + 12a^3 & + 4a^2 & & \\
 & - 2a^4 & + 4a^3 & - 6a^2 & - 2a & \\
 & & + 3a^3 & - 6a^2 & + 9a & + 3 \\
\hline
4a^5 & - 10a^4 & + 19a^3 & - 8a^2 & + 7a & + 3
\end{array}$$

43. Multiply $bc + ac + ab$ by $a - b + c$.

$$\begin{array}{l}
ab + ac + bc \\
a - b + c \\
\hline
a^2b + a^2c + abc \\
\hphantom{a^2b + a^2c\ } - abc - ab^2 - b^2c \\
\hphantom{a^2b + a^2c\ } + abc \hphantom{- ab^2 - b^2c\ } + ac^2 + bc^2 \\
\hline
a^2b + a^2c + abc - ab^2 - b^2c + ac^2 + bc^2
\end{array}$$

44. Multiply $2b^2d^2 + 3b^4 - 7bd^3 - 5b^3d + d^4$ by $3b^2 + 3d^2 - 5bd$.

$$\begin{array}{l}
3b^4 - 5b^3d + 2b^2d^2 - 7bd^3 + d^4 \\
3b^2 - 5bd + 3d^2 \\
\hline
9b^6 - 15b^5d + 6b^4d^2 - 21b^3d^3 + 3b^2d^4 \\
\hphantom{9b^6\ } - 15b^5d + 25b^4d^2 - 10b^3d^3 + 35b^2d^4 - 5bd^5 \\
\hphantom{9b^6 - 15b^5d\ } + 9b^4d^2 - 15b^3d^3 + 6b^2d^4 - 21bd^5 + 3d^6 \\
\hline
9b^6 - 30b^5d + 40b^4d^2 - 46b^3d^3 + 44b^2d^4 - 26bd^5 + 3d^6
\end{array}$$

45. Multiply $x^{m+1} - 3x^{m+2} + x^{m+3} - 2x^{m+4}$
by $2x^{m-1} + 3x^{m-2} - 4x^{m-3}$.

$$\begin{array}{l}
x^{m+1} - 3x^{m+2} + x^{m+3} - 2x^{m+4} \\
-4x^{m-3} + 3x^{m-2} + 2x^{m-1} \\
\hline
-4x^{2m-2} + 12x^{2m-1} - 4x^{2m} + 8x^{2m+1} \\
\hphantom{-4x^{2m-2}\ } + 3x^{2m-1} - 9x^{2m} + 3x^{2m+1} - 6x^{2m+2} \\
\hphantom{-4x^{2m-2} + 12x^{2m-1}\ } + 2x^{2m} - 6x^{2m+1} + 2x^{2m+2} - 4x^{2m+3} \\
\hline
-4x^{2m-2} + 15x^{2m-1} - 11x^{2m} + 5x^{2m+1} - 4x^{2m+2} - 4x^{2m+3}
\end{array}$$

46. Multiply $a^n + 3a^{n-2} - 2a^{n-1}$ by $2a^{n+1} + a^{n+2} - 3a^n$.

$$\begin{array}{l}
a^n - 2a^{n-1} + 3a^{n-2} \\
a^{n+2} + 2a^{n+1} - 3a^n \\
\hline
a^{2n+2} - 2a^{2n+1} + 3a^{2n} \\
\hphantom{a^{2n+2}\ } + 2a^{2n+1} - 4a^{2n} + 6a^{2n-1} \\
\hphantom{a^{2n+2} + 2a^{2n+1}\ } - 3a^{2n} + 6a^{2n-1} - 9a^{2n-2} \\
\hline
a^{2n+2} \hphantom{+ 2a^{2n+1}\ } - 4a^{2n} + 12a^{2n-1} - 9a^{2n-2}
\end{array}$$

47. Multiply $x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + x^{n-5}y^4$ by $x + y$.

$$\begin{array}{l}
x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + x^{n-5}y^4 \\
x + y \\
\hline
x^n - x^{n-1}y + x^{n-2}y^2 - x^{n-3}y^3 + x^{n-4}y^4 \\
\hphantom{x^n\ } + x^{n-1}y - x^{n-2}y^2 + x^{n-3}y^3 - x^{n-4}y^4 + x^{n-5}y^5 \\
\hline
x^n \hphantom{+ x^{n-1}y - x^{n-2}y^2 + x^{n-3}y^3 - x^{n-4}y^4\ } + x^{n-5}y^5
\end{array}$$

48. Multiply $a^p - 3a^{p-1} + 4a^{p-2} - 6a^{p-3} + 5a^{p-4}$ by $2a^3 - a^2 + a$.

$$\begin{array}{rrrrrrr}
a^p & -3a^{p-1} & +4a^{p-2} & -6a^{p-3} & +5a^{p-4} & & \\
2a^3 & -a^2 & +a & & & & \\
\hline
2a^{p+3} & -6a^{p+2} & +8a^{p+1} & -12a^p & +10a^{p-1} & & \\
 & -a^{p+2} & +3a^{p+1} & -4a^p & +6a^{p-1} & -5a^{p-2} & \\
 & & +a^{p+1} & -3a^p & +4a^{p-1} & -6a^{p-2} & +5a^{p-3} \\
\hline
2a^{p+3} & -7a^{p+2} & +12a^{p+1} & -19a^p & +20a^{p-1} & -11a^{p-2} & +5a^{p-3}
\end{array}$$

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1. Multiply $2x^3 + 4x^2y - 3xy^2 - y^3$ by $5x^2 - 2xy - y^2$.

$$\begin{array}{rrrrrr}
2 & +4 & -3 & -1 & & \\
5 & -2 & -1 & & & \\
\hline
10 & +20 & -15 & -5 & & \\
 & -4 & -8 & +6 & +2 & \\
 & & -2 & -4 & +3 & +1 \\
\hline
10 & +16 & -25 & -3 & +5 & +1
\end{array}$$

Hence, the product is $10x^5 + 16x^4y - 25x^3y^2 - 3x^2y^3 + 5xy^4 + y^5$.

2. Multiply $a^5 - 2a^4b + 3a^3b^2 - 2a^2b^3 + ab^4 - b^5$ by $2a^2 - ab + b^2$.

$$\begin{array}{rrrrrrrr}
1 & -2 & +3 & -2 & +1 & -1 & & \\
2 & -1 & +1 & & & & & \\
\hline
2 & -4 & +6 & -4 & +2 & -2 & & \\
 & -1 & +2 & -3 & +2 & -1 & +1 & \\
 & & +1 & -2 & +3 & -2 & +1 & -1 \\
\hline
2 & -5 & +9 & -9 & +7 & -5 & +2 & -1
\end{array}$$

Hence, the product is

$2a^7 - 5a^6b + 9a^5b^2 - 9a^4b^3 + 7a^3b^4 - 5a^2b^5 + 2ab^6 - b^7$.

3. Multiply $5c^8 - 3c^6 + 9c^4 - 10c^2 + 4$ by $9c^6 + 5c^4 - 13c^2$.

$$\begin{array}{rrrrrrr}
5 & -3 & +9 & -10 & +4 & & \\
9 & +5 & -13 & & & & \\
\hline
45 & -27 & +81 & -90 & +36 & & \\
 & +25 & -15 & +45 & -50 & +20 & \\
 & & -65 & +39 & -117 & +130 & -52 \\
\hline
45 & -2 & +1 & -6 & -131 & +150 & -52
\end{array}$$

Hence, the product is $45c^{14} - 2c^{12} + c^{10} - 6c^8 - 131c^6 + 150c^4 - 52c^2$.

4. Multiply $x^4 - 3x^3 + 2x^2 - x + 5$ by $3x^3 - 2x^2 - 5x + 3$.

$$\begin{array}{rrrrrrrr}
1 & -3 & +2 & -1 & +5 & & & \\
3 & -2 & -5 & +3 & & & & \\
\hline
3 & -9 & +6 & -3 & +15 & & & \\
 & -2 & +6 & -4 & +2 & -10 & & \\
 & & -5 & +15 & -10 & +5 & -25 & \\
 & & & +3 & -9 & +6 & -3 & +15 \\
\hline
3 & -11 & +7 & +11 & -2 & +1 & -28 & +15
\end{array}$$

Hence, the product is $3x^7 - 11x^6 + 7x^5 + 11x^4 - 2x^3 + x^2 - 28x + 15$.

5. Multiply $2 + 7a - 2a^2 + 8a^3 - 4a^4$ by $2 + 5a + 3a^2 - 7a^3$.

$$\begin{array}{rrrrrrrr}
2 & +7 & -2 & +8 & -4 & & & \\
2 & +5 & +3 & -7 & & & & \\
\hline
4 & +14 & -4 & +16 & -8 & & & \\
 & +10 & +35 & -10 & +40 & -20 & & \\
 & & +6 & +21 & -6 & +24 & -12 & \\
 & & & -14 & -49 & +14 & -56 & +28 \\
\hline
4 & +24 & +37 & +13 & -23 & +18 & -68 & +28
\end{array}$$

Hence, the product is

$$4 + 24a + 37a^2 + 13a^3 - 23a^4 + 18a^5 - 68a^6 + 28a^7.$$

6. Multiply $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x^2 + 2xy + y^2$.

$$\begin{array}{rrrrrrr}
1 & -4 & +6 & -4 & +1 & & \\
1 & +2 & +1 & & & & \\
\hline
1 & -4 & +6 & -4 & +1 & & \\
 & +2 & -8 & +12 & -8 & +2 & \\
 & & +1 & -4 & +6 & -4 & +1 \\
\hline
1 & -2 & -1 & +4 & -1 & -2 & +1
\end{array}$$

Hence, the product is $x^6 - 2x^5y - x^4y^2 + 4x^3y^3 - x^2y^4 - 2xy^5 + y^6$.

7. Multiply $3a^4b^2 - 6a^3b^3 + 4a^2b^4 + ab^5$ by $2a^2b - 5ab^2 + 2b^3$.

$$\begin{array}{rrrrrr}
3 & -6 & +4 & +1 & & \\
2 & -5 & +2 & & & \\
\hline
6 & -12 & +8 & +2 & & \\
 & -15 & +30 & -20 & -5 & \\
 & & +6 & -12 & +8 & +2 \\
\hline
6 & -27 & +44 & -30 & +3 & +2
\end{array}$$

Hence, the product is $6a^6b^3 - 27a^5b^4 + 44a^4b^5 - 30a^3b^6 + 3a^2b^7 + 2ab^8$.

8. Multiply $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x^3 - 3x^2y + 3xy^2 - y^3$.

$$\begin{array}{rrrrrrrr}
1 & -4 & +6 & -4 & +1 & & & \\
1 & -3 & +3 & -1 & & & & \\ \hline
1 & -4 & +6 & -4 & +1 & & & \\
 & -3 & +12 & -18 & +12 & -3 & & \\
 & & +3 & -12 & +18 & -12 & +3 & \\
 & & & -1 & +4 & -6 & +4 & -1 \\ \hline
1 & -7 & +21 & -35 & +35 & -21 & +7 & -1
\end{array}$$

Hence, the product is

$$x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7.$$

9. Multiply $3x^4 + 6x^3y - 9xy^3 + 12y^4$ by $2x^2 - 3xy + y^2$.

$$\begin{array}{rrrrrrr}
3 & +6 & +0 & -9 & +12 & & \\
2 & -3 & +1 & & & & \\ \hline
6 & +12 & +0 & -18 & +24 & & \\
 & -9 & -18 & -0 & +27 & -36 & \\
 & & +3 & +6 & +0 & -9 & +12 \\ \hline
6 & +3 & -15 & -12 & +51 & -45 & +12
\end{array}$$

Hence, the product is

$$6x^6 + 3x^5y - 15x^4y^2 - 12x^3y^3 + 51x^2y^4 - 45xy^5 + 12y^6.$$

10. Multiply $3a^3x - 2a^2x^2 + 5ax^3 - 8x^4$ by $-2a^2 + 6ax - 4x^2$.

$$\begin{array}{rrrrrr}
3 & -2 & +5 & -8 & & \\
-2 & +6 & -4 & & & \\ \hline
-6 & +4 & -10 & +16 & & \\
 & +18 & -12 & +30 & -48 & \\
 & & -12 & +8 & -20 & +32 \\ \hline
-6 & +22 & -34 & +54 & -68 & +32
\end{array}$$

Hence, the product is

$$-6a^5x + 22a^4x^2 - 34a^3x^3 + 54a^2x^4 - 68ax^5 + 32x^6.$$

11. Multiply $2by^5 - 3b^2y^4 + 4b^3y^3 - 5b^4y^2$ by $y^3 - 4by^2 + 6b^2y$.

$$\begin{array}{rrrrrr}
2 & -3 & +4 & -5 & & \\
1 & -4 & +6 & & & \\ \hline
1 & -3 & +4 & -5 & & \\
 & -8 & +12 & -16 & +20 & \\
 & & +12 & -18 & +24 & -30 \\ \hline
1 & -11 & +28 & -39 & +44 & -30
\end{array}$$

Hence, the product is

$$2by^8 - 11b^2y^7 + 28b^3y^6 - 39b^4y^5 + 44b^5y^4 - 30b^6y^3.$$

12. Multiply $c^3 - 8c^2z + 5cz^2 + 11z^3$ by $6c^2 - 2cz + 12z^2$.

$$\begin{array}{rrrrrr}
1 & -8 & +5 & +11 & & \\
6 & -2 & +12 & & & \\
\hline
6 & -48 & +30 & +66 & & \\
 & -2 & +16 & -10 & -22 & \\
 & & +12 & -96 & +60 & +132 \\
\hline
6 & -50 & +58 & -40 & +38 & +132
\end{array}$$

Hence, the product is $6c^5 - 50c^4z + 58c^3z^2 - 40c^2z^3 + 38cz^4 + 132z^5$.

13. Multiply $a^4 - 7a^3b + 6a^2b^2 + 8ab^3 - 2b^4$ by $a^2 - 3ab + 2b^2$.

$$\begin{array}{rrrrrrr}
1 & -7 & +6 & +8 & -2 & & \\
1 & -3 & +2 & & & & \\
\hline
1 & -7 & +6 & +8 & -2 & & \\
 & -3 & +21 & -18 & -24 & +6 & \\
 & & +2 & -14 & +12 & +16 & -4 \\
\hline
1 & -10 & +29 & -24 & -14 & +22 & -4
\end{array}$$

Hence, the product is

$$a^6 - 10a^5b + 29a^4b^2 - 24a^3b^3 - 14a^2b^4 + 22ab^5 - 4b^6.$$

14. Multiply $2x - x^2 + 4x^3 - 10 + 7x^5$ by $2x^4 - 3x^2 + 2x^3 + 2 + 3x$.

$$\begin{array}{rrrrrrrrrr}
7 & +0 & +4 & -1 & +2 & -10 & & & & \\
2 & +2 & -3 & +3 & +2 & & & & & \\
\hline
14 & +0 & +8 & -2 & +4 & -20 & & & & \\
 & +14 & +0 & +8 & -2 & +4 & -20 & & & \\
 & & -21 & -0 & -12 & +3 & -6 & +30 & & \\
 & & & +21 & +0 & +12 & -3 & +6 & -30 & \\
 & & & & +14 & +0 & +8 & -2 & +4 & -20 \\
\hline
14 & +14 & -13 & +27 & +4 & -1 & -21 & +34 & -26 & -20
\end{array}$$

Hence, the product is

$$14x^9 + 14x^8 - 13x^7 + 27x^6 + 4x^5 - x^4 - 21x^3 + 34x^2 - 26x - 20.$$

15. Multiply $2x^4 - 3x^3y + 2y^4 - 2xy^3 + 4x^2y^2$ by $2xy^2 - y^3 + x^3 - 3x^2y$.

$$\begin{array}{rrrrrrrr}
2 & -3 & +4 & -2 & +2 & & & \\
1 & -3 & +2 & -1 & & & & \\
\hline
2 & -3 & +4 & -2 & +2 & & & \\
 & -6 & +9 & -12 & +6 & -6 & & \\
 & & +4 & -6 & +8 & -4 & +4 & \\
 & & & -2 & +3 & -4 & +2 & -2 \\
\hline
2 & -9 & +17 & -22 & +19 & -14 & +6 & -2
\end{array}$$

Hence, the product is

$$2x^7 - 9x^6y + 17x^5y^2 - 22x^4y^3 + 19x^3y^4 - 14x^2y^5 + 6xy^6 - 2y^7.$$

16. Multiply $3y^m - 2y^{m+1} - 5y^{m+2} + y^{m+3}$
by $3y^{m-3} + 2y^{m-2} - 5y^{m-1} - y^m$.

$$\begin{array}{rrrrrrr}
3 & -2 & -5 & +1 & & & \\
3 & +2 & -5 & -1 & & & \\ \hline
9 & -6 & -15 & +3 & & & \\
 & +6 & -4 & -10 & +2 & & \\
 & & -15 & +10 & +25 & -5 & \\
 & & & -3 & +2 & +5 & -1 \\ \hline
9 & +0 & -34 & +0 & +29 & +0 & -1
\end{array}$$

Hence, the product is $9y^{2m-3} - 34y^{2m-1} + 29y^{2m+1} - y^{2m+3}$.

17. Multiply $a^n - a^{n+1} + a^{n+2} - a^{n+3}$ by $a^3 - a^2 - a + 1$.

$$\begin{array}{rrrrrrr}
1 & -1 & +1 & -1 & & & \\
1 & -1 & -1 & +1 & & & \\ \hline
1 & -1 & +1 & -1 & & & \\
 & -1 & +1 & -1 & +1 & & \\
 & & -1 & +1 & -1 & +1 & \\
 & & & +1 & -1 & +1 & -1 \\ \hline
1 & -2 & +1 & +0 & -1 & +2 & -1
\end{array}$$

Hence, the product is $a^n - 2a^{n+1} + a^{n+2} - a^{n+4} + 2a^{n+5} - a^{n+6}$.

18. Multiply $a^6 - 3a^5 + 6a^4 - 7a^3 + 6a^2 - 3a + 1$ by $a^2 - a + 1$.

$$\begin{array}{rrrrrrrrr}
1 & -3 & +6 & -7 & +6 & -3 & +1 & & \\
1 & -1 & +1 & & & & & & \\ \hline
1 & -3 & +6 & -7 & +6 & -3 & +1 & & \\
 & -1 & +3 & -6 & +7 & -6 & +3 & -1 & \\
 & & +1 & -3 & +6 & -7 & +6 & -3 & +1 \\ \hline
1 & -4 & +10 & -16 & +19 & -16 & +10 & -4 & +1
\end{array}$$

Hence, the product is

$$a^8 - 4a^7 + 10a^6 - 16a^5 + 19a^4 - 16a^3 + 10a^2 - 4a + 1.$$

19. Multiply $2m^4 - 6m^3 + 3m^2 - 3m + 1$ by $m^2 - 3m + 1$.

$$\begin{array}{rrrrrrr}
2 & -6 & +3 & -3 & +1 & & \\
1 & -3 & +1 & & & & \\ \hline
2 & -6 & +3 & -3 & +1 & & \\
 & -6 & +18 & -9 & +9 & -3 & \\
 & & +2 & -6 & +3 & -3 & +1 \\ \hline
2 & -12 & +23 & -18 & +13 & -6 & +1
\end{array}$$

Hence, the product is $2m^6 - 12m^5 + 23m^4 - 18m^3 + 13m^2 - 6m + 1$.

20. Multiply $6a^3x - 17a^2x^2 + 14ax^3 - 3x^4$ by $2a - 3x$.

$$\begin{array}{rrrrr}
6 & -17 & +14 & -3 & \\
2 & -3 & & & \\
\hline
12 & -34 & +28 & -6 & \\
 & -18 & +51 & -42 & +9 \\
\hline
12 & -52 & +79 & -48 & +9
\end{array}$$

Hence, the product is $12a^4x - 52a^3x^2 + 79a^2x^3 - 48ax^4 + 9x^5$.

21. Multiply $x^6 - 5x^4 + 3x^3 + 6x^2 - 7x + 2$ by $x^3 - 3x^2 + 3x - 1$.

$$\begin{array}{rrrrrrrrrr}
1 & +0 & -5 & +3 & +6 & -7 & +2 & & & \\
1 & -3 & +3 & -1 & & & & & & \\
\hline
1 & +0 & -5 & +3 & +6 & -7 & +2 & & & \\
 & -3 & -0 & +15 & -9 & -18 & +21 & -6 & & \\
 & & +3 & +0 & -15 & +9 & +18 & -21 & +6 & \\
 & & & -1 & -0 & +5 & -3 & -6 & +7 & -2 \\
\hline
1 & -3 & -2 & +17 & -18 & -11 & +38 & -33 & +13 & -2
\end{array}$$

Hence, the product is

$x^9 - 3x^8 - 2x^7 + 17x^6 - 18x^5 - 11x^4 + 38x^3 - 33x^2 + 13x - 2$.

22. Multiply $8ab^6 - 8a^6b + 24a^2b^5 + a^5b^2 + 14a^3b^4 + 12a^4b^3 + 5a^7$ by $7a^2b + 2ab^2 + 5a^3$.

$$\begin{array}{rrrrrrrrr}
8 & +24 & +14 & +12 & +1 & -8 & +5 & & \\
2 & +7 & +5 & & & & & & \\
\hline
16 & +48 & +28 & +24 & +2 & -16 & +10 & & \\
 & +56 & +168 & +98 & +84 & +7 & -56 & +35 & \\
 & & +40 & +120 & +70 & +60 & +5 & -40 & +25 \\
\hline
16 & +104 & +236 & +242 & +156 & +51 & -41 & -5 & +25
\end{array}$$

Hence, the product is

$16a^2b^8 + 104a^3b^7 + 236a^4b^6 + 242a^5b^5 + 156a^6b^4 + 51a^7b^3 - 41a^8b^2 - 5a^9b + 25a^{10}$.

23. Multiply $x^5 + 4x^3 - 92x - 3x^4 + 26x^2 + 5$ by $x^2 - 3x + 4$.

$$\begin{array}{rrrrrrrr}
1 & -3 & +4 & +26 & -92 & +5 & & \\
1 & -3 & +4 & & & & & \\
\hline
1 & -3 & +4 & +26 & -92 & +5 & & \\
 & -3 & +9 & -12 & -78 & +276 & -15 & \\
 & & +4 & -12 & +16 & +104 & -368 & +20 \\
\hline
1 & -6 & +17 & +2 & -154 & +385 & -383 & +20
\end{array}$$

Hence, the product is

$x^7 - 6x^6 + 17x^5 + 2x^4 - 154x^3 + 385x^2 - 383x + 20$.

13. Divide $3y^m - 2y^{m+1} - 5y^{m+2} + y^{m+3}$ by y^4. ✓

$$\begin{array}{r|l} y^4 & 3y^m - 2y^{m+1} - 5y^{m+2} + y^{m+3} \\ \cline{2-2} & 3y^{m-4} - 2y^{m-3} - 5y^{m-2} + y^{m-1} \end{array}$$

14. Divide $4x^n + 3x^{n+1} - 2x^{n+2} + x^{n+3}$ by x^{n-1}.

$$\begin{array}{r|l} x^{n-1} & 4x^n + 3x^{n+1} - 2x^{n+2} + x^{n+3} \\ \cline{2-2} & 4x + 3x^2 - 2x^3 + x^4 \end{array}$$

15. Divide $2a^p - 4a^{p+1} - 6a^{p+2} + 8a^{p+3}$ by $2a^{p-2}$. 2

$$\begin{array}{r|l} 2a^{p-2} & 2a^p - 4a^{p+1} - 6a^{p+2} + 8a^{p+3} \\ \cline{2-2} & a^2 - 2a^3 - 3a^4 + 4a^5 \end{array}$$

16. Divide $c^{r+3} + c^{r+4} - 2c^{r+5} - 3c^{r+6}$ by c^{r+2}.

$$\begin{array}{r|l} c^{r+2} & c^{r+3} + c^{r+4} - 2c^{r+5} - 3c^{r+6} \\ \cline{2-2} & c + c^2 - 2c^3 - 3c^4 \end{array}$$

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1. Divide $35x^3 + 47x^2 + 13x + 1$ by $5x + 1$.

$$\begin{array}{l|l} 35x^3 + 47x^2 + 13x + 1 & 5x + 1 \\ \cline{2-2} \underline{35x^3 + 7x^2} & 7x^2 + 8x + 1 \\ \qquad + 40x^2 + 13x & \\ \qquad \underline{+ 40x^2 + 8x} & \\ \qquad\qquad + 5x + 1 & \\ \qquad\qquad \underline{+ 5x + 1} & \end{array}$$

2. Divide $2a^4 - 6a^3 + 3a^2 - 3a + 1$ by $a^2 - 3a + 1$.

$$\begin{array}{l|l} 2a^4 - 6a^3 + 3a^2 - 3a + 1 & a^2 - 3a + 1 \\ \cline{2-2} \underline{2a^4 - 6a^3 + 2a^2} & 2a^2 + 1 \\ \qquad\qquad + a^2 - 3a + 1 & \\ \qquad\qquad \underline{+ a^2 - 3a + 1} & \end{array}$$

3. Divide $42a^4 + 41a^3 - 9a^2 - 9a - 1$ by $7a^2 + 8a + 1$.

$$\begin{array}{l|l} 42a^4 + 41a^3 - 9a^2 - 9a - 1 & 7a^2 + 8a + 1 \\ \cline{2-2} \underline{42a^4 + 48a^3 + 6a^2} & 6a^2 - a - 1 \\ \qquad - 7a^3 - 15a^2 - 9a & \\ \qquad \underline{- 7a^3 - 8a^2 - a} & \\ \qquad\qquad - 7a^2 - 8a - 1 & \\ \qquad\qquad \underline{- 7a^2 - 8a - 1} & \end{array}$$

4. Divide $2x^4-4x^3-3x^2-x+1$ by x^2-3x+1.

$$\begin{array}{r|l}
2x^4-4x^3-3x^2-x+1 & x^2-3x+1 \\ \cline{2-2}
\underline{2x^4-6x^3+2x^2} & 2x^2+2x+1 \\
+2x^3-5x^2-x & \\
\underline{+2x^3-6x^2+2x} & \\
+x^2-3x+1 & \\
\underline{+x^2-3x+1} &
\end{array}$$

5. Divide $6a^3x-17a^2x^2+14ax^3-3x^4$ by $2a-3x$.

$$\begin{array}{r|l}
6a^3x-17a^2x^2+14ax^3-3x^4 & 2a-3x \\ \cline{2-2}
\underline{6a^3x-9a^2x^2} & 3a^2x-4ax^2+x^3 \\
-8a^2x^2+14ax^3 & \\
\underline{-8a^2x^2+12ax^3} & \\
+2ax^3-3x^4 & \\
\underline{+2ax^3-3x^4} &
\end{array}$$

6. Divide $4x^2-28xy+49y^2$ by $2x-7y$.

$$\begin{array}{r|l}
4x^2-28xy+49y^2 & 2x-7y \\ \cline{2-2}
\underline{4x^2-14xy} & 2x-7y \\
-14xy+49y^2 & \\
\underline{-14xy+49y^2} &
\end{array}$$

7. Divide $4a^3+4a^2-29a+21$ by $2a-3$.

$$\begin{array}{r|l}
4a^3+4a^2-29a+21 & 2a-3 \\ \cline{2-2}
\underline{4a^3-6a^2} & 2a^2+5a-7 \\
+10a^2-29a & \\
\underline{+10a^2-15a} & \\
-14a+21 & \\
\underline{-14a+21} &
\end{array}$$

8. Divide $6a^3-13a^2b+4ab^2+3b^3$ by $2a-3b$.

$$\begin{array}{r|l}
6a^3-13a^2b+4ab^2+3b^3 & 2a-3b \\ \cline{2-2}
\underline{6a^3-9a^2b} & 3a^2-2ab-b^2 \\
-4a^2b+4ab^2 & \\
\underline{-4a^2b+6ab^2} & \\
-2ab^2+3b^3 & \\
\underline{-2ab^2+3b^3} &
\end{array}$$

9. Divide $6x^4 - 13ax^3 - 13a^3x + 13a^2x^2 - 5a^4$ by $2x^2 - 3ax - a^2$.

$$\begin{array}{l|l}
6x^4 - 13ax^3 + 13a^2x^2 - 13a^3x - 5a^4 & 2x^2 - 3ax - a^2 \\ \cline{2-2}
\underline{6x^4 - 9ax^3 - 3a^2x^2} & 3x^2 - 2ax + 5a^2 \\
\quad - 4ax^3 + 16a^2x^2 - 13a^3x & \\
\quad \underline{- 4ax^3 + 6a^2x^2 + 2a^3x} & \\
\qquad\qquad + 10c^2x^2 - 15a^3x - 5a^4 & \\
\qquad\qquad \underline{+ 10a^2x^2 - 15a^3x - 5a^4} &
\end{array}$$

10. Divide $45x^4 + 35x^2 - 4 + 18x^3 + 4x$ by $9x^3 + 7x - 2$.

$$\begin{array}{l|l}
45x^4 + 18x^3 + 35x^2 + 4x - 4 & 9x^3 + 7x - 2 \\ \cline{2-2}
\underline{45x^4 \qquad\quad + 35x^2 - 10x} & 5x + 2 \\
\quad + 18x^3 \qquad\quad + 14x - 4 & \\
\quad \underline{+ 18x^3 \qquad\quad + 14x - 4} &
\end{array}$$

11. Divide $6x^4 + 13xy^3 - x^3y + 4y^4 + 2x^2y^2$ by $2x^2 - 3xy + 4y^2$.

$$\begin{array}{l|l}
6x^4 - x^3y + 2x^2y^2 + 13xy^3 + 4y^4 & 2x^2 - 3xy + 4y^2 \\ \cline{2-2}
\underline{6x^4 - 9x^3y + 12x^2y^2} & 3x^2 + 4xy + y^2 \\
\quad + 8x^3y - 10x^2y^2 + 13xy^3 & \\
\quad \underline{+ 8x^3y - 12x^2y^2 + 16xy^3} & \\
\qquad\qquad + 2x^2y^2 - 3xy^3 + 4y^4 & \\
\qquad\qquad \underline{+ 2x^2y^2 - 3xy^3 + 4y^4} &
\end{array}$$

12. Divide $2x^4 + x^3y - 3xy^3 - 13x^2y^2 + y^4$ by $x^2 - 2xy - y^2$.

$$\begin{array}{l|l}
2x^4 + x^3y - 13x^2y^2 - 3xy^3 + y^4 & x^2 - 2xy - y^2 \\ \cline{2-2}
\underline{2x^4 - 4x^3y - 2x^2y^2} & 2x^2 + 5xy - y^2 \\
\quad + 5x^3y - 11x^2y^2 - 3xy^3 & \\
\quad \underline{+ 5x^3y - 10x^2y^2 - 5xy^3} & \\
\qquad\qquad - x^2y^2 + 2xy^3 + y^4 & \\
\qquad\qquad \underline{- x^2y^2 + 2xy^3 + y^4} &
\end{array}$$

13. Divide $15x^5 - 3xy^4 + 4x^3y^2 + 10x^4y + 6x^2y^3$ by $5x^3 + 3xy^2$.

$$\begin{array}{l|l}
15x^5 + 10x^4y + 4x^3y^2 + 6x^2y^3 - 3xy^4 & 5x^3 + 3xy^2 \\ \cline{2-2}
\underline{15x^5 \qquad\quad + 9x^3y^2} & 3x^2 + 2xy - y^2 \\
\quad + 10x^4y - 5x^3y^2 + 6x^2y^3 & \\
\quad \underline{+ 10x^4y \qquad\quad + 6x^2y^3} & \\
\qquad\qquad - 5x^3y^2 \qquad\quad - 3xy^4 & \\
\qquad\qquad \underline{- 5x^3y^2 \qquad\quad - 3xy^4} &
\end{array}$$

14. Divide $21a^4 - 16a^3b - 5ab^3 + 16a^2b^2 + 2b^4$ by $3a^2 - ab + b^2$.

$$\begin{array}{ll}
21a^4 - 16a^3b + 16a^2b^2 - 5ab^3 + 2b^4 & \underline{\big|\,3a^2 - ab + b^2} \\
\underline{21a^4 - 7a^3b + 7a^2b^2} & 7a^2 - 3ab + 2b^2 \\
\quad - 9a^3b + 9a^2b^2 - 5ab^3 & \\
\quad \underline{- 9a^3b + 3a^2b^2 - 3ab^3} & \\
\qquad + 6a^2b^2 - 2ab^3 + 2b^4 & \\
\qquad \underline{+ 6a^2b^2 - 2ab^3 + 2b^4} &
\end{array}$$

15. Divide $-73x^2 - 25 + 56x^4 + 95x - 59x^3$ by $-11x + 7x^3 - 3x^2 + 5$.

$$\begin{array}{ll}
56x^4 - 59x^3 - 73x^2 + 95x - 25 & \underline{\big|\,7x^3 - 3x^2 - 11x + 5} \\
\underline{56x^4 - 24x^3 - 88x^2 + 40x} & 8x - 5 \\
\quad - 35x^3 + 15x^2 + 55x - 25 & \\
\quad \underline{- 35x^3 + 15x^2 + 55x - 25} &
\end{array}$$

16. Divide $7a^4b^3 - 19a^2b^5 + a^7 - 9a^5b^2 + 13a^3b^4 - b^7 + 8ab^6$ by $b^3 - 2ab^2 + a^3$.

$$\begin{array}{ll}
a^7 - 9a^5b^2 + 7a^4b^3 + 13a^3b^4 - 19a^2b^5 + 8ab^6 - b^7 & \underline{\big|\,a^3 - 2ab^2 + b^3} \\
\underline{a^7 - 2a^5b^2 + a^4b^3} & a^4 - 7a^2b^2 + 6ab^3 - b^4 \\
\quad - 7a^5b^2 + 6a^4b^3 + 13a^3b^4 - 19a^2b^5 & \\
\quad \underline{- 7a^5b^2 + 14a^3b^4 - 7a^2b^5} & \\
\qquad + 6a^4b^3 - a^3b^4 - 12a^2b^5 + 8ab^6 & \\
\qquad \underline{+ 6a^4b^3 - 12a^2b^5 + 6ab^6} & \\
\qquad\quad - a^3b^4 + 2ab^6 - b^7 & \\
\qquad\quad \underline{- a^3b^4 + 2ab^6 - b^7} &
\end{array}$$

17. Divide $4x^5y - 4x^3y^3 + 4x^2y^4 - xy^5$ by $2x^2 - 2xy + y^2$.

$$\begin{array}{ll}
4x^5y - 4x^3y^3 + 4x^2y^4 - xy^5 & \underline{\big|\,2x^2 - 2xy + y^2} \\
\underline{4x^5y - 4x^4y^2 + 2x^3y^3} & 2x^3y + 2x^2y^2 - xy^3 \\
\quad + 4x^4y^2 - 6x^3y^3 + 4x^2y^4 & \\
\quad \underline{+ 4x^4y^2 - 4x^3y^3 + 2x^2y^4} & \\
\qquad - 2x^3y^3 + 2x^2y^4 - xy^5 & \\
\qquad \underline{- 2x^3y^3 + 2x^2y^4 - xy^5} &
\end{array}$$

18. Divide $4x^4 - x^2y^2 + 6xy^3 - 9y^4$ by $2x^2 - xy + 3y^2$.

$$\begin{array}{ll}
4x^4 - x^2y^2 + 6xy^3 - 9y^4 & \underline{\big|\,2x^2 - xy + 3y^2} \\
\underline{4x^4 - 2x^3y + 6x^2y^2} & 2x^2 + xy - 3y^2 \\
\quad + 2x^3y - 7x^2y^2 + 6xy^3 & \\
\quad \underline{+ 2x^3y - x^2y^2 + 3xy^3} & \\
\qquad - 6x^2y^2 + 3xy^3 - 9y^4 & \\
\qquad \underline{- 6x^2y^2 + 3xy^3 - 9y^4} &
\end{array}$$

19. Divide $x^6 - 5x^4 + 3x^3 + 6x^2 - 7x + 2$ by $x^3 - 3x^2 + 3x - 1$.

$$\begin{array}{ll}
x^6 \qquad\quad - 5x^4 + 3x^3 + 6x^2 - 7x + 2 & \multicolumn{1}{|l}{x^3 - 3x^2 + 3x - 1} \\ \cline{2-2}
\underline{x^6 - 3x^5 + 3x^4 - x^3} & x^3 + 3x^2 + x - 2 \\
\quad + 3x^5 - 8x^4 + 4x^3 + 6x^2 & \\
\quad \underline{+ 3x^5 - 9x^4 + 9x^3 - 3x^2} & \\
\qquad\qquad + x^4 - 5x^3 + 9x^2 - 7x & \\
\qquad\qquad \underline{+ x^4 - 3x^3 + 3x^2 - x} & \\
\qquad\qquad\qquad - 2x^3 + 6x^2 - 6x + 2 & \\
\qquad\qquad\qquad \underline{- 2x^3 + 6x^2 - 6x + 2} &
\end{array}$$

20. Divide $a^4 - b^4$ by $a^3 + a^2b + ab^2 + b^3$.

$$\begin{array}{ll}
a^4 \qquad\qquad\qquad\qquad - b^4 & \multicolumn{1}{|l}{a^3 + a^2b + ab^2 + b^3} \\ \cline{2-2}
\underline{a^4 + a^3b + a^2b^2 + ab^3} & a - b \\
\quad - a^3b - a^2b^2 - ab^3 - b^4 & \\
\quad \underline{- a^3b - a^2b^2 - ab^3 - b^4} &
\end{array}$$

21. Divide $3x^5 + 4x^3y^2 + 4x^2y^3 + xy^4 + 4y^5$ by $x^3 + x^2y + xy^2 + y^3$.

$$\begin{array}{ll}
3x^5 \qquad\quad + 4x^3y^2 + 4x^2y^3 + xy^4 + 4y^5 & \multicolumn{1}{|l}{x^3 + x^2y + xy^2 + y^3} \\ \cline{2-2}
\underline{3x^5 + 3x^4y + 3x^3y^2 + 3x^2y^3} & 3x^2 - 3xy + 4y^2 \\
\quad - 3x^4y + x^3y^2 + x^2y^3 + xy^4 & \\
\quad \underline{- 3x^4y - 3x^3y^2 - 3x^2y^3 - 3xy^4} & \\
\qquad\qquad + 4x^3y^2 + 4x^2y^3 + 4xy^4 + 4y^5 & \\
\qquad\qquad \underline{+ 4x^3y^2 + 4x^2y^3 + 4xy^4 + 4y^5} &
\end{array}$$

22. Divide $a^5 - 5a^3b^2 - 5a^2b^3 + b^5$ by $a^2 - 3ab + b^2$.

$$\begin{array}{ll}
a^5 \qquad\quad - 5a^3b^2 - 5a^2b^3 \qquad\quad + b^5 & \multicolumn{1}{|l}{a^2 - 3ab + b^2} \\ \cline{2-2}
\underline{a^5 - 3a^4b + a^3b^2} & a^3 + 3a^2b + 3ab^2 + b^3 \\
\quad + 3a^4b - 6a^3b^2 - 5a^2b^3 & \\
\quad \underline{+ 3a^4b - 9a^3b^2 + 3a^2b^3} & \\
\qquad\qquad + 3a^3b^2 - 8a^2b^3 & \\
\qquad\qquad \underline{+ 3a^3b^2 - 9a^2b^3 + 3ab^4} & \\
\qquad\qquad\qquad + a^2b^3 - 3ab^4 + b^5 & \\
\qquad\qquad\qquad \underline{+ a^2b^3 - 3ab^4 + b^5} &
\end{array}$$

23. Divide $a^4 + 4ab^3 - 4abc + 3b^4 + 2b^2c - c^2$ by $a^2 - 2ab + 3b^2 - c$.

$$\begin{array}{l|l}
a^4 \qquad\qquad\qquad + 4ab^3 - 4abc + 3b^4 + 2b^2c - c^2 & a^2 - 2ab + 3b^2 - c \\ \cline{2-2}
a^4 - 2a^3b + 3a^2b^2 - a^2c & a^2 + 2ab + b^2 + c \\ \cline{1-1}
\quad + 2a^3b - 3a^2b^2 + a^2c + 4ab^3 - 4abc & \\
\quad + 2a^3b - 4a^2b^2 \qquad + 6ab^3 - 2abc & \\ \cline{1-1}
\qquad + a^2b^2 + a^2c - 2ab^3 - 2abc + 3b^4 + 2b^2c & \\
\qquad + a^2b^2 \qquad - 2ab^3 \qquad + 3b^4 - b^2c & \\ \cline{1-1}
\qquad\qquad + a^2c \qquad - 2abc \qquad + 3b^2c - c^2 & \\
\qquad\qquad + a^2c \qquad - 2abc \qquad + 3b^2c - c^2 & \\ \cline{1-1}
\end{array}$$

24. Divide $4x^4 - 5x^2y^2 - 8x^2 + y^4 - 4y^2 + 4$ by $2x^2 - 3xy + y^2 - 2$.

$$\begin{array}{l|l}
4x^4 \qquad - 5x^2y^2 \qquad - 8x^2 \qquad + y^4 - 4y^2 + 4 & 2x^2 - 3xy + y^2 - 2 \\ \cline{2-2}
4x^4 - 6x^3y + 2x^2y^2 \qquad - 4x^2 & 2x^2 + 3xy + y^2 - 2 \\ \cline{1-1}
\quad + 6x^3y - 7x^2y^2 \qquad - 4x^2 & \\
\quad + 6x^3y - 9x^2y^2 + 3xy^3 \qquad - 6xy & \\ \cline{1-1}
\qquad + 2x^2y^2 - 3xy^3 - 4x^2 + 6xy + y^4 - 4y^2 & \\
\qquad + 2x^2y^2 - 3xy^3 \qquad\qquad + y^4 - 2y^2 & \\ \cline{1-1}
\qquad\qquad - 4x^2 + 6xy \qquad - 2y^2 + 4 & \\
\qquad\qquad - 4x^2 + 6xy \qquad - 2y^2 + 4 & \\ \cline{1-1}
\end{array}$$

25. Divide $a - 4a^3 + 12a^4 - 9a^5$ by $1 + 2a - 3a^2$.

$$\begin{array}{l|l}
a \qquad\quad - 4a^3 + 12a^4 - 9a^5 & 1 + 2a - 3a^2 \\
\cline{2-2}
\underline{a + 2a^2 - 3a^3} & a - 2a^2 + 3a^3 \\
\quad - 2a^2 - a^3 + 12a^4 & \\
\quad \underline{- 2a^2 - 4a^3 + 6a^4} & \\
\qquad\qquad + 3a^3 + 6a^4 - 9a^5 & \\
\qquad\qquad \underline{+ 3a^3 + 6a^4 - 9a^5} &
\end{array}$$

26. Divide $a^{m+2} - a^{m+3} + 37a^{m+5} - 55a^{m+6} + 50a^{m+7}$ by $a^2 - 3a^3 + 10a^4$.

$$\begin{array}{l|l}
a^{m+2} - a^{m+3} \qquad + 37a^{m+5} - 55a^{m+6} + 50a^{m+7} & a^2 - 3a^3 + 10a^4 \\
\cline{2-2}
\underline{a^{m+2} - 3a^{m+3} + 10a^{m+4}} & a^m + 2a^{m+1} - 4a^{m+2} + 5a^{m+3} \\
\quad + 2a^{m+3} - 10a^{m+4} + 37a^{m+5} & \\
\quad \underline{+ 2a^{m+3} - 6a^{m+4} + 20a^{m+5}} & \\
\qquad\qquad - 4a^{m+4} + 17a^{m+5} - 55a^{m+6} & \\
\qquad\qquad \underline{- 4a^{m+4} + 12a^{m+5} - 40a^{m+6}} & \\
\qquad\qquad\qquad + 5a^{m+5} - 15a^{m+6} + 50a^{m+7} & \\
\qquad\qquad\qquad \underline{+ 5a^{m+5} - 15a^{m+6} + 50a^{m+7}} &
\end{array}$$

27. Divide $6d^{x+3}-5d^{x+2}-6d^{x+1}+19d^{x}-21d^{x-1}+4d^{x-2}$ by $2d^3+d^2-4d$.

$$\begin{array}{l|l}
6d^{x+3}-5d^{x+2}-6d^{x+1}+19d^{x}-21d^{x-1}+4d^{x-2} & 2d^3+d^2-4d \\ \cline{2-2}
\underline{6d^{x+3}+3d^{x+2}-12d^{x+1}} & 3d^{x}-4d^{x-1}+5d^{x-2}-d^{x-3} \\
\qquad -8d^{x+2}+6d^{x+1}+19d^{x} & \\
\qquad \underline{-8d^{x+2}-4d^{x+1}+16d^{x}} & \\
\qquad\qquad +10d^{x+1}+3d^{x}-21d^{x-1} & \\
\qquad\qquad \underline{+10d^{x+1}+5d^{x}-20d^{x-1}} & \\
\qquad\qquad\qquad -2d^{x}-d^{x-1}+4d^{x-2} & \\
\qquad\qquad\qquad \underline{-2d^{x}-d^{x-1}+4d^{x-2}} &
\end{array}$$

28. Divide $4a^{x+1}-30a^{x}+19a^{x-1}+5a^{x-2}+9a^{x-4}$ by $a^{x-3}-7a^{x-4}+2a^{x-5}-3a^{x-6}$.

$$\begin{array}{l|l}
4a^{x+1}-30a^{x}+19a^{x-1}+5a^{x-2}\qquad+9a^{x-4} & a^{x-3}-7a^{x-4}+2a^{x-5}-3a^{x-6} \\ \cline{2-2}
\underline{4a^{x+1}-28a^{x}+8a^{x-1}-12a^{x-2}} & 4a^4-2a^3-3a^2 \\
\qquad -2a^{x}+11a^{x-1}+17a^{x-2} & \\
\qquad \underline{-2a^{x}+14a^{x-1}-4a^{x-2}+6a^{x-3}} & \\
\qquad\qquad -3a^{x-1}+21a^{x-2}-6a^{x-3}+9a^{x-4} & \\
\qquad\qquad \underline{-3a^{x-1}+21a^{x-2}-6a^{x-3}+9a^{x-4}} &
\end{array}$$

29. Divide $6x^{m-n+2}+x^{m-n+1}-22x^{m-n}+19x^{m-n-1}-4x^{m-n-2}$ by $3x^{3-n}-4x^{2-n}+x^{1-n}$.

$$\begin{array}{l|l}
6x^{m-n+2}+x^{m-n+1}-22x^{m-n}+19x^{m-n-1}-4x^{m-n-2} & 3x^{3-n}-4x^{2-n}+x^{1-n} \\ \cline{2-2}
\underline{6x^{m-n+2}-8x^{m-n+1}+2x^{m-n}} & 2x^{m-1}+3x^{m-2}-4x^{m-3} \\
\qquad +9x^{m-n+1}-24x^{m-n}+19x^{m-n-1} & \\
\qquad \underline{+9x^{m-n+1}-12x^{m-n}+3x^{m-n-1}} & \\
\qquad\qquad -12x^{m-n}+16x^{m-n-1}-4x^{m-n-2} & \\
\qquad\qquad \underline{-12x^{m-n}+16x^{m-n-1}-4x^{m-n-2}} &
\end{array}$$

30. Divide $6b^{x+y+2}+b^{x+y+1}-9b^{x+y}+11b^{x+y-1}-6b^{x+y-2}+b^{x+y-3}$ by $2b^{y+2}+3b^{y+1}-b^{y}$.

$$\begin{array}{l|l}
6b^{x+y+2}+b^{x+y+1}-9b^{x+y}+11b^{x+y-1}-6b^{x+y-2}+b^{x+y-3} & 2b^{y+2}+3b^{y+1}-b^{y} \\ \cline{2-2}
\underline{6b^{x+y+2}+9b^{x+y+1}-3b^{x+y}} & 3b^{x}-4b^{x-1}+3b^{x-2}-b^{x-3} \\
\qquad -8b^{x+y+1}-6b^{x+y}+11b^{x+y-1} & \\
\qquad \underline{-8b^{x+y+1}-12b^{x+y}+4b^{x+y-1}} & \\
\qquad\qquad +6b^{x+y}+7b^{x+y-1}-6b^{x+y-2} & \\
\qquad\qquad \underline{+6b^{x+y}+9b^{x+y-1}-3b^{x+y-2}} & \\
\qquad\qquad\qquad -2b^{x+y-1}-3b^{x+y-2}+b^{x+y-3} & \\
\qquad\qquad\qquad \underline{-2b^{x+y-1}-3b^{x+y-2}+b^{x+y-3}} &
\end{array}$$

31. Divide $a^{m+n}b^n - 4\,a^{m+n-1}b^{2n} - 27\,a^{m+n-2}b^{3n} + 42\,a^{m+n-3}b^{4n}$ by $a^nb^n - 7\,a^{n-1}b^{2n}$.

$$\begin{array}{l|l}
a^{m+n}b^n - 4\,a^{m+n-1}b^{2n} - 27\,a^{m+n-2}b^{3n} + 42\,a^{m+n-3}b^{4n} & a^nb^n - 7\,a^{n-1}b^{2n} \\ \cline{2-2}
\underline{a^{m+n}b^n - 7\,a^{m+n-1}b^{2n}\phantom{- 27\,a^{m+n-2}b^{3n}}} & a^m + 3\,a^{m-1}b^n - 6\,a^{m-2}b^{2n} \\
\qquad + 3\,a^{m+n-1}b^{2n} - 27\,a^{m+n-2}b^{3n} & \\
\qquad \underline{+ 3\,a^{m+n-1}b^{2n} - 21\,a^{m+n-2}b^{3n}\phantom{+ 42\,a^{m+n-3}b^{4n}}} & \\
\qquad\qquad\qquad - 6\,a^{m+n-2}b^{3n} + 42\,a^{m+n-3}b^{4n} & \\
\qquad\qquad\qquad \underline{- 6\,a^{m+n-2}b^{3n} + 42\,a^{m+n-3}b^{4n}} &
\end{array}$$

32. Divide $x^{3n} - 3\,x^{2n}y^n + 3\,x^ny^{2n} - y^{3n}$ by $x^n - y^n$.

$$\begin{array}{l|l}
x^{3n} - 3\,x^{2n}y^n + 3\,x^ny^{2n} - y^{3n} & x^n - y^n \\ \cline{2-2}
\underline{x^{3n} - x^{2n}y^n\phantom{+ 3\,x^ny^{2n}}} & x^{2n} - 2\,x^ny^n + y^{2n} \\
\quad - 2\,x^{2n}y^n + 3\,x^ny^{2n} & \\
\quad \underline{- 2\,x^{2n}y^n + 2\,x^ny^{2n}\phantom{- y^{3n}}} & \\
\qquad\qquad + x^ny^{2n} - y^{3n} & \\
\qquad\qquad \underline{+ x^ny^{2n} - y^{3n}} &
\end{array}$$

33. Divide $x^2y^2z + 3x^2y^3 - 3xyz^3 + yz^4 - x^2z^3 + y^5 - 2y^3z^2 + 3xy^3z - 3x^2yz^2$ by $y^2 - z^2$.

$$
\begin{array}{llllllllll|l}
x^2y^2z & +3x^2y^3 & -x^2z^3 & -3x^2yz^2 & +3xy^3z & -3xyz^3 & +y^5 & -2y^3z^2 & +yz^4 & & y^2-z^2 \\
\cline{11-11}
\underline{x^2y^2z} & & \underline{-x^2z^3} & & & & & & & & x^2z+3x^2y+3xyz+y^3-yz^2 \\
 & +3x^2y^3 & & -3x^2yz^2 & & & & & & & \\
 & \underline{+3x^2y^3} & & \underline{-3x^2yz^2} & & & & & & & \\
 & & & & +3xy^3z & -3xyz^3 & & & & & \\
 & & & & \underline{+3xy^3z} & \underline{-3xyz^3} & & & & & \\
 & & & & & & +y^5 & -2y^3z^2 & & & \\
 & & & & & & \underline{+y^5} & \underline{-\ \ y^3z^2} & & & \\
 & & & & & & & -\ \ y^3z^2 & +yz^4 & & \\
 & & & & & & & \underline{-\ \ y^3z^2} & \underline{+yz^4} & & \\
\end{array}
$$

Exercise 26. Page 72

1. Divide $2a^4 - 3a^3b + 6a^2b^2 - ab^3 + 6b^4$ by $a^2 - 2ab + 3b^2$.

$$
\begin{array}{rl}
2-3+6-1+6 & \big|\,\underline{1-2+3} \\
\underline{2-4+6} & \,2+1+2 \\
+1+0-1 & \\
\underline{+1-2+3} & \\
+2-4+6 & \\
\underline{+2-4+6} &
\end{array}
$$

Hence, the quotient is $2a^2 + ab + 2b^2$.

2. Divide $2x^4 + 3x^3y + x^2y^2 - 4xy^3 - 2y^4$ by $x^2 + 2xy + 2y^2$.

$$
\begin{array}{rl}
2+3+1-4-2 & \big|\,\underline{1+2+2} \\
\underline{2+4+4} & \,2-1-1 \\
-1-3-4 & \\
\underline{-1-2-2} & \\
-1-2-2 & \\
\underline{-1-2-2} &
\end{array}
$$

Hence, the quotient is $2x^2 - xy - y^2$.

3. Divide $a^5 - 2a^4b - a^3b^2 + 5a^2b^3 - 4ab^4 + b^5$ by $a^3 - 2ab^2 + b^3$.

$$
\begin{array}{rl}
1-2-1+5-4+1 & \big|\,\underline{1+0-2+1} \\
\underline{1+0-2+1} & \,1-2+1 \\
-2+1+4-4 & \\
\underline{-2+0+4-2} & \\
+1+0-2+1 & \\
\underline{+1+0-2+1} &
\end{array}
$$

Hence, the quotient is $a^2 - 2ab + b^2$.

4. Divide $c^5 + 2c^4d - 5c^3d^2 + 15c^2d^3 - 6cd^4 + 9d^5$ by $c^2 - 2cd + 3d^2$.

$$
\begin{array}{rl}
1+2-5+15-6+9 & \big|\,\underline{1-2+3} \\
\underline{1-2+3} & \,1+4+0+3 \\
+4-8+15 & \\
\underline{+4-8+12} & \\
+\ 3-6+9 & \\
\underline{+\ 3-6+9} &
\end{array}
$$

Hence, the quotient is $c^3 + 4c^2d + 3d^3$.

5. Divide $9x^5+5x^3y^2-4x^2y^3-8xy^4-2y^5$ by $3x^3+2x^2y+4xy^2+2y^3$.

$$\begin{array}{l|l}
9+0+5-4-8-2 & \underline{3+2+4+2} \\
\underline{9+6+12+6} & 3-2-1 \\
\quad -6-7-10-8 & \\
\quad \underline{-6-4-8-4} & \\
\qquad -3-2-4-2 & \\
\qquad \underline{-3-2-4-2} &
\end{array}$$

Hence, the quotient is $3x^2-2xy-y^2$.

6. Divide $2a^6-5a^5+6a^4-6a^3+6a^2-4a+1$ by $a^4-a^3+a^2-a+1$.

$$\begin{array}{l|l}
2-5+6-6+6-4+1 & \underline{1-1+1-1+1} \\
\underline{2-2+2-2+2} & 2-3+1 \\
\quad -3+4-4+4-4 & \\
\quad \underline{-3+3-3+3-3} & \\
\qquad +1-1+1-1+1 & \\
\qquad \underline{+1-1+1-1+1} &
\end{array}$$

Hence, the quotient is $2a^2-3a+1$.

7. Divide $x^7-2x^6+x^5+2x^3-3x^2+1$ by x^5+2x+1.

$$\begin{array}{l|l}
1-2+1+0+2-3+0+1 & \underline{1+0+0+0+2+1} \\
\underline{1+0+0+0+2+1} & 1-2+1 \\
\quad -2+1+0+0-4+0 & \\
\quad \underline{-2+0+0+0-4-2} & \\
\qquad +1+0+0+0+2+1 & \\
\qquad \underline{+1+0+0+0+2+1} &
\end{array}$$

Hence, the quotient is x^2-2x+1.

8. Divide $1+2b-2b^2-3b^3-b^4-4b^5-b^7$ by $1-2b^2-b^4$.

$$\begin{array}{l|l}
1+2-2-3-1-4+0-1 & \underline{1+0-2+0-1} \\
\underline{1+0-2+0-1} & 1+2+0+1 \\
\quad +2+0-3+0-4 & \\
\quad \underline{+2+0-4+0-2} & \\
\qquad\qquad +1+0-2+0-1 & \\
\qquad\qquad \underline{+1+0-2+0-1} &
\end{array}$$

Hence, the quotient is $1+2b+b^3$.

9. Divide $x^4y^4 - 4x^2y^2z^2 + 12xyz^3 - 9z^4$ by $x^2y^2 - 2xyz + 3z^2$.

$$\begin{array}{ll}
1+0-4+12-9 \,\big| & \underline{1-2+3} \\
\underline{1-2+3} & 1+2-3 \\
\quad +2-7+12 & \\
\quad \underline{+2-4+\ 6} & \\
\qquad -3+\ 6-9 & \\
\qquad \underline{-3+\ 6-9} &
\end{array}$$

Hence, the quotient is $x^2y^2 + 2xyz - 3z^2$.

10. Divide $6a^5 - 17a^4b + 14a^2b^3 + 3ab^4 + 4b^5$ by $3a^3 - 4a^2b - b^3$.

$$\begin{array}{ll}
6-17+\ 0+14+3+4 \,\big| & \underline{3-4+0-1} \\
\underline{6-\ 8+\ 0-\ 2} & 2-3-4 \\
\quad -\ 9+\ 0+16+3 & \\
\quad \underline{-\ 9+12+\ 0+3} & \\
\qquad -12+16+0+4 & \\
\qquad \underline{-12+16+0+4} &
\end{array}$$

Hence, the quotient is $2a^2 - 3ab - 4b^2$.

11. Divide $26x^3y^3 + x^6 + 6y^6 - 17xy^5 - 5x^5y - 2x^4y^2 - x^2y^4$ by $x^2 - 3y^2 - 2xy$.

$$\begin{array}{ll}
1-5-2+26-\ 1-17+6 \,\big| & \underline{1-2-3\qquad\qquad} \\
\underline{1-2-3} & 1-3-5+7-2 \\
\quad -3+1+26 & \\
\quad \underline{-3+6+\ 9} & \\
\qquad -5+17-\ 1 & \\
\qquad \underline{-5+10+15} & \\
\qquad\quad +\ 7-16-17 & \\
\qquad\quad \underline{+\ 7-14-21} & \\
\qquad\qquad -\ 2+\ 4+6 & \\
\qquad\qquad \underline{-\ 2+\ 4+6} &
\end{array}$$

Hence, the quotient is $x^4 - 3x^3y - 5x^2y^2 + 7xy^3 - 2y^4$.

12. Divide $48x^2y^3 + 26xy^4 - 17x^3y^2 + 8x^5 - 22x^4y - 8y^5$ by $2x^2 - 4y^2 - 3xy$.

$$\begin{array}{ll}
8-22-17+48+26-8 & \underline{|\,2-3-4\qquad} \\
\underline{8-12-16\qquad\qquad} & 4-5-8+2 \\
\quad -10-\ \ 1+48 & \\
\quad \underline{-10+15+20\qquad} & \\
\qquad\quad -16+28+26 & \\
\qquad\quad \underline{-16+24+32\quad} & \\
\qquad\qquad\quad +\ 4-\ \ 6-8 & \\
\qquad\qquad\quad \underline{+\ 4-\ \ 6-8} &
\end{array}$$

Hence, the quotient is $4x^3 - 5x^2y - 8xy^2 + 2y^3$.

13. Divide $8ab^6 - 8a^6b + 24a^2b^5 + a^5b^2 + 14a^3b^4 + 12a^4b^3 + 5a^7$ by $7a^2b + 2ab^2 + 5a^3$.

$$\begin{array}{ll}
8+24+14+12+\ 1-\ 8+5 & \underline{|\,2+7+5\qquad\qquad} \\
\underline{8+28+20\qquad\qquad} & 4-2+4-3+1 \\
\ -\ \ 4-\ \ 6+12 & \\
\ \underline{-\ \ 4-14-10\qquad} & \\
\qquad\ +\ \ 8+22+\ 1 & \\
\qquad\ \underline{+\ \ 8+28+20\qquad} & \\
\qquad\qquad\quad -\ \ 6-19-\ 8 & \\
\qquad\qquad\quad \underline{-\ \ 6-21-15\qquad} & \\
\qquad\qquad\qquad\quad +\ 2+\ \ 7+5 & \\
\qquad\qquad\qquad\quad \underline{+\ 2+\ \ 7+5} &
\end{array}$$

Hence, the quotient is $4b^4 - 2ab^3 + 4a^2b^2 - 3a^3b + a^4$.

14. Divide $2x^8 + 5x^7y + 3x^5y^3 - 3x^4y^4 - 5x^3y^5 - 2x^2y^6 - 2xy^7 + 4y^8$ by $2x^5 - x^4y + 3x^3y^2 - 2x^2y^3 + xy^4 - 2y^5$.

$$\begin{array}{ll}
2+5+0+3-3-5-2-2+4 & \underline{|\,2-1+3-2+1-2} \\
\underline{2-1+3-2+1-2\qquad} & 1+3+0-2 \\
\quad +6-3+5-4-3-2 & \\
\quad \underline{+6-3+9-6+3-6\qquad} & \\
\qquad\qquad -4+2-6+4-2+4 & \\
\qquad\qquad \underline{-4+2-6+4-2+4} &
\end{array}$$

Hence, the quotient is $x^3 + 3x^2y - 2y^3$.

15. Divide $6a^{n+6}-20a^{n+5}+25a^{n+4}-19a^{n+3}+10a^{n+2}-4a^{n+1}+a^n$ by $3a^4-7a^3+4a^2-2a+1$.

$$\begin{array}{ll}
6-20+25-19+10-4+1 & \big|\,\underline{3-7+4-2+1} \\
\underline{6-14+\ \,8-\ \,4+\ 2\qquad} & \quad 2-2+1 \\
\quad -\ 6+17-15+\ 8-4 & \\
\quad \underline{-\ 6+14-\ \,8+\ 4-2\quad} & \\
\qquad\quad +\ 3-\ 7+\ 4-2+1 & \\
\qquad\quad \underline{+\ 3-\ 7+\ 4-2+1} &
\end{array}$$

Hence, the quotient is $2a^{n+2}-2a^{n+1}+a^n$.

16. Divide $6x^{m+5}+11x^{m+4}y-21x^{m+3}y^2-5x^{m+2}y^3+10x^{m+1}y^4-x^{m-1}y^6$ by $3x^m-2x^{m-1}y-x^{m-2}y^2$.

$$\begin{array}{ll}
6+11-21-5+10+0-1 & \big|\,\underline{3-2-1\qquad\qquad} \\
\underline{6-\ 4-\ 2\qquad} & \quad 2+5-3-2+1 \\
\quad +15-19-5 & \\
\quad \underline{+15-10-5\quad} & \\
\qquad\quad -\ 9+0+10 & \\
\qquad\quad \underline{-\ 9+6+\ 3\quad} & \\
\qquad\qquad\quad -6+\ 7+0 & \\
\qquad\qquad\quad \underline{-6+\ 4+2\quad} & \\
\qquad\qquad\qquad +\ 3-2-1 & \\
\qquad\qquad\qquad \underline{+\ 3-2-1} &
\end{array}$$

Hence, the quotient is $2x^5+5x^4y-3x^3y^2-2x^2y^3+xy^4$.

Exercise 27. Page 73

1. Add $\frac{3}{8}x^2+\frac{2}{3}xy+\frac{3}{10}y^2$; $\frac{1}{4}x^2+\frac{2}{9}xy-\frac{2}{5}y^2$; $\frac{1}{2}x^2-\frac{5}{6}xy-\frac{1}{2}y^2$.

$$\begin{array}{rrr}
\frac{3}{8}x^2 & +\ \frac{2}{3}xy & +\ \frac{3}{10}y^2 \\
\frac{1}{4}x^2 & +\ \frac{2}{9}xy & -\ \frac{2}{5}y^2 \\
\frac{1}{2}x^2 & -\ \frac{5}{6}xy & -\ \frac{1}{2}y^2 \\
\hline
\frac{9}{8}x^2 & +\ \frac{1}{18}xy & -\ \frac{3}{5}y^2
\end{array}$$

2. Add $\frac{2}{7}a^2-\frac{1}{5}ab+\frac{2}{3}b^2$; $\frac{8}{14}a^2+\frac{7}{10}ab-\frac{1}{9}b^2$; $-\frac{8}{21}a^2-\frac{3}{5}ab-\frac{17}{27}b^2$; $\frac{13}{28}a^2+\frac{4}{15}ab+\frac{4}{3}b^2$.

$$\begin{array}{rrr}
\frac{2}{7}a^2 & -\ \frac{1}{5}ab & +\ \frac{2}{3}b^2 \\
\frac{3}{14}a^2 & +\ \frac{7}{10}ab & -\ \frac{1}{9}b^2 \\
-\frac{8}{21}a^2 & -\ \frac{3}{5}ab & -\ \frac{17}{27}b^2 \\
\frac{13}{28}a^2 & +\ \frac{4}{15}ab & +\ \frac{4}{3}b^2 \\
\hline
\frac{7}{12}a^2 & +\ \frac{1}{6}ab & +\ \frac{34}{27}b^2
\end{array}$$

3. From $\frac{1}{2}x - \frac{13}{6}a + \frac{1}{3}b - \frac{1}{4}y$ take $-\frac{1}{2}y - \frac{3}{5}b + \frac{1}{6}a + \frac{11}{4}x$.

$$\begin{array}{r}
\frac{1}{2}x - \frac{13}{6}a + \frac{1}{3}b - \frac{1}{4}y \\
\frac{11}{4}x + \frac{1}{6}a - \frac{3}{5}b - \frac{1}{2}y \\
\hline
-\frac{9}{4}x - \frac{7}{3}a + \frac{14}{15}b + \frac{1}{4}y
\end{array}$$

4. From $\frac{5}{2}a - 76 - 3bc + \frac{1}{2}c$ take $\frac{3}{4}a - 56\frac{4}{5} + \frac{1}{4}c - 3bc$.

$$\begin{array}{r}
\frac{5}{2}a - 76 \quad - 3bc + \frac{1}{2}c \\
\frac{3}{4}a - 56\frac{4}{5} - 3bc + \frac{1}{4}c \\
\hline
\frac{7}{4}a - 19\frac{1}{5} \qquad\quad + \frac{1}{4}c
\end{array}$$

5. From $\frac{5}{3}a^6 + \frac{7}{2}a^5 - \frac{9}{4}a^4 - \frac{11}{6}a^3 + \frac{13}{3}a^2 - \frac{5}{8}a + \frac{1}{4}$ take $\frac{7}{2}a^6 - \frac{9}{4}a^5 - \frac{11}{6}a^4 + \frac{13}{3}a^3 - \frac{5}{3}a^2 - \frac{7}{12}a - \frac{5}{12}$.

$$\begin{array}{r}
\frac{5}{3}a^6 + \frac{7}{2}a^5 - \frac{9}{4}a^4 - \frac{11}{6}a^3 + \frac{13}{3}a^2 - \frac{5}{8}a + \frac{1}{4} \\
\frac{7}{2}a^6 - \frac{9}{4}a^5 - \frac{11}{6}a^4 + \frac{13}{3}a^3 - \frac{5}{3}a^2 - \frac{7}{12}a - \frac{5}{12} \\
\hline
-\frac{11}{6}a^6 + \frac{23}{4}a^5 - \frac{5}{12}a^4 - \frac{37}{6}a^3 + 6a^2 - \frac{1}{24}a + \frac{2}{3}
\end{array}$$

6. Multiply $24x^5 - 36x^4y - 16x^3y^2 + 20x^2y^3 - 16xy^4 + \frac{1}{2}y^5$ by $\frac{1}{4}x^3 + 8x^2y + 7xy^2 - 11y^3$.

$$\begin{array}{l}
24x^5 - 36x^4y - 16x^3y^2 + 20x^2y^3 - 16xy^4 + \frac{1}{2}y^5 \\
\frac{1}{4}x^3 + 8x^2y + 7xy^2 - 11y^3 \\
\hline
6x^8 - 9x^7y - 4x^6y^2 + 5x^5y^3 - 4x^4y^4 + \frac{1}{8}x^3y^5 \\
\qquad + 192x^7y - 288x^6y^2 - 128x^5y^3 + 160x^4y^4 - 128x^3y^5 + 4x^2y^6 \\
\qquad\qquad + 168x^6y^2 - 252x^5y^3 - 112x^4y^4 + 140x^3y^5 - 112x^2y^6 + 3\frac{1}{2}xy^7 \\
\qquad\qquad\qquad - 264x^5y^3 + 396x^4y^4 + 176x^3y^5 - 220x^2y^6 + 176xy^7 - 5\frac{1}{2}y^8 \\
\hline
6x^8 + 183x^7y - 124x^6y^2 - 639x^5y^3 + 440x^4y^4 + 188\frac{1}{8}x^3y^5 - 328x^2y^6 + 179\frac{1}{2}xy^7 - 5\frac{1}{2}y^8
\end{array}$$

7. Multiply $\frac{1}{3}a^3 + \frac{3}{5}a^2x - \frac{5}{6}ax^2 + \frac{1}{2}x^3$ by $\frac{1}{2}a^2 - \frac{8}{11}ax - \frac{6}{11}x^2$.

$$\begin{array}{l}
\frac{1}{3}a^3 + \frac{3}{5}a^2x - \frac{5}{6}ax^2 + \frac{1}{2}x^3 \\
\frac{1}{2}a^2 - \frac{8}{11}ax - \frac{6}{11}x^2 \\
\hline
\frac{1}{6}a^5 + \frac{3}{10}a^4x - \frac{5}{12}a^3x^2 + \frac{1}{4}a^2x^3 \\
\qquad - \frac{8}{33}a^4x - \frac{24}{55}a^3x^2 + \frac{20}{33}a^2x^3 - \frac{4}{11}ax^4 \\
\qquad\qquad - \frac{2}{11}a^3x^2 - \frac{18}{55}a^2x^3 + \frac{5}{11}ax^4 - \frac{3}{11}x^5 \\
\hline
\frac{1}{6}a^5 + \frac{19}{330}a^4x - \frac{683}{660}a^3x^2 + \frac{349}{660}a^2x^3 + \frac{1}{11}ax^4 - \frac{3}{11}x^5
\end{array}$$

8. Multiply $\frac{1}{4} - \frac{1}{4}x^3 - \frac{1}{6}x + \frac{1}{5}x^2$ by $\frac{1}{4} + \frac{1}{4}x + \frac{1}{2}x^2$.

$$\begin{array}{l}
\frac{1}{4} - \frac{1}{6}x + \frac{1}{5}x^2 - \frac{1}{4}x^3 \\
\frac{1}{4} + \frac{1}{4}x + \frac{1}{2}x^2 \\
\hline
\frac{1}{16} - \frac{1}{24}x + \frac{1}{20}x^2 - \frac{1}{16}x^3 \\
\qquad + \frac{1}{16}x - \frac{1}{24}x^2 + \frac{1}{20}x^3 - \frac{1}{16}x^4 \\
\qquad\qquad + \frac{1}{8}x^2 - \frac{1}{12}x^3 + \frac{1}{10}x^4 - \frac{1}{8}x^5 \\
\hline
\frac{1}{16} + \frac{1}{48}x + \frac{2}{15}x^2 - \frac{23}{240}x^3 + \frac{3}{80}x^4 - \frac{1}{8}x^5
\end{array}$$

9. Divide $\frac{2}{3}a^3 - \frac{443}{270}a^2d + \frac{11}{8}ad^2 - \frac{5}{12}d^3$ by $\frac{3}{2}a - \frac{5}{3}d$.

$$\begin{array}{l|l}
\frac{2}{3}a^3 - \frac{443}{270}a^2d + \frac{11}{8}ad^2 - \frac{5}{12}d^3 & \frac{3}{2}a - \frac{5}{3}d \\
\cline{2-2}
\frac{2}{3}a^3 - \frac{20}{27}a^2d & \frac{4}{9}a^2 - \frac{3}{5}ad + \frac{1}{4}d^2 \\
\cline{1-1}
\qquad - \frac{9}{10}a^2d + \frac{11}{8}ad^2 & \\
\qquad - \frac{9}{10}a^2d + ad^2 & \\
\cline{1-1}
\qquad\qquad + \frac{3}{8}ad^2 - \frac{5}{12}d^3 & \\
\qquad\qquad + \frac{3}{8}ad^2 - \frac{5}{12}d^3 & \\
\cline{1-1}
\end{array}$$

10. Divide $x^5 - \frac{23}{30}x^4 + \frac{31}{10}x^3 + \frac{5}{3} - \frac{181}{18}x - \frac{7}{3}x^2$ by $x^2 - \frac{1}{6}x + 5$.

$$\begin{array}{l|l}
x^5 - \frac{23}{30}x^4 + \frac{31}{10}x^3 - \frac{7}{3}x^2 - \frac{181}{18}x + \frac{5}{3} & x^2 - \frac{1}{6}x + 5 \\
\cline{2-2}
x^5 - \frac{1}{6}x^4 + 5x^3 & x^3 - \frac{3}{5}x^2 - 2x + \frac{1}{3} \\
\cline{1-1}
\quad - \frac{3}{5}x^4 - \frac{19}{10}x^3 - \frac{7}{3}x^2 & \\
\quad - \frac{3}{5}x^4 + \frac{1}{10}x^3 - 3x^2 & \\
\cline{1-1}
\qquad - 2x^3 + \frac{2}{3}x^2 - \frac{181}{18}x & \\
\qquad - 2x^3 + \frac{1}{3}x^2 - 10x & \\
\cline{1-1}
\qquad\qquad + \frac{1}{3}x^2 - \frac{1}{18}x + \frac{5}{3} & \\
\qquad\qquad + \frac{1}{3}x^2 - \frac{1}{18}x + \frac{5}{3} & \\
\cline{1-1}
\end{array}$$

11. Divide $\frac{1}{3}a^5 + \frac{11}{45}a^4b - \frac{73}{30}a^3b^2 - \frac{31}{15}a^2b^3 + \frac{7}{10}ab^4 - 3b^5$ by $\frac{2}{3}a^2 - \frac{2}{5}ab - 4b^2$.

$$\begin{array}{l|l}
\frac{1}{3}a^5 + \frac{11}{45}a^4b - \frac{73}{30}a^3b^2 - \frac{31}{15}a^2b^3 + \frac{7}{10}ab^4 - 3b^5 & \frac{2}{3}a^2 - \frac{2}{5}ab - 4b^2 \\ \cline{2-2}
\underline{\frac{1}{3}a^5 - \frac{1}{5}a^4b - 2a^3b^2} & \frac{1}{2}a^3 + \frac{2}{3}a^2b - \frac{1}{4}ab^2 + \frac{3}{4}b^3 \\
\quad + \frac{4}{9}a^4b - \frac{13}{30}a^3b^2 - \frac{31}{15}a^2b^3 & \\
\quad \underline{+ \frac{4}{9}a^4b - \frac{4}{15}a^3b^2 - \frac{8}{3}a^2b^3} & \\
\qquad - \frac{1}{6}a^3b^2 + \frac{3}{5}a^2b^3 + \frac{7}{10}ab^4 & \\
\qquad \underline{- \frac{1}{6}a^3b^2 + \frac{1}{10}a^2b^3 + ab^4} & \\
\qquad\quad + \frac{1}{2}a^2b^3 - \frac{3}{10}ab^4 - 3b^5 & \\
\qquad\quad \underline{+ \frac{1}{2}a^2b^3 - \frac{3}{10}ab^4 - 3b^5} &
\end{array}$$

12. Divide $\frac{16}{15}a^7 + \frac{2}{15}a^6b - \frac{91}{30}a^5b^2 + \frac{233}{180}a^4b^3 + \frac{5}{3}a^3b^4 - \frac{7}{12}a^2b^5 - \frac{5}{8}ab^6$ by $\frac{8}{9}a^3 + \frac{2}{3}a^2b - \frac{5}{3}ab^2$.

$$\begin{array}{l|l}
\frac{16}{15}a^7 + \frac{2}{15}a^6b - \frac{91}{30}a^5b^2 + \frac{233}{180}a^4b^3 + \frac{5}{3}a^3b^4 - \frac{7}{12}a^2b^5 - \frac{5}{8}ab^6 & \frac{8}{9}a^3 + \frac{2}{3}a^2b - \frac{5}{3}ab^2 \\ \cline{2-2}
\underline{\frac{16}{15}a^7 + \frac{4}{5}a^6b - 2a^5b^2} & \frac{6}{5}a^4 - \frac{3}{4}a^3b - \frac{3}{5}a^2b^2 + \frac{1}{2}ab^3 + \frac{3}{8}b^4 \\
\quad - \frac{2}{3}a^6b - \frac{31}{30}a^5b^2 + \frac{233}{180}a^4b^3 & \\
\quad \underline{- \frac{2}{3}a^6b - \frac{1}{2}a^5b^2 + \frac{5}{4}a^4b^3} & \\
\qquad - \frac{8}{15}a^5b^2 + \frac{2}{45}a^4b^3 + \frac{5}{3}a^3b^4 & \\
\qquad \underline{- \frac{8}{15}a^5b^2 - \frac{2}{5}a^4b^3 + a^3b^4} & \\
\qquad\quad + \frac{4}{9}a^4b^3 + \frac{2}{3}a^3b^4 - \frac{7}{12}a^2b^5 & \\
\qquad\quad \underline{+ \frac{4}{9}a^4b^3 + \frac{1}{3}a^3b^4 - \frac{5}{6}a^2b^5} & \\
\qquad\qquad + \frac{1}{3}a^3b^4 + \frac{1}{4}a^2b^5 - \frac{5}{8}ab^6 & \\
\qquad\qquad \underline{+ \frac{1}{3}a^3b^4 + \frac{1}{4}a^2b^5 - \frac{5}{8}ab^6} &
\end{array}$$

7. Multiply $\frac13 a^3 + \frac35 a^2x - \frac56 ax^2 + \frac12 x^3$ by $\frac12 a^2 - \frac{8}{11} ax - \frac{6}{11} x^2$.

$$\begin{array}{rrrrrr}
\frac13 a^3 & +\frac35 a^2x & -\frac56 ax^2 & +\frac12 x^3 & & \\
\frac12 a^2 & -\frac{8}{11} ax & -\frac{6}{11} x^2 & & & \\
\hline
\frac16 a^5 & +\frac{3}{10} a^4x & -\frac{5}{12} a^3x^2 & +\frac14 a^2x^3 & & \\
 & -\frac{8}{33} a^4x & -\frac{24}{55} a^3x^2 & +\frac{20}{33} a^2x^3 & -\frac{4}{11} ax^4 & \\
 & & -\frac{2}{11} a^3x^2 & -\frac{18}{55} a^2x^3 & +\frac{5}{11} ax^4 & -\frac{3}{11} x^5 \\
\hline
\frac16 a^5 & +\frac{19}{330} a^4x & -\frac{683}{660} a^3x^2 & +\frac{349}{660} a^2x^3 & +\frac{1}{11} ax^4 & -\frac{3}{11} x^5
\end{array}$$

8. Multiply $\frac14 - \frac14 x^3 - \frac16 x + \frac15 x^2$ by $\frac14 + \frac14 x + \frac12 x^2$.

$$\begin{array}{rrrrrr}
\frac14 & -\frac16 x & +\frac15 x^2 & -\frac14 x^3 & & \\
\frac14 & +\frac14 x & +\frac12 x^2 & & & \\
\hline
\frac{1}{16} & -\frac{1}{24} x & +\frac{1}{20} x^2 & -\frac{1}{16} x^3 & & \\
 & +\frac{1}{16} x & -\frac{1}{24} x^2 & +\frac{1}{20} x^3 & -\frac{1}{16} x^4 & \\
 & & +\frac18 x^2 & -\frac{1}{12} x^3 & +\frac{1}{10} x^4 & -\frac18 x^5 \\
\hline
\frac{1}{16} & +\frac{1}{48} x & +\frac{2}{15} x^2 & -\frac{23}{240} x^3 & +\frac{3}{80} x^4 & -\frac18 x^5
\end{array}$$

9. Divide $\frac23 a^3 - \frac{443}{270} a^2d + \frac{11}{8} ad^2 - \frac{5}{12} d^3$ by $\frac32 a - \frac53 d$.

$$\begin{array}{rrrrl}
\frac23 a^3 & -\frac{443}{270} a^2d & +\frac{11}{8} ad^2 & -\frac{5}{12} d^3 & \big|\,\underline{\frac32 a - \frac53 d} \\
\frac23 a^3 & -\frac{20}{27} a^2d & & & \;\;\frac49 a^2 - \frac35 ad + \frac14 d^2 \\
\cline{1-3}
 & -\frac{9}{10} a^2d & +\frac{11}{8} ad^2 & & \\
 & -\frac{9}{10} a^2d & + ad^2 & & \\
\cline{2-4}
 & & +\frac38 ad^2 & -\frac{5}{12} d^3 & \\
 & & +\frac38 ad^2 & -\frac{5}{12} d^3 & \\
\cline{3-4}
\end{array}$$

10. Divide $x^5 - \frac{23}{30} x^4 + \frac{31}{10} x^3 + \frac53 - \frac{181}{18} x - \frac73 x^2$ by $x^2 - \frac16 x + 5$.

$$\begin{array}{rrrrrrl}
x^5 & -\frac{23}{30} x^4 & +\frac{31}{10} x^3 & -\frac73 x^2 & -\frac{181}{18} x & +\frac53 & \big|\,\underline{x^2 - \frac16 x + 5} \\
x^5 & -\frac16 x^4 & +5x^3 & & & & \;\;x^3 - \frac35 x^2 - 2x + \frac13 \\
\cline{1-4}
 & -\frac35 x^4 & -\frac{19}{10} x^3 & -\frac73 x^2 & & & \\
 & -\frac35 x^4 & +\frac{1}{10} x^3 & -3x^2 & & & \\
\cline{2-5}
 & & -2x^3 & +\frac23 x^2 & -\frac{181}{18} x & & \\
 & & -2x^3 & +\frac13 x^2 & -10x & & \\
\cline{3-6}
 & & & +\frac13 x^2 & -\frac{1}{18} x & +\frac53 & \\
 & & & +\frac13 x^2 & -\frac{1}{18} x & +\frac53 & \\
\cline{4-6}
\end{array}$$

17. Divide $0.2\,m^6 - 0.46\,m^5n + 0.32\,m^4n^2 - 0.08\,m^3n^3 - 2\,m^2n^4 + 0.2\,mn^5 + 1.12\,n^6$ by $0.5\,m^4 - 0.4\,m^3n + 1.2\,m^2n^2 + 0.8\,mn^3 - 1.4\,n^4$.

$$\begin{array}{r|l}
0.2\,m^6 - 0.46\,m^5n + 0.32\,m^4n^2 - 0.08\,m^3n^3 - 2\,m^2n^4 \quad + 0.2\,mn^5 \quad + 1.12\,n^6 & 0.5\,m^4 - 0.4\,m^3n + 1.2\,m^2n^2 + 0.8\,mn^3 - 1.4\,n^4 \\ \cline{2-2}
\underline{0.2\,m^6 - 0.16\,m^5n + 0.48\,m^4n^2 + 0.32\,m^3n^3 - 0.56\,m^2n^4} \qquad\qquad\qquad & 0.4\,m^2 - 0.6\,mn - 0.8\,n^2 \\
-0.3\,m^5n \quad -0.16\,m^4n^2 - 0.4\,m^3n^3 \quad - 1.44\,m^2n^4 + 0.2\,mn^5 \qquad\qquad & \\
\underline{-0.3\,m^5n \quad +0.24\,m^4n^2 - 0.72\,m^3n^3 - 0.48\,m^2n^4 + 0.84\,mn^5} \qquad\qquad & \\
-0.4\,m^4n^2 \quad + 0.32\,m^3n^3 - 0.96\,m^2n^4 - 0.64\,mn^5 + 1.12\,n^6 & \\
\underline{-0.4\,m^4n^2 \quad + 0.32\,m^3n^3 - 0.96\,m^2n^4 - 0.64\,mn^5 + 1.12\,n^6} &
\end{array}$$

18. Divide $\frac{6}{5}x^5 + \frac{11}{144}a^4x + \frac{299}{480}a^2x^3 - \frac{7}{8}ax^4 - \frac{3}{2}a^3x^2 + \frac{5}{9}a^5$ by $\frac{5}{8}ax - \frac{3}{2}x^2 + \frac{2}{3}a^2$.

$$\begin{array}{r|l}
\frac{5}{9}a^5 + \frac{11}{144}a^4x - \frac{3}{2}a^3x^2 + \frac{299}{480}a^2x^3 - \frac{7}{8}ax^4 + \frac{6}{5}x^5 & \frac{2}{3}a^2 + \frac{5}{8}ax - \frac{3}{2}x^2 \\ \cline{2-2}
\underline{\frac{5}{9}a^5 + \frac{25}{48}a^4x - \frac{5}{4}a^3x^2} \qquad\qquad\qquad\qquad\qquad & \frac{5}{6}a^3 - \frac{2}{3}a^2x + \frac{1}{4}ax^2 - \frac{4}{5}x^3 \\
- \frac{4}{9}a^4x - \frac{1}{4}a^3x^2 + \frac{299}{480}a^2x^3 \qquad\qquad\qquad & \\
\underline{- \frac{4}{9}a^4x - \frac{5}{12}a^3x^2 + a^2x^3} \qquad\qquad\qquad & \\
+ \frac{1}{6}a^3x^2 - \frac{181}{480}a^2x^3 - \frac{7}{8}ax^4 \qquad\qquad & \\
\underline{+ \frac{1}{6}a^3x^2 + \frac{5}{32}a^2x^3 - \frac{3}{8}ax^4} \qquad\qquad & \\
- \frac{8}{15}a^2x^3 - \frac{1}{2}ax^4 + \frac{6}{5}x^5 & \\
\underline{- \frac{8}{15}a^2x^3 - \frac{1}{2}ax^4 + \frac{6}{5}x^5} &
\end{array}$$

Exercise 28. Page 76

Write the product of:

1. $(x + a)^2 = x^2 + 2ax + a^2$.
2. $(y - b)^2 = y^2 - 2by + b^2$.
3. $(1 + 2x^2)^2 = 1 + 4x^2 + 4x^4$.
4. $(2x - a)^2 = 4x^2 - 4ax + a^2$.
5. $(2a - 5b)^2 = 4a^2 - 20ab + 25b^2$.
6. $(4z - 3)^2 = 16z^2 - 24z + 9$.
7. $(3 + 2d)^2 = 9 + 12d + 4d^2$.
8. $(3x + 5y)^2 = 9x^2 + 30xy + 25y^2$.
9. $(-2m + 3n)^2 = 4m^2 - 12mn + 9n^2$.
10. $(2 + 4m^4)^2 = 4 + 16m^4 + 16m^8$.
11. $(2a - 5a^2)^2 = 4a^2 - 20a^3 + 25a^4$.
12. $(x + a)(x - a) = x^2 - a^2$.
13. $(1 + x)(1 - x) = 1 - x^2$.
14. $(3 - y)(3 + y) = 9 - y^2$.
15. $(3x - 2y)(3x + 2y) = 9x^2 - 4y^2$.
16. $(3ab + 1)(3ab - 1) = 9a^2b^2 - 1$.
17. $(5x^2 - 3y^3)(5x^2 + 3y^3) = 25x^4 - 9y^6$.
18. $(a^2 + 3x)(a^2 - 3x) = a^4 - 9x^2$.
19. $(5 - bx^3)(bx^3 + 5) = 25 - b^2x^6$.
20. $(6m + 7n^4)(7n^4 - 6m) = 49n^8 - 36m^2$.
21. $(8 - ab^2)(ab^2 + 8) = 64 - a^2b^4$.
22. $(a^4x + ax^4)(ax^4 - a^4x) = a^2x^8 - a^8x^2$.
23. $(3ab^2 - 4a^2b)(3ab^2 + 4a^2b) = 9a^2b^4 - 16a^4b^2$.
24. $(2x^2y + 5xy^2)(2x^2y - 5xy^2) = 4x^4y^2 - 25x^2y^4$.

Exercise 29. Page 77

Write the product of:

1. $(x + y + z)(x - y - z) = x^2 - y^2 - 2yz - z^2$.
2. $(x - y + z)(x - y - z) = x^2 - 2xy + y^2 - z^2$.
3. $(x^2 + xy + y^2)(x^2 - xy + y^2) = x^4 + 2x^2y^2 + y^4 - x^2y^2 = x^4 + x^2y^2 + y^4$.
4. $(a^2 + 2ab + b^2)(a^2 - 2ab + b^2) = a^4 + 2a^2b^2 + b^4 - 4a^2b^2 = a^4 - 2a^2b^2 + b^4$
5. $(a^2 + a - 1)(a^2 - a + 1) = a^4 - a^2 + 2a - 1$.
6. $(2x - y - 3z)(2x - y + 3z) = 4x^2 - 4xy + y^2 - 9z^2$.
7. $(x - 2y + 3z)(x + 2y + 3z) = x^2 + 6xz + 9z^2 - 4y^2$.
8. $(bc + ac + ab)(bc - ac - ab) = b^2c^2 - a^2c^2 - 2a^2bc - a^2b^2$.
9. $(ax^2 + bx + c)(ax^2 - bx - c) = a^2x^4 - b^2x^2 - 2bcx - c^2$.

10. $(a^3 + 3a^2 - 4a)(a^3 - 3a^2 + 4a) = a^6 - 9a^4 + 24a^3 - 16a^2.$

11. $(3x^2 - 2xy + y^2)(3x^2 + 2xy - y^2) = 9x^4 - 4x^2y^2 + 4xy^3 - y^4.$

12. $(x^2 + 2ax + 3b)(x^2 - 2ax - 3b) = x^4 - 4a^2x^2 - 12abx - 9b^2.$

13. $(x^2 + 6x - 7)(x^2 - 6x + 7) = x^4 - 36x^2 + 84x - 49.$

14. $(a^3b - 2a^2b^2 - 2ab^3)(a^3b + 2a^2b^2 - 2ab^3)$
$= a^6b^2 - 4a^4b^4 + 4a^2b^6 - 4a^4b^4 = a^6b^2 - 8a^4b^4 + 4a^2b^6.$

Exercise 30. Page 78

Perform the operations indicated:

1. $(3a - 4b)^2 = 9a^2 - 24ab + 16b^2.$

2. $(x - y + z)^2 = x^2 + y^2 + z^2 - 2xy + 2xz - 2yz.$

3. $(x + y - z)^2 = x^2 + y^2 + z^2 + 2xy - 2xz - 2yz.$

4. $(y^2 - 3yz + 6z^2)^2 = y^4 + 9y^2z^2 + 36z^4 - 6y^3z + 12y^2z^2 - 36yz^3$
$= y^4 + 21y^2z^2 + 36z^4 - 6y^3z - 36yz^3.$

5. $(m^2 + 2mn + n^2)^2 = m^4 + 4m^2n^2 + n^4 + 4m^3n + 2m^2n^2 + 4mn^3$
$= m^4 + 6m^2n^2 + n^4 + 4m^3n + 4mn^3.$

6. $(a^2 - 2ab + b^2)^2 = a^4 + 4a^2b^2 + b^4 - 4a^3b + 2a^2b^2 - 4ab^3$
$= a^4 + 6a^2b^2 + b^4 - 4a^3b - 4ab^3.$

7. $(x^2 - 3x - 4)^2 = x^4 + 9x^2 + 16 - 6x^3 - 8x^2 + 24x$
$= x^4 + x^2 + 16 - 6x^3 + 24x.$

8. $(x^2 - x - 1)^2 = x^4 + x^2 + 1 - 2x^3 - 2x^2 + 2x = x^4 - x^2 + 1 - 2x^3 + 2x.$

9. $(x^2 - 3a^2 + 2b^2)^2 = x^4 + 9a^4 + 4b^4 - 6a^2x^2 + 4b^2x^2 - 12a^2b^2.$

10. $(a^2 - 5a + 2)^2 = a^4 + 25a^2 + 4 - 10a^3 + 4a^2 - 20a$
$= a^4 + 29a^2 + 4 - 10a^3 - 20a.$

11. $(a^3b - 2a^2b^2 - 2ab^3)^2 = a^6b^2 + 4a^4b^4 + 4a^2b^6 - 4a^5b^3 - 4a^4b^4 + 8a^3b^5$
$= a^6b^2 + 4a^2b^6 - 4a^5b^3 + 8a^3b^5.$

12. $(x^2 - 3ax - 2a^2)^2 = x^4 + 9a^2x^2 + 4a^4 - 6ax^3 - 4a^2x^2 + 12a^3x$
$= x^4 + 5a^2x^2 + 4a^4 - 6ax^3 + 12a^3x.$

13. $(a + 2b + 3c + d)^2$
$= a^2 + 4b^2 + 9c^2 + d^2 + 4ab + 6ac + 2ad + 12bc + 4bd + 6cd.$

14. $(a - 2b + 3c - d)^2$
$= a^2 + 4b^2 + 9c^2 + d^2 - 4ab + 6ac - 2ad - 12bc + 4bd - 6cd.$

15. $(1 + x + 2x^2 + 3x^3)^2$
$= 1 + x^2 + 4x^4 + 9x^6 + 2x + 4x^2 + 6x^3 + 4x^3 + 6x^4 + 12x^5$
$= 1 + 5x^2 + 10x^4 + 9x^6 + 2x + 10x^3 + 12x^5.$

16. $(1 - x - 2x^2 - 3x^3)^2$
$= 1 + x^2 + 4x^4 + 9x^6 - 2x - 4x^2 - 6x^3 + 4x^3 + 6x^4 + 12x^5$
$= 1 - 3x^2 + 10x^4 + 9x^6 - 2x - 2x^3 + 12x^5.$

17. $(2 - a^2 + 3a^3 - d^2)^2$
$= 4 + a^4 + 9a^6 + d^4 - 4a^2 + 12a^3 - 4d^2 - 6a^5 + 2a^2d^2 - 6a^3d^2.$

18. $(\frac{1}{2}x^2 - 2y + \frac{2}{3}y^2 + 2z^3)^2$
$= \frac{1}{4}x^4 + 4y^2 + \frac{4}{9}y^4 + 4z^6 - 2x^2y + \frac{2}{3}x^2y^2 + 2x^2z^3 - \frac{8}{3}y^3 - 8yz^3 + \frac{8}{3}y^2z^3.$

Exercise 31. Page 80

Write by inspection the product of:

1. $(x + 1)(x + 2) = x^2 + 3x + 2.$
2. $(c - 3)(c - 2) = c^2 - 5c + 6.$
3. $(a - 7)(a + 4) = a^2 - 3a - 28.$
4. $(x - 11)(x + 5) = x^2 - 6x - 55.$
5. $(a - 6)(a - 11) = a^2 - 17a + 66.$
6. $(x - 3y)(x + 7y) = x^2 + 4xy - 21y^2.$
7. $(a + 5b)(a - 8b) = a^2 - 3ab - 40b^2.$
8. $(c - 3d)(c + 2d) = c^2 - cd - 6d^2.$
9. $(x^2 - a^3)(x^2 + 3a^3) = x^4 + 2a^3x^2 - 3a^6.$
10. $(a + 1)(a + 4) = a^2 + 5a + 4.$
11. $(c - 3d)(c + d) = c^2 - 2cd - 3d^2.$
12. $(a^2 + 2b^2)(a^2 - 7b^2) = a^4 - 5a^2b^2 - 14b^4.$
13. $(x^4 - 2)(x^4 + 5) = x^8 + 3x^4 - 10.$
14. $(x^2 + 3yz)(x^2 - 2yz) = x^4 + x^2yz - 6y^2z^2.$
15. $(a^2 + 2ab)(a^2 + 5ab) = a^4 + 7a^3b + 10a^2b^2.$
16. $(x^2 + 4ax)(x^2 - 3ax) = x^4 + ax^3 - 12a^2x^2.$
17. $(a^6 - 3b^6)(a^6 - 5b^6) = a^{12} - 8a^6b^6 + 15b^{12}.$
18. $(x - 3y)(x - 7y) = x^2 - 10xy + 21y^2.$
19. $(x^2 + 2y^2)(x^2 + y^2) = x^4 + 3x^2y^2 + 2y^4.$
20. $(a - 4b)(a - 5b) = a^2 - 9ab + 20b^2.$
21. $(x + y^2)(x + 7y^2) = x^2 + 8xy^2 + 7y^4.$
22. $(ax + 5by)(ax - 4by) = a^2x^2 + abxy - 20b^2y^2.$
23. $(x^2 + 3xy)(x^2 - 5xy) = x^4 - 2x^3y - 15x^2y^2.$
24. $(xy - 7yz)(xy + 6yz) = x^2y^2 - xy^2z - 42y^2z^2.$

25. $\{a-(2b+c)\}\{a-2(2b+c)\}=a^2-3a(2b+c)+2(2b+c)^2$
$=a^2-6ab-3ac+8b^2+8bc+2c^2.$

26. $\{x+3(y+2z)\}\{x-(y+2z)\}=x^2+2x(y+2z)-3(y+2z)^2$
$=x^2+2xy+4xz-3y^2-12yz-12z^2.$

27. $\{a+4(b+c)\}\{a-7(b+c)\}=a^2-3a(b+c)-28(b+c)^2$
$=a^2-3ab-3ac-28b^2-56bc-28c^2.$

Exercise 32. Page 81

Write by inspection the product of:

1. $(3a-4b)(a-b)=3a^2-7ab+4b^2.$
2. $(4b-5c)(3b+2c)=12b^2-7bc-10c^2.$
3. $(4x+9y)(x-5y)=4x^2-11xy-45y^2.$
4. $(3a-2b)(2a-b)=6a^2-7ab+2b^2.$
5. $(5a+2b)(3a-4b)=15a^2-14ab-8b^2.$ ✓
6. $(a-2b)(3a+4b)=3a^2-2ab-8b^2.$ ✓
7. $(a^3-4a)(3a^3-6a)=3a^6-18a^4+24a^2.$
8. $(a+3)(3a+5)=3a^2+14a+15.$
9. $(7a-2b)(3a-b)=21a^2-13ab+2b^2.$
10. $(9a-4c)(3a-4c)=27a^2-48ac+16c^2.$
11. $(3b+2c)(c+2b)=6b^2+7bc+2c^2.$
12. $(3x-a)(2x-3a)=6x^2-11ax+3a^2.$
13. $(8-9x)(3-6x)=24-75x+54x^2.$ ✓ 1
14. $(2x+4y)(y-9x)=4y^2-34xy-18x^2.$
15. $(9x-2y)(4x-3y)=36x^2-35xy+6y^2.$
16. $(7x-8a)(2x-3a)=14x^2-37ax+24a^2.$ ✓ 2
17. $(3a+2b)(5a+9b)=15a^2+37ab+18b^2.$
18. $(2a+5c)(4a+c)=8a^2+22ac+5c^2.$
19. $(7c-4d)(8c+4d)=56c^2-4cd-16d^2.$
20. $(9x-3y)(5x-2y)=45x^2-33xy+6y^2.$
21. $(1+2x^2)(3-5x^2)=3+x^2-10x^4.$
22. $(9m^3-5n)(m^3-n)=9m^6-14m^3n+5n^2.$
23. $(10-x^2)(6+5x^2)=60+44x^2-5x^4.$ 1
24. $(7a-2)(8a-9)=56a^2-79a+18.$ ✓
25. $(3x-2y)(3x+y)=9x^2-3xy-2y^2.$
26. $(5x^2-2y^2)(3x^2-7y^2)=15x^4-41x^2y^2+14y^4.$
27. $(a^2-5d)(3a^2+9d)=3a^4-6a^2d-45d^2.$
28. $(4m^2-y^4)(7m^2-8y^4)=28m^4-39m^2y^4+8y^8.$
29. $(3m-a)(2m+5a)=6m^2+13am-5a^2.$
30. $(\frac{1}{5}a+\frac{1}{4}b)(\frac{1}{8}a-\frac{1}{5}b)=\frac{1}{40}a^2-\frac{7}{800}ab-\frac{1}{20}b^2.$

Exercise 33. Page 82

Write by inspection the quotient of:

1. $\dfrac{x^4-64}{x^2+8}=x^2-8.$

2. $\dfrac{x^2-4y^2}{x-2y}=x+2y.$

3. $\dfrac{a^8-81b^4}{a^4-9b^2}=a^4+9b^2.$

4. $\dfrac{a^6-b^6}{a^3+b^3}=a^3-b^3.$

5. $\dfrac{16y^2-4x^2}{4y+2x}=4y-2x.$

6. $\dfrac{64a^2b^2-49b^2c^2}{8ab-7bc}=8ab+7bc.$

7. $\dfrac{4b^2-9m^2n^2}{2b-3mn}=2b+3mn.$

8. $\dfrac{4a^2b^2-25c^2d^2}{2ab+5cd}=2ab-5cd.$

9. $\dfrac{16m^2n^2-49a^2b^2}{4mn+7ab}=4mn-7ab.$

10. $\dfrac{25a^2b^2c^2-36x^2y^4z^6}{5abc-6xy^2z^3}=5abc+6xy^2z^3.$

Exercise 34. Page 83

Write by inspection the quotient of:

1. $\dfrac{8a^3-27b^3}{2a-3b}=4a^2+6ab+9b^2.$

2. $\dfrac{8a^3+27b^3}{2a+3b}=4a^2-6ab+9b^2.$

3. $\dfrac{a^9-27b^3}{a^3-3b}=a^6+3a^3b+9b^2.$

4. $\dfrac{8a^3-125b^3}{2a-5b}=4a^2+10ab+25b^2.$

5. $\dfrac{216+125a^9}{6+5a^3}=36-30a^3+25a^6.$

6. $\dfrac{343c^3+64d^3}{7c+4d}=49c^2-28cd+16d^2.$

7. $\dfrac{1-64c^6}{1-4c^2}=1+4c^2+16c^4.$

8. $\dfrac{216m^3+343n^9}{6m+7n^3}=36m^2-42mn^3+49n^6.$

9. $\frac{27x^3y^3 + 125a^3b^3}{3xy + 5ab} = 9x^2y^2 - 15abxy + 25a^2b^2.$

10. $\frac{b^3x^9 - 125}{bx^3 - 5} = b^2x^6 + 5bx^3 + 25.$

11. $\frac{27y^6 - 512z^9}{3y^2 - 8z^3} = 9y^4 + 24y^2z^3 + 64z^6.$

12. $\frac{64x^3 + 729y^6}{4x + 9y^2} = 16x^2 - 36xy^2 + 81y^4.$

13. $\frac{125d^3 - 1331b^3}{5d - 11b} = 25d^2 + 55bd + 121b^2.$

14. $\frac{8a^6b^3 + 125x^3y^9}{2a^2b + 5xy^3} = 4a^4b^2 - 10a^2bxy^3 + 25x^2y^6.$

Exercise 35. Page 84

Write by inspection the quotient of:

1. $\frac{a^4b^4 - x^4y^4}{ab - xy} = a^3b^3 + a^2b^2xy + abx^2y^2 + x^3y^3.$

2. $\frac{b^5x^5 + c^5y^5}{bx + cy} = b^4x^4 - b^3x^3cy + b^2x^2c^2y^2 - bxc^3y^3 + c^4y^4.$

3. $\frac{32a^5 - 243b^5}{2a - 3b} = 16a^4 + 24a^3b + 36a^2b^2 + 54ab^3 + 81b^4.$

4. $\frac{x^4 - 81y^4}{x + 3y} = x^3 - 3x^2y + 9xy^2 - 27y^3.$

5. $\frac{16x^4 - 81y^4}{2x - 3y} = 8x^3 + 12x^2y + 18xy^2 + 27y^3.$

6. $\frac{x^6 - 729y^6}{x - 3y} = x^5 + 3x^4y + 9x^3y^2 + 27x^2y^3 + 81xy^4 + 243y^5.$

7. $\frac{x^6 - 729y^6}{x + 3y} = x^5 - 3x^4y + 9x^3y^2 - 27x^2y^3 + 81xy^4 - 243y^5.$

8. $\frac{625x^4 - 81a^4}{5x - 3a} = 125x^3 + 75ax^2 + 45a^2x + 27a^3.$

9. $\frac{625x^4 - 81x^4}{5x + 3a} = 125x^3 - 75ax^2 + 45a^2x - 27a^3.$

Exercise 36. Page 86

Resolve into factors:

1. $x^3 + x^2 - 5x = x(x^2 + x - 5)$.
2. $a^2x^2 - 3a^2x + 5a^2 = a^2(x^2 - 3x + 5)$.
3. $a^2x^2 + 3ay + 4az^2 = a(ax^2 + 3y + 4z^2)$.
4. $a^2cx - abcy = ac(ax - by)$.
5. $a^3 - 2a^2b - a^2 = a^2(a - 2b - 1)$.
6. $2abxy - 3acxz = ax(2by - 3cz)$.
7. $6a^3x - a^2x^2 - 9a^4 = a^2(6ax - x^2 - 9a^2)$.
8. $x^2yz + xy^2z + xyz^2 = xyz(x + y + z)$.
9. $ab^2 + abc - a^2b = ab(b + c - a)$.
10. $b^2c^2x^2 - bcd^2x = bcx(bcx - d^2)$.
11. $abx^4 + 2ax^3 + 6acx = ax(bx^3 + 2x^2 + 6c)$.
12. $acx^2 - bc^2x + b^2c = c(ax^2 - bcx + b^2)$.
13. $3x^3y^2 + 4x^2y^3 + 5xy^4 = xy^2(3x^2 + 4xy + 5y^2)$.
14. $2x^5y^2 + 4x^4y^3 + 6x^3y^4 = 2x^3y^2(x^2 + 2xy + 3y^2)$.
15. $a^4bc + a^3b^2c^2 + 2a^2b^3c^3 = a^2bc(a^2 + abc + 2b^2c^2)$.
16. $8x^3y^2z^2 + 4x^2y^3z^3 + 8xy^4z^4 = 4xy^2z^2(2x^2 + xyz + 2y^2z^2)$.
17. $3a^4x^6 - 2a^3x^4 + 7a^2x^2 = a^2x^2(3a^2x^4 - 2ax^2 + 7)$.
18. $5abx^3 - 4a^2b^2x^2 - 4a^3b^3x = abx(5x^2 - 4abx - 4a^2b^2)$.
19. $6a^4b + 12a^3b^2 - 2a^2b^3 = 2a^2b(3a^2 + 6ab - b^2)$.
20. $9ady + 60ad^2 - 12a^2dx = 3ad(3y + 20d - 4ax)$.

Exercise 37. Page 87

Resolve into factors:

1.
$$\begin{aligned} ax - ay + bx - by &= (ax - ay) + (bx - by) \\ &= a(x - y) + b(x - y) \\ &= (a + b)(x - y). \end{aligned}$$

2.
$$\begin{aligned} 2ac + bc - 2ad - bd &= (2ac - 2ad) + (bc - bd) \\ &= 2a(c - d) + b(c - d) \\ &= (2a + b)(c - d). \end{aligned}$$

3.
$$\begin{aligned} ax - 2ay - 3bx + 6by &= (ax - 2ay) - (3bx - 6by) \\ &= a(x - 2y) - 3b(x - 2y) \\ &= (a - 3b)(x - 2y). \end{aligned}$$

4.
$$\begin{aligned} 8mx - 7my - 8nx + 7ny &= (8mx - 8nx) - (7my - 7ny) \\ &= 8x(m - n) - 7y(m - n) \\ &= (8x - 7y)(m - n). \end{aligned}$$

5
$$\begin{aligned}3mx + 2cy - 3cx - 2my &= (3mx - 3cx) - (2my - 2cy)\\ &= 3x(m - c) - 2y(m - c)\\ &= (3x - 2y)(m - c).\end{aligned}$$

6.
$$\begin{aligned}mx + nx + m + n &= (mx + nx) + (m + n)\\ &= x(m + n) + (m + n)\\ &= (x + 1)(m + n).\end{aligned}$$

7.
$$\begin{aligned}y^2 + b^2 + ay^2 + ab^2 &= (y^2 + b^2) + (ay^2 + ab^2)\\ &= (y^2 + b^2) + a(y^2 + b^2)\\ &= (1 + a)(y^2 + b^2).\end{aligned}$$

8.
$$\begin{aligned}ax - bx - 2ay + 2by - 2cy + cx &= (ax - bx + cx) - (2ay - 2by + 2cy)\\ &= x(a - b + c) - 2y(a - b + c)\\ &= (x - 2y)(a - b + c).\end{aligned}$$

9.
$$\begin{aligned}3b - a^2 - a^2x^4 + 3bx^4 &= (3b + 3bx^4) - (a^2 + a^2x^4)\\ &= 3b(1 + x^4) - a^2(1 + x^4)\\ &= (3b - a^2)(1 + x^4).\end{aligned}$$

10.
$$\begin{aligned}y - z^2 + xz^2 - xy &= (y - xy) - (z^2 - xz^2)\\ &= y(1 - x) - z^2(1 - x)\\ &= (y - z^2)(1 - x).\end{aligned}$$

11.
$$\begin{aligned}3x^3 - 6ax^2 - x + 2a &= (3x^3 - 6ax^2) - (x - 2a)\\ &= 3x^2(x - 2a) - (x - 2a)\\ &= (3x^2 - 1)(x - 2a).\end{aligned}$$

12.
$$\begin{aligned}2a^2x - 9 - 6x + 3a^2 &= (2a^2x + 3a^2) - (6x + 9)\\ &= a^2(2x + 3) - 3(2x + 3)\\ &= (a^2 - 3)(2x + 3).\end{aligned}$$

13.
$$\begin{aligned}3x^2 + ax + 6bx + 2ab &= (3x^2 + ax) + (6bx + 2ab)\\ &= x(3x + a) + 2b(3x + a)\\ &= (x + 2b)(3x + a).\end{aligned}$$

14.
$$\begin{aligned}6a^2 - 3ab - 4ac + 2bc &= (6a^2 - 3ab) - (4ac - 2bc)\\ &= 3a(2a - b) - 2c(2a - b)\\ &= (3a - 2c)(2a - b).\end{aligned}$$

15.
$$\begin{aligned}4y + 9x^3 - 6x^2y - 6x &= (4y - 6x) - (6x^2y - 9x^3)\\ &= 2(2y - 3x) - 3x^2(2y - 3x)\\ &= (2 - 3x^2)(2y - 3x).\end{aligned}$$

16.
$$\begin{aligned}a^2c^2 - 2a^3b - 2c^3d + 4abcd &= (a^2c^2 - 2c^3d) - (2a^3b - 4abcd)\\ &= c^2(a^2 - 2cd) - 2ab(a^2 - 2cd)\\ &= (c^2 - 2ab)(a^2 - 2cd).\end{aligned}$$

17.
$$\begin{aligned} 2xy + 3yz + 6y^2 + xz &= (2xy + 6y^2) + (xz + 3yz) \\ &= 2y(x + 3y) + z(x + 3y) \\ &= (2y + z)(x + 3y). \end{aligned}$$

18.
$$\begin{aligned} c^2m - abm + abn - c^2n &= (c^2m - c^2n) - (abm - abn) \\ &= c^2(m - n) - ab(m - n) \\ &= (c^2 - ab)(m - n). \end{aligned}$$

19.
$$\begin{aligned} a^2 + 3ad - 2ab - 6bd &= (a^2 - 2ab) + (3ad - 6bd) \\ &= a(a - 2b) + 3d(a - 2b) \\ &= (a + 3d)(a - 2b). \end{aligned}$$

20.
$$\begin{aligned} 12bx + 3ay - 9by - 4ax &= (12bx - 4ax) - (9by - 3ay) \\ &= 4x(3b - a) - 3y(3b - a) \\ &= (4x - 3y)(3b - a). \end{aligned}$$

21.
$$\begin{aligned} 3x^2 + bx + 9ax + 3ab &= (3x^2 + bx) + (9ax + 3ab) \\ &= x(3x + b) + 3a(3x + b) \\ &= (x + 3a)(3x + b). \end{aligned}$$

22.
$$\begin{aligned} 6a^2 - 3ab - 2bm + 4am &= (6a^2 + 4am) - (3ab + 2bm) \\ &= 2a(3a + 2m) - b(3a + 2m) \\ &= (2a - b)(3a + 2m). \end{aligned}$$

23.
$$\begin{aligned} xy - xz - 2z^2 + 2yz &= (xy - xz) + (2yz - 2z^2) \\ &= x(y - z) + 2z(y - z) \\ &= (x + 2z)(y - z). \end{aligned}$$

24.
$$\begin{aligned} 6a^2 + 9ac - 6bc - 4ab &= (6a^2 - 4ab) + (9ac - 6bc) \\ &= 2a(3a - 2b) + 3c(3a - 2b) \\ &= (2a + 3c)(3a - 2b). \end{aligned}$$

25. $x^4 + x^2y^2 + x^2z^2 - x^3y - xyz^2 + y^2z^2$
$$\begin{aligned} &= (x^4 + x^2z^2) - (x^3y + xyz^2) + (x^2y^2 + y^2z^2) \\ &= x^2(x^2 + z^2) - xy(x^2 + z^2) + y^2(x^2 + z^2) \\ &= (x^2 + z^2)(x^2 - xy + y^2). \end{aligned}$$

26. $3x^3 + 5axy - 5b^2y - 3ax^2 + 3b^2x - 5x^2y$
$$\begin{aligned} &= (3x^3 - 3ax^2 + 3b^2x) - (5x^2y - 5axy + 5b^2y) \\ &= 3x(x^2 - ax + b^2) - 5y(x^2 - ax + b^2) \\ &= (3x - 5y)(x^2 - ax + b^2). \end{aligned}$$

27. $a^2b + by^2 + 6ax^2 - 3xy^2 - 3a^2x - 2abx$
$$\begin{aligned} &= (a^2b - 3a^2x) - (2abx - 6ax^2) + (by^2 - 3xy^2) \\ &= a^2(b - 3x) - 2ax(b - 3x) + y^2(b - 3x) \\ &= (b - 3x)(a^2 - 2ax + y^2). \end{aligned}$$

28. $a^2 - 4cx - 2ax + 2ac + 2bx - 3ad + 6dx - ab$
$= (a^2 - 2ax) - (ab - 2bx) + (2ac - 4cx) - (3ad - 6dx)$
$= a(a - 2x) - b(a - 2x) + 2c(a - 2x) - 3d(a - 2x)$
$= (a - 2x)(a - b + 2c - 3d).$

29. $6a^2x + a^2xy + 9a^2bz - 3abxz - 2ax^2 - 3a^3y$
$= a(6ax + axy + 9abz - 3bxz - 2x^2 - 3a^2y)$
$= a[(6ax - 2x^2) - (3a^2y - axy) + (9abz - 3bxz)]$
$= a[2x(3a - x) - ay(3a - x) + 3bz(3a - x)]$
$= a(3a - x)(2x - ay + 3bz).$

Exercise 38. Page 89

Resolve into factors:

1. $x^2 + 2abx + a^2b^2 = (x + ab)(x + ab).$
2. $x^2 + 6xy + 9y^2 = (x + 3y)(x + 3y).$
3. $x^2 - 4ax + 4a^2 = (x - 2a)(x - 2a).$
4. $x^2 - 10xy + 25y^2 = (x - 5y)(x - 5y).$
5. $9x^2 + 48xy + 64y^2 = (3x + 8y)(3x + 8y).$
6. $25x^2 - 70xy + 49y^2 = (5x - 7y)(5x - 7y).$
7. $16x^2 + 40x + 25 = (4x + 5)(4x + 5).$
8. $4x^2 - 12x + 9 = (2x - 3)(2x - 3).$
9. $9a^2 + 30ab + 25b^2 = (3a + 5b)(3a + 5b).$
10. $1 - 16x + 64x^2 = (1 - 8x)(1 - 8x).$
11. $4a^4 + 20a^2b + 25b^2 = (2a^2 + 5b)(2a^2 + 5b).$
12. $36x^4 + 60x^2 + 25 = (6x^2 + 5)(6x^2 + 5).$
13. $x^6y^6 - 6x^3y^3z + 9z^2 = (x^3y^3 - 3z)(x^3y^3 - 3z).$
14. $x^6 - 8x^3y^2 + 16y^4 = (x^3 - 4y^2)(x^3 - 4y^2).$
15. $25x^2 + 40xy + 16y^2 = (5x + 4y)(5x + 4y).$
16. $49a^4 - 42a^2b^2c^2 + 9b^4c^4 = (7a^2 - 3b^2c^2)(7a^2 - 3b^2c^2).$
17. $9x^4 + 24x^2y^2 + 16y^4 = (3x^2 + 4y^2)(3x^2 + 4y^2).$
18. $4x^4y^4 - 20x^3y^3 + 25x^2y^2 = x^2y^2(4x^2y^2 - 20xy + 25)$
$= x^2y^2(2xy - 5)(2xy - 5).$
19. $25a^{10}b^{10} - 20a^5b^5c^2d^2 + 4c^4d^4 = (5a^5b^5 - 2c^2d^2)(5a^5b^5 - 2c^2d^2).$
20. $9m^4 - 66m^2n + 121n^2 = (3m^2 - 11n)(3m^2 - 11n).$
21. $49n^{16} + 84m^3n^8 + 36m^6 = (7n^8 + 6m^3)(7n^8 + 6m^3).$
22. $4x^4 + 52ax^2 + 169a^2 = (2x^2 + 13a)(2x^2 + 13a).$
23. $144x^2y^2 - 168ab^2xy + 49a^2b^4 = (12xy - 7ab^2)(12xy - 7ab^2).$
24. $1 - 8abc + 16a^2b^2c^2 = (1 - 4abc)(1 - 4abc).$
25. $49x^2 + 126xy^2 + 81y^4 = (7x + 9y^2)(7x + 9y^2).$

26. $64a^4b^2 - 16a^2bx^3 + x^6 = (8a^2b - x^3)(8a^2b - x^3)$.
27. $121a^2 + 176ab + 64b^2 = (11a + 8b)(11a + 8b)$.
28. $144x^2 + 600xy + 625y^2 = (12x + 25y)(12x + 25y)$.
29. $(a + b)^2 - 6c(a + b) + 9c^2 = (a + b - 3c)(a + b - 3c)$.
30. $(x - y)^2 + 12z(x - y) + 36z^2 = (x - y + 6z)(x - y + 6z)$.

Exercise 39. Page 90

Resolve into factors:

1. $a^2 - b^2 = (a + b)(a - b)$.
2. $1 - a^2 = (1 + a)(1 - a)$.
3. $4x^2 - 81y^2 = (2x + 9y)(2x - 9y)$.
4. $1 - 36a^2 = (1 + 6a)(1 - 6a)$.
5. $4 - 49x^2 = (2 + 7x)(2 - 7x)$.
6. $49x^2 - 64 = (7x + 8)(7x - 8)$.
7. $36a^2 - 25b^2 = (6a + 5b)(6a - 5b)$.
8. $4m^2 - 1 = (2m + 1)(2m - 1)$.
9. $9 - 16n^2 = (3 + 4n)(3 - 4n)$.
10. $25x^2 - 4 = (5x + 2)(5x - 2)$.
11. $y^2 - 25z^2 = (y + 5z)(y - 5z)$.
12. $49a^2 - 25b^2 = (7a + 5b)(7a - 5b)$.
13. $64x^2 - 25 = (8x + 5)(8x - 5)$.
14. $4a^4 - 9x^2 = (2a^2 + 3x)(2a^2 - 3x)$.
15. $1 - a^4b^4c^2 = (1 + a^2b^2c)(1 - a^2b^2c)$.
16. $256x^2y^2 - 625 = (16xy + 25)(16xy - 25)$.
17. $a^{10} - 4b^4 = (a^5 + 2b^2)(a^5 - 2b^2)$.
18. $64a^4b^2 - x^6 = (8a^2b + x^3)(8a^2b - x^3)$.
19. $121a^4 - 64b^2 = (11a^2 + 8b)(11a^2 - 8b)$.
20. $49x^2 - 100y^4 = (7x + 10y^2)(7x - 10y^2)$.
21. $144x^2y^2 - 49a^2b^4 = (12xy + 7ab^2)(12xy - 7ab^2)$.
22. $49x^2 - 9b^4c^2 = (7x + 3b^2c)(7x - 3b^2c)$.
23. $256x^2y^2 - 1 = (16xy + 1)(16xy - 1)$.
24. $a^4b^2 - 4c^2d^4 = (a^2b + 2cd^2)(a^2b - 2cd^2)$.
25. $169x^2y^2 - 144a^2 = (13xy + 12a)(13xy - 12a)$.
26. $4a^2b - 25b^3 = b(4a^2 - 25b^2) = b(2a + 5b)(2a - 5b)$.
27. $4x^4 - 9y^2z^2 = (2x^2 + 3yz)(2x^2 - 3yz)$.
28. $a^8 - b^6 = (a^4 + b^3)(a^4 - b^3)$.
29. $144x^2 - 625y^2 = (12x + 25y)(12x - 25y)$.
30. $4a^2b^2 - 169c^2 = (2ab + 13c)(2ab - 13c)$.
31. $4 - 361a^4b^6c^2 = (2 + 19a^2b^3c)(2 - 19a^2b^3c)$.
32. $9m^4 - 121n^2p^4 = (3m^2 + 11np^2)(3m^2 - 11np^2)$.
33. $25a^{10}b^{10} - 4c^4d^4 = (5a^5b^5 + 2c^2d^2)(5a^5b^5 - 2c^2d^2)$.
34. $36x^4y^8 - 25a^4b^6 = (6x^2y^4 + 5a^2b^3)(6x^2y^4 - 5a^2b^3)$.

35. $81x^2y^6 - 49c^4 = (9xy^3 + 7c^2)(9xy^3 - 7c^2)$.
36. $a^{2m} - b^{2n} = (a^m + b^n)(a^m - b^n)$.
37. $4x^{2m} - 9y^{2n} = (2x^m + 3y^n)(2x^m - 3y^n)$.
38. $81a^{4m} - 49b^{2m} = (9a^{2m} + 7b^m)(9a^{2m} - 7b^m)$.
39. $a^{4c} - 36b^{2c}c^{4c} = (a^{2c} + 6b^cc^{2c})(a^{2c} - 6b^cc^{2c})$.
40. $121x^2 - 144z^{2m} = (11x + 12z^m)(11x - 12z^m)$.
41. $9a^{10n} - 64b^{12n} = (3a^{5n} + 8b^{6n})(3a^{5n} - 8b^{6n})$.
42. $49x^{2m} - 36y^{4m}z^{6m} = (7x^m + 6y^{2m}z^{3m})(7x^m - 6y^{2m}z^{3m})$.

Exercise 40. Page 91

Resolve into factors:

1. $(x + 3y)^2 - 4z^2$
$= (x + 3y + 2z)(x + 3y - 2z)$.

2. $(x - 5y)^2 - (a - 2b)^2$
$= (x - 5y + a - 2b)(x - 5y - a + 2b)$.

3. $(4a + 5)^2 - (2x - 3)^2$
$= (4a + 5 + 2x - 3)(4a + 5 - 2x + 3)$
$= (4a + 2x + 2)(4a - 2x + 8)$
$= 4(2a + x + 1)(2a - x + 4)$.

4. $(3a + 5b)^2 - (1 + 8x)^2$
$= (3a + 5b + 1 + 8x)(3a + 5b - 1 - 8x)$.

5. $(x^3 - 4y^2)^2 - (3a^4 - 2b^2)^2$
$= (x^3 - 4y^2 + 3a^4 - 2b^2)(x^3 - 4y^2 - 3a^4 + 2b^2)$.

6. $(x + y)^2 - (x - 3y)^2$
$= (x + y + x - 3y)(x + y - x + 3y)$
$= (2x - 2y)(4y)$
$= 8y(x - y)$.

7. $(2a - 5b)^2 - (a + 4b)^2$
$= (2a - 5b + a + 4b)(2a - 5b - a - 4b)$
$= (3a - b)(a - 9b)$.

8. $(7a - 3bc)^2 - (4x - 5yz)^2$
$= (7a - 3bc + 4x - 5yz)(7a - 3bc - 4x + 5yz)$.

9. $(2x^2 + 13a)^2 - (3y + 7b)^2$
$= (2x^2 + 13a + 3y + 7b)(2x^2 + 13a - 3y - 7b)$.

10. $(3x - y)^2 - (2x - 3y)^2$

$$\begin{aligned}&= (3x - y + 2x - 3y)(3x - y - 2x + 3y)\\&= (5x - 4y)(x + 2y).\end{aligned}$$

11. $(4a + 3b)^2 - (3a + 2b)^2$

$$\begin{aligned}&= (4a + 3b + 3a + 2b)(4a + 3b - 3a - 2b)\\&= (7a + 5b)(a + b).\end{aligned}$$

12. $(2x - 1)^2 - (3x + 1)^2$

$$\begin{aligned}&= (2x - 1 + 3x + 1)(2x - 1 - 3x - 1)\\&= 5x(-x - 2)\\&= -5x(x + 2).\end{aligned}$$

13. $(x + y - z)^2 - (x - y - z)^2$

$$\begin{aligned}&= (x + y - z + x - y - z)(x + y - z - x + y + z)\\&= (2x - 2z)(2y)\\&= 4y(x - z).\end{aligned}$$

14. $36(a + b)^2 - 25(c - d)^2$

$$\begin{aligned}&= [6(a + b) + 5(c - d)][6(a + b) - 5(c - d)]\\&= (6a + 6b + 5c - 5d)(6a + 6b - 5c + 5d).\end{aligned}$$

15. $49(a - b)^2 - 36(c + d)^2$

$$\begin{aligned}&= [7(a - b) + 6(c + d)][7(a - b) - 6(c + d)]\\&= (7a - 7b + 6c + 6d)(7a - 7b - 6c - 6d).\end{aligned}$$

16. $(5x^2 - 4y)^2 - (3x^2 - 2y)^2$

$$\begin{aligned}&= (5x^2 - 4y + 3x^2 - 2y)(5x^2 - 4y - 3x^2 + 2y)\\&= (8x^2 - 6y)(2x^2 - 2y)\\&= 4(4x^2 - 3y)(x^2 - y).\end{aligned}$$

17. $64(x - y)^2 - 81(y - z)^2$

$$\begin{aligned}&= [8(x - y) + 9(y - z)][8(x - y) - 9(y - z)]\\&= (8x - 8y + 9y - 9z)(8x - 8y - 9y + 9z)\\&= (8x + y - 9z)(8x - 17y + 9z).\end{aligned}$$

18. $49(x + y)^2 - 25(x - y)^2$

$$\begin{aligned}&= [7(x + y) + 5(x - y)][7(x + y) - 5(x - y)]\\&= (7x + 7y + 5x - 5y)(7x + 7y - 5x + 5y)\\&= (12x + 2y)(2x + 12y)\\&= 4(6x + y)(x + 6y).\end{aligned}$$

Exercise 41. Page 92

Resolve into factors:

1. $$\begin{aligned} x^2 - 2xy + y^2 - 16z^2 &= (x^2 - 2xy + y^2) - (16z^2) \\ &= (x - y)^2 - (4z)^2 \\ &= (x - y + 4z)(x - y - 4z). \end{aligned}$$

2. $$\begin{aligned} a^2 - 4ab + 4b^2 - 9c^2 &= (a^2 - 4ab + 4b^2) - (9c^2) \\ &= (a - 2b)^2 - (3c)^2 \\ &= (a - 2b + 3c)(a - 2b - 3c). \end{aligned}$$

3. $$\begin{aligned} x^2 - 6xy + 9y^2 - 16z^2 &= (x^2 - 6xy + 9y^2) - (16z^2) \\ &= (x - 3y)^2 - (4z)^2 \\ &= (x - 3y + 4z)(x - 3y - 4z). \end{aligned}$$

4. $$\begin{aligned} x^2 - 2ax + a^2 - 49y^2 &= (x^2 - 2ax + a^2) - (49y^2) \\ &= (x - a)^2 - (7y)^2 \\ &= (x - a + 7y)(x - a - 7y). \end{aligned}$$

5. $$\begin{aligned} 36b^2 - 4 + 20a - 25a^2 &= (36b^2) - (4 - 20a + 25a^2) \\ &= (6b)^2 - (2 - 5a)^2 \\ &= (6b + 2 - 5a)(6b - 2 + 5a). \end{aligned}$$

6. $$\begin{aligned} 4c^2 - a^4 - 9b^2 - 6a^2b &= (4c^2) - (a^4 + 6a^2b + 9b^2) \\ &= (2c)^2 - (a^2 + 3b)^2 \\ &= (2c + a^2 + 3b)(2c - a^2 - 3b). \end{aligned}$$

7. $$\begin{aligned} 4x^2 - 25y^2 + 9 - 12x &= (4x^2 - 12x + 9) - (25y^2) \\ &= (2x - 3)^2 - (5y)^2 \\ &= (2x - 3 + 5y)(2x - 3 - 5y). \end{aligned}$$

8. $$\begin{aligned} 25x^2 - 4a^2 + 16y^2 + 40xy &= (25x^2 + 40xy + 16y^2) - (4a^2) \\ &= (5x + 4y)^2 - (2a)^2 \\ &= (5x + 4y + 2a)(5x + 4y - 2a). \end{aligned}$$

9. $$\begin{aligned} 16a^4 - x^2 - y^2 + 2xy &= (16a^4) - (x^2 - 2xy + y^2) \\ &= (4a^2)^2 - (x - y)^2 \\ &= (4a^2 + x - y)(4a^2 - x + y). \end{aligned}$$

10. $$\begin{aligned} 9x^2 - a^2 - 4ab - 4b^2 &= (9x^2) - (a^2 + 4ab + 4b^2) \\ &= (3x)^2 - (a + 2b)^2 \\ &= (3x + a + 2b)(3x - a - 2b). \end{aligned}$$

11. $$\begin{aligned} a^2 + 2ab + b^2 - c^2 - 2cd - d^2 &= (a^2 + 2ab + b^2) - (c^2 + 2cd + d^2) \\ &= (a + b)^2 - (c + d)^2 \\ &= (a + b + c + d)(a + b - c - d). \end{aligned}$$

12. $a^2 + b^2 - c^2 - d^2 - 2\,ab + 2\,cd$
$$\begin{aligned}&= (a^2 - 2\,ab + b^2) - (c^2 - 2\,cd + d^2)\\&= (a - b)^2 - (c - d)^2\\&= (a - b + c - d)(a - b - c + d).\end{aligned}$$

13. $a^2 - c^2 + 4\,b^2 - 4\,d^2 - 4\,ab - 4\,cd$
$$\begin{aligned}&= (a^2 - 4\,ab + 4\,b^2) - (c^2 + 4\,cd + 4\,d^2)\\&= (a - 2\,b)^2 - (c + 2\,d)^2\\&= (a - 2\,b + c + 2\,d)(a - 2\,b - c - 2\,d).\end{aligned}$$

14. $x^2 - 9\,b^2 + 9\,y^2 - a^2 + 6\,ab - 6\,xy$
$$\begin{aligned}&= (x^2 - 6\,xy + 9\,y^2) - (a^2 - 6\,ab + 9\,b^2)\\&= (x - 3\,y)^2 - (a - 3\,b)^2\\&= (x - 3\,y + a - 3\,b)(x - 3\,y - a + 3\,b).\end{aligned}$$

15. $49\,x^2 + 4\,y^2 - 9\,a^2 - 16\,b^2 - 28\,xy - 24\,ab$
$$\begin{aligned}&= (49\,x^2 - 28\,xy + 4\,y^2) - (9\,a^2 + 24\,ab + 16\,b^2)\\&= (7\,x - 2\,y)^2 - (3\,a + 4\,b)^2\\&= (7\,x - 2\,y + 3\,a + 4\,b)(7\,x - 2\,y - 3\,a - 4\,b).\end{aligned}$$

16. $16\,a^2 - 9\,x^2y^2 - 25\,z^4 + 25\,b^2 + 30\,xyz^2 + 40\,ab$
$$\begin{aligned}&= (16\,a^2 + 40\,ab + 25\,b^2) - (9\,x^2y^2 - 30\,xyz^2 + 25\,z^4)\\&= (4\,a + 5\,b)^2 - (3\,xy - 5\,z^2)^2\\&= (4\,a + 5\,b + 3\,xy - 5\,z^2)(4\,a + 5\,b - 3\,xy + 5\,z^2).\end{aligned}$$

17. $4\,a^2 + 9\,x^2 - 12\,ax + 12\,by - 9\,b^2 - 4\,y^2$
$$\begin{aligned}&= (4\,a^2 - 12\,ax + 9\,x^2) - (9\,b^2 - 12\,by + 4\,y^2)\\&= (2\,a - 3\,x)^2 - (3\,b - 2\,y)^2\\&= (2\,a - 3\,x + 3\,b - 2\,y)(2\,a - 3\,x - 3\,b + 2\,y).\end{aligned}$$

18. $4\,x^4 + 9\,y^6 - 12\,x^2y^3 - 64\,a^2b^4c^6$
$$\begin{aligned}&= (4\,x^4 - 12\,x^2y^3 + 9\,y^6) - (64\,a^2b^4c^6)\\&= (2\,x^2 - 3\,y^3)^2 - (8\,ab^2c^3)^2\\&= (2\,x^2 - 3\,y^3 + 8\,ab^2c^3)(2\,x^2 - 3\,y^3 - 8\,ab^2c^3).\end{aligned}$$

19. $9\,x^2 + 16\,y^2 - 49\,a^2 - 4\,b^2 + 28\,ab + 24\,xy$
$$\begin{aligned}&= (9\,x^2 + 24\,xy + 16\,y^2) - (49\,a^2 - 28\,ab + 4\,b^2)\\&= (3\,x + 4\,y)^2 - (7\,a - 2\,b)^2\\&= (3\,x + 4\,y + 7\,a - 2\,b)(3\,x + 4\,y - 7\,a + 2\,b).\end{aligned}$$

20. $81\,x^2 - m^2n^2 - 64\,a^4 + 126\,xy + 16\,a^2mn + 49\,y^2$
$$\begin{aligned}&= (81\,x^2 + 126\,xy + 49\,y^2) - (m^2n^2 - 16\,a^2mn + 64\,a^4)\\&= (9\,x + 7\,y)^2 - (mn - 8\,a^2)\\&= (9\,x + 7\,y + mn - 8\,a^2)(9\,x + 7\,y - mn + 8\,a^2).\end{aligned}$$

21.
$$\begin{aligned}&121 - 110x^2 - y^2 - 625z^2 - 50yz + 25x^4\\&= (121 - 110x^2 + 25x^4) - (y^2 + 50yz + 625z^2)\\&= (11 - 5x^2)^2 - (y + 25z)^2\\&= (11 - 5x^2 + y + 25z)(11 - 5x^2 - y - 25z).\end{aligned}$$

22.
$$\begin{aligned}&x^6 + 225 - 30x^3 + 28abc - 49a^2b^2 - 4c^2\\&= (x^6 - 30x^3 + 225) - (49a^2b^2 - 28abc + 4c^2)\\&= (x^3 - 15)^2 - (7ab - 2c)^2\\&= (x^3 - 15 + 7ab - 2c)(x^3 - 15 - 7ab + 2c).\end{aligned}$$

23.
$$\begin{aligned}&4x^{2m} - p^{2n} - a^{2n} + 9 - 12x^m - 2a^np^n\\&= (4x^{2m} - 12x^m + 9) - (p^{2n} + 2a^np^n + a^{2n})\\&= (2x^m - 3)^2 - (p^n + a^n)^2\\&= (2x^m - 3 + p^n + a^n)(2x^m - 3 - p^n - a^n).\end{aligned}$$

24.
$$\begin{aligned}&36a^{2n+2} - 4y^{2n+4} - 12x^ny^{n+2} + 16b^{2n-2} - 48a^{n+1}b^{n-1} - 9x^{2n}\\&= (36a^{2n+2} - 48a^{n+1}b^{n-1} + 16b^{2n-2})\\&\quad - (9x^{2n} + 12x^ny^{n+2} + 4y^{2n+4})\\&= (6a^{n+1} - 4b^{n-1})^2 - (3x^n + 2y^{n+2})^2\\&= (6a^{n+1} - 4b^{n-1} + 3x^n + 2y^{n+2})(6a^{n+1} - 4b^{n-1} - 3x^n - 2y^{n+2})\end{aligned}$$

Exercise 42. Page 94

Resolve into factors:

1.
$$\begin{aligned}x^4 + 4x^2 + 16 &= (x^4 + 8x^2 + 16) - (4x^2)\\&= (x^2 + 4)^2 - (2x)^2\\&= (x^2 + 4 + 2x)(x^2 + 4 - 2x)\\&= (x^2 + 2x + 4)(x^2 - 2x + 4).\end{aligned}$$

2.
$$\begin{aligned}9x^4 + 2x^2y^2 + y^4 &= (9x^4 + 6x^2y^2 + y^4) - (4x^2y^2)\\&= (3x^2 + y^2)^2 - (2xy)^2\\&= (3x^2 + y^2 + 2xy)(3x^2 + y^2 - 2xy)\\&= (3x^2 + 2xy + y^2)(3x^2 - 2xy + y^2).\end{aligned}$$

3.
$$\begin{aligned}4a^4 + 11a^2b^2 + 25b^4 &= (4a^4 + 20a^2b^2 + 25b^4) - (9a^2b^2)\\&= (2a^2 + 5b^2) - (3ab)^2\\&= (2a^2 + 5b^2 + 3ab)(2a^2 + 5b^2 - 3ab)\\&= (2a^2 + 3ab + 5b^2)(2a^2 - 3ab + 5b^2).\end{aligned}$$

4.
$$\begin{aligned}a^4 + 64b^4 &= (a^4 + 16a^2b^2 + 64b^4) - (16a^2b^2)\\&= (a^2 + 8b^2) - (4ab)^2\\&= (a^2 + 8b^2 + 4ab)(a^2 + 8b^2 - 4ab)\\&= (a^2 + 4ab + 8b^2)(a^2 - 4ab + 8b^2).\end{aligned}$$

5.
$$\begin{aligned}4x^4-16a^2x^2+9a^4&=(4x^4-12a^2x^2+9a^4)-(4a^2x^2)\\&=(2x^2-3a^2)-(2ax)^2\\&=(2x^2-3a^2+2ax)(2x^2-3a^2-2ax)\\&=(2x^2+2ax-3a^2)(2x^2-2ax-3a^2).\end{aligned}$$

6.
$$\begin{aligned}4a^4-13a^2+1&=(4a^4-4a^2+1)-(9a^2)\\&=(2a^2-1)^2-(3a)^2\\&=(2a^2-1+3a)(2a^2-1-3a)\\&=(2a^2+3a-1)(2a^2-3a-1).\end{aligned}$$

7.
$$\begin{aligned}9x^4+15x^2y^2+16y^4&=(9x^4+24x^2y^2+16y^4)-(9x^2y^2)\\&=(3x^2+4y^2)^2-(3xy)^2\\&=(3x^2+4y^2+3xy)(3x^2+4y^2-3xy)\\&=(3x^2+3xy+4y^2)(3x^2-3xy+4y^2).\end{aligned}$$

8.
$$\begin{aligned}25a^4+71a^2x^2+64x^4&=(25a^4+80a^2x^2+64x^4)-(9a^2x^2)\\&=(5a^2+8x^2)^2-(3ax)^2\\&=(5a^2+8x^2+3ax)(5a^2+8x^2-3ax)\\&=(5a^2+3ax+8x^2)(5a^2-3ax+8x^2).\end{aligned}$$

9.
$$\begin{aligned}49x^4-44x^2y^2+4y^4&=(49x^4-28x^2y^2+4y^4)-(16x^2y^2)\\&=(7x^2-2y^2)^2-(4xy)^2\\&=(7x^2-2y^2+4xy)(7x^2-2y^2-4xy)\\&=(7x^2+4xy-2y^2)(7x^2-4xy-2y^2).\end{aligned}$$

10.
$$\begin{aligned}81a^4+45a^2b^2+49b^4&=(81a^4+126a^2b^2+49b^4)-(81a^2b^2)\\&=(9a^2+7b^2)^2-(9ab)^2\\&=(9a^2+7b^2+9ab)(9a^2+7b^2-9ab)\\&=(9a^2+9ab+7b^2)(9a^2-9ab+7b^2).\end{aligned}$$

11.
$$\begin{aligned}4a^4-29a^2x^2+25x^4&=(4a^4-20a^2x^2+25x^4)-(9a^2x^2)\\&=(2a^2-5x^2)^2-(3ax)^2\\&=(2a^2-5x^2+3ax)(2a^2-5x^2-3ax)\\&=(2a^2+3ax-5x^2)(2a^2-3ax-5x^2).\end{aligned}$$

12.
$$\begin{aligned}121a^4+54a^2y^2+81y^4&=(121a^4+198a^2y^2+81y^4)-(144a^2y^2)\\&=(11a^2+9y^2)^2-(12ay)^2\\&=(11a^2+9y^2+12ay)(11a^2+9y^2-12ay)\\&=(11a^2+12ay+9y^2)(11a^2-12ay+9y^2).\end{aligned}$$

13.
$$\begin{aligned}49-221x^2+100x^4&=(49-140x^2+100x^4)-(81x^2)\\&=(7-10x^2)^2-(9x)^2\\&=(7-10x^2+9x)(7-10x^2-9x)\\&=(7+9x-10x^2)(7-9x-10x^2).\end{aligned}$$

14.
$$\begin{aligned} 49a^4+6a^2c^4+9c^8 &= (49a^4+42a^2c^4+9c^8)-(36a^2c^4)\\ &= (7a^2+3c^4)^2-(6ac^2)^2\\ &= (7a^2+3c^4+6ac^2)(7a^2+3c^4-6ac^2)\\ &= (7a^2+6ac^2+3c^4)(7a^2-6ac^2+3c^4)\end{aligned}$$

15.
$$\begin{aligned} 81c^4-115c^2+25 &= (81c^4-90c^2+25)-(25c^2)\\ &= (9c^2-5)^2-(5c)^2\\ &= (9c^2-5+5c)(9c^2-5-5c)\\ &= (9c^2+5c-5)(9c^2-5c-5).\end{aligned}$$

16.
$$\begin{aligned} 121x^4+29x^2y^2+25y^4 &= (121x^4+110x^2y^2+25y^4)-(81x^2y^2)\\ &= (11x^2+5y^2)^2-(9xy)^2\\ &= (11x^2+5y^2+9xy)(11x^2+5y^2-9xy)\\ &= (11x^2+9xy+5y^2)(11x^2-9xy+5y^2).\end{aligned}$$

17.
$$\begin{aligned} 144a^4+47a^2c^2+49c^4 &= (144a^4+168a^2c^2+49c^4)-(121a^2c^2)\\ &= (12a^2+7c^2)^2-(11ac)^2\\ &= (12a^2+7c^2+11ac)(12a^2+7c^2-11ac)\\ &= (12a^2+11ac+7c^2)(12a^2-11ac+7c^2).\end{aligned}$$

18.
$$\begin{aligned} 36x^4-141x^2y^2+25y^4 &= (36x^4-60x^2y^2+25y^4)-(81x^2y^2)\\ &= (6x^2-5y^2)^2-(9xy)^2\\ &= (6x^2-5y^2+9xy)(6x^2-5y^2-9xy)\\ &= (6x^2+9xy-5y^2)(6x^2-9xy-5y^2).\end{aligned}$$

19.
$$\begin{aligned} 49x^4-121x^2y^2+64y^4 &= (49x^4-112x^2y^2+64y^4)-(9x^2y^2)\\ &= (7x^2-8y^2)^2-(3xy)^2\\ &= (7x^2-8y^2+3xy)(7x^2-8y^2-3xy)\\ &= (7x^2+3xy-8y^2)(7x^2-3xy-8y^2).\end{aligned}$$

20.
$$\begin{aligned} &49a^4b^4+61a^2b^2y^2+169y^4\\ &= (49a^4b^4+182a^2b^2y^2+169y^4)-(121a^2b^2y^2)\\ &= (7a^2b^2+13y^2)^2-(11aby)^2\\ &= (7a^2b^2+13y^2+11aby)(7a^2b^2+13y^2-11aby)\\ &= (7a^2b^2+11aby+13y^2)(7a^2b^2-11aby+13y^2).\end{aligned}$$

21.
$$\begin{aligned} &121x^4-302x^2y^2+169y^4\\ &= (121x^4-286x^2y^2+169y^4)-(16x^2y^2)\\ &= (11x^2-13y^2)^2-(4xy)^2\\ &= (11x^2-13y^2+4xy)(11x^2-13y^2-4xy)\\ &= (11x^2+4xy-13y^2)(11x^2-4xy-13y^2).\end{aligned}$$

22.
$$\begin{aligned} 4a^4 + 4a^2x^2 + 25x^4 &= (4a^4 + 20a^2x^2 + 25x^4) - (16a^2x^2) \\ &= (2a^2 + 5x^2)^2 - (4ax)^2 \\ &= (2a^2 + 5x^2 + 4ax)(2a^2 + 5x^2 - 4ax) \\ &= (2a^2 + 4ax + 5x^2)(2a^2 - 4ax + 5x^2). \end{aligned}$$

23.
$$\begin{aligned} a^4 - 16b^4 &= (a^2 + 4b^2)(a^2 - 4b^2) \\ &= (a^2 + 4b^2)(a + 2b)(a - 2b). \end{aligned}$$

24.
$$\begin{aligned} x^8 - 256y^6 &= (x^4 + 16y^4)(x^4 - 16y^4) \\ &= (x^4 + 16y^4)(x^2 + 4y^2)(x^2 - 4y^2) \\ &= (x^4 + 16y^4)(x^2 + 4y^2)(x + 2y)(x - 2y). \end{aligned}$$

25.
$$\begin{aligned} 16c^4 - 81d^4 &= (4c^2 + 9d^2)(4c^2 - 9d^2) \\ &= (4c^2 + 9d^2)(2c + 3d)(2c - 3d). \end{aligned}$$

26.
$$\begin{aligned} 256x^4 - 625y^4 &= (16x^2 + 25y^2)(16x^2 - 25y^2) \\ &= (16x^2 + 25y^2)(4x + 5y)(4x - 5y). \end{aligned}$$

27.
$$\begin{aligned} x^4 + 324y^4 &= (x^4 + 36x^2y^2 + 324y^4) - (36x^2y^2) \\ &= (x^2 + 18y^2)^2 - (6xy)^2 \\ &= (x^2 + 18y^2 + 6xy)(x^2 + 18y^2 - 6xy) \\ &= (x^2 + 6xy + 18y^2)(x^2 - 6xy + 18y^2). \end{aligned}$$

28.
$$\begin{aligned} 81a^4 - 207a^2b^2 + 49b^4 &= (81a^4 - 126a^2b^2 + 49b^4) - (81a^2b^2) \\ &= (9a^2 - 7b^2)^2 - (9ab)^2 \\ &= (9a^2 - 7b^2 + 9ab)(9a^2 - 7b^2 - 9ab) \\ &= (9a^2 + 9ab - 7b^2)(9a^2 - 9ab - 7b^2). \end{aligned}$$

29. $64a^4b^4 + 81x^4 + 63a^2b^2x^2$
$$\begin{aligned} &= (64a^4b^4 + 144a^2b^2x^2 + 81x^4) - (81a^2b^2x^2) \\ &= (8a^2b^2 + 9x^2)^2 - (9abx)^2 \\ &= (8a^2b^2 + 9x^2 + 9abx)(8a^2b^2 + 9x^2 - 9abx) \\ &= (8a^2b^2 + 9abx + 9x^2)(8a^2b^2 - 9abx + 9x^2). \end{aligned}$$

30. $100a^4 + 9b^4x^4 + 11a^2b^2x^2$
$$\begin{aligned} &= (100a^4 + 60a^2b^2x^2 + 9b^4x^4) - (49a^2b^2x^2) \\ &= (10a^2 + 3b^2x^2)^2 - (7abx)^2 \\ &= (10a^2 + 3b^2x^2 + 7abx)(10a^2 + 3b^2x^2 - 7abx) \\ &= (10a^2 + 7abx + 3b^2x^2)(10a^2 - 7abx + 3b^2x^2). \end{aligned}$$

31. $81a^8b^8 + 25x^4 - 171a^4b^4x^2$
$$\begin{aligned} &= (81a^8b^8 - 90a^4b^4x^2 + 25x^4) - (81a^4b^4x^2) \\ &= (9a^4b^4 - 5x^2)^2 - (9a^2b^2x)^2 \\ &= (9a^4b^4 - 5x^2 + 9a^2b^2x)(9a^4b^4 - 5x^2 - 9a^2b^2x) \\ &= (9a^4b^4 + 9a^2b^2x - 5x^2)(9a^4b^4 - 9a^2b^2x - 5x^2). \end{aligned}$$

32. $121\,a^4 + 144\,x^8y^4 + 260\,a^2x^4y^2$

$$\begin{aligned}&= (121\,a^4 + 264\,a^2x^4y^2 + 144\,x^8y^4) - (4\,a^2x^4y^2)\\&= (11\,a^2 + 12\,x^4y^2)^2 - (2\,ax^2y)^2\\&= (11\,a^2 + 12\,x^4y^2 + 2\,ax^2y)\,(11\,a^2 + 12\,x^4y^2 - 2\,ax^2y)\\&= (11\,a^2 + 2\,ax^2y + 12\,x^4y^2)\,(11\,a^2 - 2\,ax^2y + 12\,x^4y^2).\end{aligned}$$

33. $49\,a^4 - 193\,a^2b^2 + 64\,b^4$

$$\begin{aligned}&= (49\,a^4 - 112\,a^2b^2 + 64\,b^4) - (81\,a^2b^2)\\&= (7\,a^2 - 8\,b^2) - (9\,ab)^2\\&= (7\,a^2 - 8\,b^2 + 9\,ab)\,(7\,a^2 - 8\,b^2 - 9\,ab)\\&= (7\,a^2 + 9\,ab - 8\,b^2)\,(7\,a^2 - 9\,ab - 8\,b^2).\end{aligned}$$

34. $4\,b^4x^4 - 133\,b^2x^2y^4 + 9\,y^8$

$$\begin{aligned}&= (4\,b^4x^4 - 12\,b^2x^2y^4 + 9\,y^8) - (121\,b^2x^2y^4)\\&= (2\,b^2x^2 - 3\,y^4)^2 - (11\,bxy^2)^2\\&= (2\,b^2x^2 - 3\,y^4 + 11\,bxy^2)\,(2\,b^2x^2 - 3\,y^4 - 11\,bxy^2)\\&= (2\,b^2x^2 + 11\,bxy^2 - 3\,y^4)\,(2\,b^2x^2 - 11\,bxy^2 - 3\,y^4).\end{aligned}$$

35. $36\,x^4y^4 + 11\,x^2y^2z^2 + 25\,z^4$

$$\begin{aligned}&= (36\,x^4y^4 + 60\,x^2y^2z^2 + 25\,z^4) - (49\,x^2y^2z^2)\\&= (6\,x^2y^2 + 5\,z^2) - (7\,xyz)^2\\&= (6\,x^2y^2 + 5\,z^2 + 7\,xyz)\,(6\,x^2y^2 + 5\,z^2 - 7\,xyz)\\&= (6\,x^2y^2 + 7\,xyz + 5\,z^2)\,(6\,x^2y^2 - 7\,xyz + 5\,z^2).\end{aligned}$$

36. $64\,x^4y^8 + 79\,x^2y^4z^2 + 100\,z^4$

$$\begin{aligned}&= (64\,x^4y^8 + 160\,x^2y^4z^2 + 100\,z^4) - (81\,x^2y^4z^2)\\&= (8\,x^2y^4 + 10\,z^2)^2 - (9\,xy^2z)^2\\&= (8\,x^2y^4 + 10\,z^2 + 9\,xy^2z)\,(8\,x^2y^4 + 10\,z^2 - 9\,xy^2z)\\&= (8\,x^2y^4 + 9\,xy^2z + 10\,z^2)\,(8\,x^2y^4 - 9\,xy^2z + 10\,z^2).\end{aligned}$$

37. $49\,a^4b^4 - 53\,a^2b^2x^2 + 4\,x^4$

$$\begin{aligned}&= (49\,a^4b^4 - 28\,a^2b^2x^2 + 4\,x^4) - (25\,a^2b^2x^2)\\&= (7\,a^2b^2 - 2\,x^2)^2 - (5\,abx)^2\\&= (7\,a^2b^2 - 2\,x^2 + 5\,abx)\,(7\,a^2b^2 - 2\,x^2 - 5\,abx)\\&= (7\,a^2b^2 + 5\,abx - 2\,x^2)\,(7\,a^2b^2 - 5\,abx - 2\,x^2).\end{aligned}$$

38. $121\,x^4 - 295\,x^2y^2 + 169\,y^4$

$$\begin{aligned}&= (121\,x^4 - 286\,x^2y^2 + 169\,y^4) - (9\,x^2y^2)\\&= (11\,x^2 - 13\,y^2)^2 - (3\,xy)^2\\&= (11\,x^2 - 13\,y^2 + 3\,xy)\,(11\,x^2 - 13\,y^2 - 3\,xy)\\&= (11\,x^2 + 3\,xy - 13\,y^2)\,(11\,x^2 - 3\,xy - 13\,y^2).\end{aligned}$$

39. $289\,x^4y^4 + 21\,x^2y^2d^2 + 9\,d^4$
$= (289\,x^4y^4 + 102\,x^2y^2d^2 + 9\,d^4) - (81\,x^2y^2d^2)$
$= (17\,x^2y^2 + 3\,d^2)^2 - (9\,xyd)^2$
$= (17\,x^2y^2 + 3\,d^2 + 9\,xyd)\,(17\,x^2y^2 + 3\,d^2 - 9\,xyd)$
$= (17\,x^2y^2 + 9\,xyd + 3\,d^2)\,(17\,x^2y^2 - 9\,xyd + 3\,d^2).$

40. $361\,x^4 - 92\,a^2c^2x^2 + 4\,a^4c^4$
$= (361\,x^4 - 76\,a^2c^2x^2 + 4\,a^4c^4) - (16\,a^2c^2x^2)$
$= (19\,x^2 - 2\,a^2c^2)^2 - (4\,acx)^2$
$= (19\,x^2 - 2\,a^2c^2 + 4\,acx)\,(19\,x^2 - 2\,a^2c^2 - 4\,acx)$
$= (19\,x^2 + 4\,acx - 2\,a^2c^2)\,(19\,x^2 - 4\,acx - 2\,a^2c^2).$

41. $4\,m^4n^8 + 3\,a^2m^2n^4 + 9\,a^4$
$= (4\,m^4n^8 + 12\,a^2m^2n^4 + 9\,a^4) - (9\,a^2m^2n^4)$
$= (2\,m^2n^4 + 3\,a^2)^2 - (3\,amn^2)^2$
$= (2\,m^2n^4 + 3\,a^2 + 3\,amn^2)\,(2\,m^2n^4 + 3\,a^2 - 3\,amn^2)$
$= (2\,m^2n^4 + 3\,amn^2 + 3\,a^2)\,(2\,m^2n^4 - 3\,amn^2 + 3\,a^2).$

42. $9\,a^4 + 5\,a^2x^2 + 25\,x^4$
$= (9\,a^4 + 30\,a^2x^2 + 25\,x^4) - (25\,a^2x^2)$
$= (3\,a^2 + 5\,x^2)^2 - (5\,ax)^2$
$= (3\,a^2 + 5\,x^2 + 5\,ax)\,(3\,a^2 + 5\,x^2 - 5\,ax)$
$= (3\,a^2 + 5\,ax + 5\,x^2)\,(3\,a^2 - 5\,ax + 5\,x^2).$

43. $49\,m^4 + 110\,m^2n^2 + 81\,n^4$
$= (49\,m^4 + 126\,m^2n^2 + 81\,n^4) - (16\,m^2n^2)$
$= (7\,m^2 + 9\,n^2)^2 - (4\,mn)^2$
$= (7\,m^2 + 9\,n^2 + 4\,mn)\,(7\,m^2 + 9\,n^2 - 4\,mn)$
$= (7\,m^2 + 4\,mn + 9\,n^2)\,(7\,m^2 - 4\,mn + 9\,n^2).$

44. $49\,x^4 - 15\,x^2y^2 + 121\,y^4$
$= (49\,x^4 + 154\,x^2y^2 + 121\,y^4) - (169\,x^2y^2)$
$= (7\,x^2 + 11\,y^2)^2 - (13\,xy)^2$
$= (7\,x^2 + 11\,y^2 + 13\,xy)\,(7\,x^2 + 11\,y^2 - 13\,xy)$
$= (7\,x^2 + 13\,xy + 11\,y^2)\,(7\,x^2 - 13\,xy + 11\,y^2).$

45. $64\,a^4 + 128\,a^2b^2c^2 + 81\,b^4c^4$
$= (64\,a^4 + 144\,a^2b^2c^2 + 81\,b^4c^4) - (16\,a^2b^2c^2)$
$= (8\,a^2 + 9\,b^2c^2)^2 - (4\,abc)^2$
$= (8\,a^2 + 9\,b^2c^2 + 4\,abc)\,(8\,a^2 + 9\,b^2c^2 - 4\,abc)$
$= (8\,a^2 + 4\,abc + 9\,b^2c^2)\,(8\,a^2 - 4\,abc + 9\,b^2c^2).$

46. $4\,m^4n^4 - 37\,m^2n^2x^2 + 9\,x^4$
$= (4\,m^4n^4 - 12\,m^2n^2x^2 + 9\,x^4) - (25\,m^2n^2x^2)$
$= (2\,m^2n^2 - 3\,x^2)^2 - (5\,mnx)^2$
$= (2\,m^2n^2 - 3\,x^2 + 5\,mnx)\,(2\,m^2n^2 - 3\,x^2 - 5\,mnx)$
$= (2\,m^2n^2 + 5\,mnx - 3\,x^2)\,(2\,m^2n^2 - 5\,mnx - 3\,x^2).$

Exercise 43. Page 95

Resolve into two factors:

1.
$$\begin{aligned} a^2 + 4b^2 &= (a^2 + 4ab + 4b^2) - (4ab) \\ &= (a + 2b)^2 - (2\sqrt{ab})^2 \\ &= (a + 2b + 2\sqrt{ab})(a + 2b - 2\sqrt{ab}) \\ &= (a + 2\sqrt{ab} + 2b)(a - 2\sqrt{ab} + 2b). \end{aligned}$$

2.
$$\begin{aligned} 4x^2 + 9y^2 &= (4x^2 + 12xy + 9y^2) - (12xy) \\ &= (2x + 3y)^2 - (2\sqrt{3xy})^2 \\ &= (2x + 3y + 2\sqrt{3xy})(2x + 3y - 2\sqrt{3xy}) \\ &= (2x + 2\sqrt{3xy} + 3y)(2x - 2\sqrt{3xy} + 3y). \end{aligned}$$

3.
$$\begin{aligned} 25m^2 + 16n^2 &= (25m^2 + 40mn + 16n^2) - (40mn) \\ &= (5m + 4n)^2 - (2\sqrt{10mn})^2 \\ &= (5m + 4n + 2\sqrt{10mn})(5m + 4n - 2\sqrt{mn}) \\ &= (5m + 2\sqrt{10mn} + 4n)(5m - 2\sqrt{mn} + 4n). \end{aligned}$$

4.
$$\begin{aligned} 64c^2 + 49d^2 &= (64c^2 + 112cd + 49d^2) - (112cd) \\ &= (8c + 7d)^2 - (4\sqrt{7cd})^2 \\ &= (8c + 7d + 4\sqrt{7cd})(8c + 7d - 4\sqrt{7cd}) \\ &= (8c + 4\sqrt{7cd} + 7d)(8c - 4\sqrt{7cd} + 7d). \end{aligned}$$

5.
$$\begin{aligned} 36a^4 + 49b^2c^2 &= (36a^4 + 84a^2bc + 49b^2c^2) - (84a^2bc) \\ &= (6a^2 + 7bc)^2 - (2a\sqrt{21bc})^2 \\ &= (6a^2 + 7bc + 2a\sqrt{21bc})(6a^2 + 7bc - 2a\sqrt{21bc}) \\ &= (6a^2 + 2a\sqrt{21bc} + 7bc)(6a^2 - 2a\sqrt{21bc} + 7bc). \end{aligned}$$

6.
$$\begin{aligned} 9x^2 + 25y^2 &= (9x^2 + 30xy + 25y^2) - (30xy) \\ &= (3x + 5y)^2 - (\sqrt{30xy})^2 \\ &= (3x + 5y + \sqrt{30xy})(3x + 5y - \sqrt{30xy}) \\ &= (3x + \sqrt{30xy} + 5y)(3x - \sqrt{30xy} + 5y). \end{aligned}$$

7.
$$\begin{aligned} 81m^2 + 4n^4 &= (81m^2 + 36mn^2 + 4n^4) - (36mn^2) \\ &= (9m + 2n^2)^2 - (6n\sqrt{m})^2 \\ &= (9m + 2n^2 + 6n\sqrt{m})(9m + 2n^2 - 6n\sqrt{m}) \\ &= (9m + 6n\sqrt{m} + 2n^2)(9m - 6n\sqrt{m} + 2n^2). \end{aligned}$$

8.
$$\begin{aligned} 9c^2 + 121d^2 &= (9c^2 + 66cd + 121d^2) - (66cd) \\ &= (3c + 11d)^2 - (\sqrt{66cd})^2 \\ &= (3c + 11d + \sqrt{66cd})(3c + 11d - \sqrt{66cd}) \\ &= (3c + \sqrt{66cd} + 11d)(3c - \sqrt{66cd} + 11d). \end{aligned}$$

9. $169a^2 + 4b^2c^2 = (169a^2 + 52abc + 4b^2c^2) - (52abc)$
$= (13a + 2bc)^2 - (2\sqrt{13abc})^2$
$= (13a + 2bc + 2\sqrt{13abc})(13a + 2bc - 2\sqrt{13abc})$
$= (13a + 2\sqrt{13abc} + 2bc)(13a - 2\sqrt{13abc} + 2bc).$

10. $144x^2 + 49y^2z^2 = (144x^2 + 168xyz + 49y^2z^2) - (168xyz)$
$= (12x + 7yz)^2 - (2\sqrt{42xyz})^2$
$= (12x + 7yz + 2\sqrt{42xyz})(12x + 7yz - 2\sqrt{42xyz})$
$= (12x + 2\sqrt{42xyz} + 7yz)(12x - 2\sqrt{42xyz} + 7yz).$

Exercise 44. Page 98

Resolve into factors:

1. $x^2+11x+24 = (x+3)(x+8)$.
2. $x^2+5x-14 = (x+7)(x-2)$.
3. $a^2-3a-40 = (a+5)(a-8)$.
4. $c^2-12c+35 = (c-5)(c-7)$.
5. $y^2-4y-12 = (y+2)(y-6)$.
6. $x^2-10x+16 = (x-2)(x-8)$.
7. $a^2+10a+24 = (a+4)(a+6)$.
8. $a^2+2a-15 = (a+5)(a-3)$.
9. $a^2-3a-28 = (a+4)(a-7)$.
10. $x^2-10x+21 = (x-3)(x-7)$.
11. $x^2 - x - 30 = (x+5)(x-6)$.
12. $b^2+11b+30 = (b+5)(b+6)$.
13. $d^2+11d+28 = (d+4)(d+7)$.
14. $a^2-15a+54 = (a-6)(a-9)$.
15. $x^2+3x-54 = (x+9)(x-6)$.
16. $x^2-4x-77 = (x+7)(x-11)$.
17. $c^2+3c-40 = (c+8)(c-5)$.
18. $c^2-14c+45 = (c-5)(c-9)$.
19. $x^2+13x+40 = (x+5)(x+8)$.
20. $x^2-3x-54 = (x+6)(x-9)$.
21. $x^2-16x+63 = (x-7)(x-9)$.
22. $a^2+4a-60 = (a+10)(a-6)$.
23. $a^2 - 7a - 18 = (a+2)(a-9)$.
24. $x^2 + 15x + 44 = (x+4)(x+11)$.
25. $x^2 - 20xy + 96y^2 = (x-8y)(x-12y)$.
26. $x^2 + 8xy - 65y^2 = (x+13y)(x-5y)$.
27. $x^2 + 17xy + 66y^2 = (x+6y)(x+11y)$.
28. $c^2 - 4cd - 96d^2 = (c+8d)(c-12d)$.
29. $c^2 + 17cd + 72d^2 = (c+8d)(c+9d)$.
30. $x^2 - xy - 72y^2 = (x+8y)(x-9y)$.
31. $y^2 - 18yz + 72z^2 = (y-6z)(y-12z)$.
32. $m^2 + 6mn - 27n^2 = (m+9n)(m-3n)$.
33. $a^2 + 3ab - 154b^2 = (a+14b)(a-11b)$.
34. $x^2 - 20xy + 91y^2 = (x-7y)(x-13y)$.
35. $a^2 + 18ad + 56d^2 = (a+4d)(a+14d)$.
36. $x^2 - 4xz - 45z^2 = (x+5z)(x-9z)$.
37. $x^2 + 20xy + 75y^2 = (x+5y)(x+15y)$.

38. $x^2 - 38xy + 105y^2 = (x - 3y)(x - 35y).$
39. $m^2 + 10mn - 24n^2 = (m + 12n)(m - 2n).$
40. $c^2 - 8cd - 105d^2 = (c + 7d)(c - 15d).$
41. $a^2 + 13ab - 140b^2 = (a + 20b)(a - 7b).$
42. $a^2 + 22ac + 96c^2 = (a + 6c)(a + 16c).$
43. $x^2 - 13xyz - 90y^2z^2 = (x + 5yz)(x - 18yz).$
44. $a^2 - 23abc + 120b^2c^2 = (a - 8bc)(a - 15bc).$
45. $a^2 - 17abc + 72b^2c^2 = (a - 8bc)(a - 9bc).$
46. $m^2 + 5mnp - 84n^2p^2 = (m + 12np)(m - 7np).$
47. $x^2y^2 + 30xyz + 209z^2 = (xy + 11z)(xy + 19z).$
48. $a^2b^2 - abc - 110c^2 = (ab + 10c)(ab - 11c).$

Exercise 45. Page 99

1. Resolve into factors $6x^2 - 19x + 15$.

Multiply by 6, $(6x)^2 - 19(6x) + 90.$
Put z for $6x$, $z^2 - 19z + 90.$
Resolve into factors, $(z - 9)(z - 10).$
Put $6x$ for z, $(6x - 9)(6x - 10).$
Divide by 3×2, $(2x - 3)(3x - 5).$

$$\therefore 6x^2 - 19x + 15 = (2x - 3)(3x - 5).$$

2. Resolve into factors $6x^2 + 17x + 12$.

Multiply by 6, $(6x)^2 + 17(6x) + 72.$
Put z for $6x$, $z^2 + 17z + 72.$
Resolve into factors, $(z + 8)(z + 9).$
Put $6x$ for z, $(6x + 8)(6x + 9).$
Divide by 2×3, $(3x + 4)(2x + 3).$

$$\therefore 6x^2 + 17x + 12 = (3x + 4)(2x + 3).$$

3. Resolve into factors $12x^2 + 7x - 12$.

Multiply by 12, $(12x)^2 + 7(12x) - 144.$
Put z for $12x$, $z^2 + 7z - 144.$
Resolve into factors, $(z + 16)(z - 9).$
Put $12x$ for z, $(12x + 16)(12x - 9).$
Divide by 4×3, $(3x + 4)(4x - 3).$

$$\therefore 12x^2 + 7x - 12 = (3x + 4)(4x - 3).$$

4. Resolve into factors $15x^2 - 29x - 14$.

Multiply by 15, $(15x)^2 - 29(15x) - 210.$
Put z for $15x$, $z^2 - 29z - 210.$
Resolve into factors, $(z + 6)(z - 35).$
Put $15x$ for z, $(15x + 6)(15x - 35).$
Divide by 3×5, $(5x + 2)(3x - 7).$

$$\therefore\ 15x^2 - 29x - 14 = (5x + 2)(3x - 7).$$

5. Resolve into factors $12x^2 + 8x - 15$.

Multiply by 12, $(12x)^2 + 8(12x) - 180.$
Put z for $12x$, $z^2 + 8z - 180.$
Resolve into factors, $(z + 18)(z - 10).$
Put $12x$ for z, $(12x + 18)(12x - 10).$
Divide by 6×2, $(2x + 3)(6x - 5).$

$$\therefore\ 12x^2 + 8x - 15 = (2x + 3)(6x - 5).$$

6. Resolve into factors $20x^2 - 41x + 20$.

Multiply by 20, $(20x)^2 - 41(20x) + 400.$
Put z for $20x$, $z^2 - 41z + 400.$
Resolve into factors, $(z - 16)(z - 25).$
Put $20x$ for z, $(20x - 16)(20x - 25).$
Divide by 4×5, $(5x - 4)(4x - 5).$

$$\therefore\ 20x^2 - 41x + 20 = (5x - 4)(4x - 5).$$

7. Resolve into factors $6x^2 - 17x - 14$.

Multiply by 6, $(6x)^2 - 17(6x) - 84.$
Put z for $6x$, $z^2 - 17z - 84.$
Resolve into factors, $(z + 4)(z - 21).$
Put $6x$ for z, $(6x + 4)(6x - 21).$
Divide by 2×3, $(3x + 2)(2x - 7).$

$$\therefore\ 6x^2 - 17x - 14 = (3x + 2)(2x - 7).$$

8. Resolve into factors $6x^2 + 13x + 6$.

Multiply by 6, $(6x)^2 + 13(6x) + 36.$
Put z for $6x$, $z^2 + 13z + 36.$
Resolve into factors, $(z + 4)(z + 9).$
Put $6x$ for z, $(6x + 4)(6x + 9).$
Divide by 2×3, $(3x + 2)(2x + 3).$

$$\therefore\ 6x^2 + 13x + 6 = (3x + 2)(2x + 3).$$

9. Resolve into factors $8x^2 + 2x - 15$.

Multiply by 8, $(8x)^2 + 2(8x) - 120.$
Put z for $8x$, $z^2 + 2z - 120.$
Resolve into factors, $(z + 12)(z - 10).$
Put $8x$ for z, $(8x + 12)(8x - 10).$
Divide by 4×2, $(2x + 3)(4x - 5).$

$$\therefore 8x^2 + 2x - 15 = (2x + 3)(4x - 5).$$

10. Resolve into factors $14x^2 + 31xy - 10y^2$.

Multiply by 14, $(14x)^2 + 31y(14x) - 140y^2.$
Put z for $14x$, $z^2 + 31yz - 140y^2.$
Resolve into factors, $(z + 35y)(z - 4y).$
Put $14x$ for z, $(14x + 35y)(14x - 4y).$
Divide by 7×2, $(2x + 5y)(7x - 2y).$

$$\therefore 14x^2 + 31xy - 10y^2 = (2x + 5y)(7x - 2y).$$

11. Resolve into factors $6x^2 + 11xy - 35y^2$.

Multiply by 6, $(6x)^2 + 11y(6x) - 210y^2.$
Put z for $6x$, $z^2 + 11yz - 210y^2.$
Resolve into factors, $(z + 21y)(z - 10y).$
Put $6x$ for z, $(6x + 21y)(6x - 10y).$
Divide by 3×2, $(2x + 7y)(3x - 5y).$

$$\therefore 6x^2 + 11xy - 35y^2 = (2x + 7y)(3x - 5y).$$

12. Resolve into factors $21x^2 + 26xy - 15y^2$.

Multiply by 21, $(21x)^2 + 26y(21x) - 315y^2.$
Put z for $21x$, $z^2 + 26yz - 315y^2.$
Resolve into factors, $(z + 35y)(z - 9y).$
Put $21x$ for z, $(21x + 35y)(21x - 9y).$
Divide by 7×3, $(3x + 5y)(7x - 3y).$

$$\therefore 21x^2 + 26xy - 15y^2 = (3x + 5y)(7x - 3y).$$

13. Resolve into factors $42x^2 - 59xy + 20y^2$.

Multiply by 42, $(42x)^2 - 59y(42x) + 840y^2.$
Put z for $42x$, $z^2 - 59yz + 840y^2.$
Resolve into factors, $(z - 24y)(z - 35y).$
Put $42x$ for z, $(42x - 24y)(42x - 35y).$
Divide by 6×7, $(7x - 4y)(6x - 5y).$

$$\therefore 42x^2 - 59xy + 20y^2 = (7x - 4y)(6x - 5y).$$

14. Resolve into factors $21x^2 - 37xy + 12y^2$.

Multiply by 21, $(21x)^2 - 37y(21x) + 252y^2$.
Put z for $21x$, $z^2 - 37yz + 252y^2$.
Resolve into factors, $(z - 9y)(z - 28y)$.
Put $21x$ for z, $(21x - 9y)(21x - 28y)$.
Divide by 3×7, $(7x - 3y)(3x - 4y)$.

$$\therefore 21x^2 - 37xy + 12y^2 = (7x - 3y)(3x - 4y).$$

15. Resolve into factors $14x^2 + 53xy + 14y^2$.

Multiply by 14, $(14x)^2 + 53y(14x) + 196y^2$.
Put z for $14x$, $z^2 + 53yz + 196y^2$.
Resolve into factors, $(z + 4y)(z + 49y)$.
Put $14x$ for z, $(14x + 4y)(14x + 49y)$.
Divide by 2×7, $(7x + 2y)(2x + 7y)$.

$$\therefore 14x^2 + 53xy + 14y^2 = (7x + 2y)(2x + 7y).$$

16. Resolve into factors $21x^2 - 40xy + 16y^2$.

Multiply by 21, $(21x)^2 - 40y(21x) + 336y^2$.
Put z for $21x$, $z^2 - 40yz + 336y^2$.
Resolve into factors, $(z - 12y)(z - 28y)$.
Put $21x$ for z, $(21x - 12y)(21x - 28y)$.
Divide by 3×7, $(7x - 4y)(3x - 4y)$.

$$\therefore 21x^2 - 40xy + 16y^2 = (7x - 4y)(3x - 4y).$$

17. Resolve into factors $30x^2 + 43xy + 15y^2$.

Multiply by 30, $(30x)^2 + 43y(30x) + 450y^2$.
Put z for $30x$, $z^2 + 43yz + 450y^2$.
Resolve into factors, $(z + 18y)(z + 25y)$.
Put $30x$ for z, $(30x + 18y)(30x + 25y)$.
Divide by 6×5, $(5x + 3y)(6x + 5y)$.

$$\therefore 30x^2 + 43xy + 15y^2 = (5x + 3y)(6x + 5y).$$

18. Resolve into factors $21x^2 + 41xy + 10y^2$.

Multiply by 21, $(21x)^2 + 41y(21x) + 210y^2$.
Put z for $21x$, $z^2 + 41yz + 210y^2$.
Resolve into factors, $(z + 6y)(z + 35y)$.
Put $21x$ for z, $(21x + 6y)(21x + 35y)$.
Divide by 3×7, $(7x + 2y)(3x + 5y)$.

$$\therefore 21x^2 + 41xy + 10y^2 = (7x + 2y)(3x + 5y).$$

19. Resolve into factors $21x^2 + 29xy - 10y^2$.

Multiply by 21, $(21x)^2 + 29y(21x) - 210y^2$.
Put z for $21x$, $z^2 + 29yz - 210y^2$.
Resolve into factors, $(z + 35y)(z - 6y)$.
Put $21x$ for z, $(21x + 35y)(21x - 6y)$.
Divide by 7×3, $(3x + 5y)(7x - 2y)$.

$$\therefore\ 21x^2 + 29xy - 10y^2 = (3x + 5y)(7x - 2y).$$

20. Resolve into factors $10x^2 + 19xy - 15y^2$.

Multiply by 10, $(10x)^2 + 19y(10x) - 150y^2$.
Put z for $10x$, $z^2 + 19yz - 150y^2$.
Resolve into factors, $(z + 25y)(z - 6y)$.
Put $10x$ for z, $(10x + 25y)(10x - 6y)$.
Divide by 5×2, $(2x + 5y)(5x - 3y)$.

$$\therefore\ 10x^2 + 19xy - 15y^2 = (2x + 5y)(5x - 3y).$$

21. Resolve into factors $6a^2 + 11ab - 35b^2$.

Multiply by 6, $(6a)^2 + 11b(6a) - 210b^2$.
Put z for $6a$, $z^2 + 11bz - 210b^2$.
Resolve into factors, $(z + 21b)(z - 10b)$.
Put $6a$ for z, $(6a + 21b)(6a - 10b)$.
Divide by 3×2, $(2a + 7b)(3a - 5b)$.

$$\therefore\ 6a^2 + 11ab - 35b^2 = (2a + 7b)(3a - 5b).$$

22. Resolve into factors $12c^2 - 11cd - 15d^2$.

Multiply by 12, $(12c)^2 - 11d(12c) - 180d^2$.
Put z for $12c$, $z^2 - 11dz - 180d^2$.
Resolve into factors, $(z + 9d)(z - 20d)$.
Put $12c$ for z, $(12c + 9d)(12c - 20d)$.
Divide by 3×4, $(4c + 3d)(3c - 5d)$.

$$\therefore\ 12c^2 - 11cd - 15d^2 = (4c + 3d)(3c - 5d).$$

23. Resolve into factors $12m^2 + mn - 20n^2$.

Multiply by 12, $(12m)^2 + n(12m) - 240n^2$.
Put z for $12m$, $z^2 + nz - 240n^2$.
Resolve into factors, $(z + 16n)(z - 15n)$.
Put $12m$ for z, $(12m + 16n)(12m - 15n)$.
Divide by 4×3, $(3m + 4n)(4m - 5n)$.

$$\therefore\ 12m^2 + mn - 20n^2 = (3m + 4n)(4m - 5n).$$

24. Resolve into factors $12y^2 + 25yz + 12z^2$.

Multiply by 12, $(12y)^2 + 25z(12y) + 144z^2$.

Put x for $12y$, $x^2 + 25xz + 144z^2$.

Resolve into factors, $(x + 9z)(x + 16z)$.

Put $12y$ for x, $(12y + 9z)(12y + 16z)$.

Divide by 3×4, $(4y + 3z)(3y + 4z)$.

$$\therefore\ 12y^2 + 25yz + 12z^2 = (4y + 3z)(3y + 4z).$$

25. Resolve into factors $15a^2 - 14ab - 8b^2$.

Multiply by 15, $(15a)^2 - 14b(15a) - 120b^2$.

Put z for $15a$, $z^2 - 14bz - 120b^2$.

Resolve into factors, $(z + 6b)(z - 20b)$.

Put $15a$ for z, $(15a + 6b)(15a - 20b)$.

Divide by 3×5, $(5a + 2b)(3a - 4b)$.

$$\therefore\ 15a^2 - 14ab - 8b^2 = (5a + 2b)(3a - 4b).$$

26. Resolve into factors $15a^2 - 16ab + 4b^2$.

Multiply by 15, $(15a)^2 - 16b(15a) + 60b^2$.

Put z for $15a$, $z^2 - 16bz + 60b^2$.

Resolve into factors, $(z - 6b)(z - 10b)$.

Put $15a$ for z, $(15a - 6b)(15a - 10b)$.

Divide by 3×5, $(5a - 2b)(3a - 2b)$.

$$\therefore\ 15a^2 - 16ab + 4b^2 = (5a - 2b)(3a - 2b).$$

27. Resolve into factors $30c^2 - cd - 20d^2$.

Multiply by 30, $(30c)^2 - d(30c) - 600d^2$.

Put z for $30c$, $z^2 - dz - 600d^2$.

Resolve into factors, $(z + 24d)(z - 25d)$.

Put $30c$ for z, $(30c + 24d)(30c - 25d)$.

Divide by 6×5, $(5c + 4d)(6c - 5d)$.

$$\therefore\ 30c^2 - cd - 20d^2 = (5c + 4d)(6c - 5d).$$

28. Resolve into factors $30c^2 - 11cd - 30d^2$.

Multiply by 30, $(30c)^2 - 11d(30c) - 900d^2$.

Put z for $30c$, $z^2 - 11dz - 900d^2$.

Resolve into factors, $(z + 25d)(z - 36d)$.

Put $30c$ for z, $(30c + 25d)(30c - 36d)$.

Divide by 5×6, $(6c + 5d)(5c - 6d)$.

$$\therefore\ 30c^2 - 11cd - 30d^2 = (6c + 5d)(5c - 6d).$$

29. Resolve into factors $10a^2 - 51ab + 56b^2$.

Multiply by 10, $(10a)^2 - 51b(10a) + 560b^2$.

Put z for $10a$, $z^2 - 51bz + 560b^2$.

Resolve into factors, $(z - 16b)(z - 35b)$.

Put $10a$ for z, $(10a - 16b)(10a - 35b)$.

Divide by 2×5, $(5a - 8b)(2a - 7b)$.

$$\therefore\ 10a^2 - 51ab + 56b^2 = (5a - 8b)(2a - 7b).$$

30. Resolve into factors $15b^2 + 2bc - 45c^2$.

Multiply by 15, $(15b)^2 + 2c(15b) - 675c^2$.

Put z for $15b$, $z^2 + 2cz - 675c^2$.

Resolve into factors, $(z + 27c)(z - 25c)$.

Put $15b$ for z, $(15b + 27c)(15b - 25c)$.

Divide by 3×5, $(5b + 9c)(3b - 5c)$.

$$\therefore\ 15b^2 + 2bc - 45c^2 = (5b + 9c)(3b - 5c).$$

31. Resolve into factors $12a^2 - 25ab - 50b^2$.

Multiply by 12, $(12a)^2 - 25b(12a) - 600b^2$.

Put z for $12a$, $z^2 - 25bz - 600b^2$.

Resolve into factors, $(z - 40b)(z + 15b)$.

Put $12a$ for z, $(12a - 40b)(12a + 15b)$.

Divide by 4×3, $(3a - 10b)(4a + 5b)$.

$$\therefore\ 12a^2 - 25ab - 50b^2 = (3a - 10b)(4a + 5b).$$

32. Resolve into factors $6m^2 + 11mn - 72n^2$.

Multiply by 6, $(6m)^2 + 11n(6m) - 432n^2$.

Put z for $6m$, $z^2 + 11nz - 432n^2$.

Resolve into factors, $(z + 27n)(z - 16n)$.

Put $6m$ for z, $(6m + 27n)(6m - 16n)$.

Divide by 3×2, $(2m + 9n)(3m - 8n)$.

$$\therefore\ 6m^2 + 11mn - 72n^2 = (2m + 9n)(3m - 8n).$$

33. Resolve into factors $15x^2 + 44xy - 20y^2$.

Multiply by 15, $(15x)^2 + 44y(15x) - 300y^2$.

Put z for $15x$, $z^2 + 44yz - 300y^2$.

Resolve into factors, $(z + 50y)(z - 6y)$.

Put $15x$ for z, $(15x + 50y)(15x - 6y)$.

Divide by 5×3, $(3x + 10y)(5x - 2y)$.

$$\therefore\ 15x^2 + 44xy - 20y^2 = (3x + 10y)(5x - 2y).$$

34. Resolve into factors $20x^2 - xy - 99y^2$.

Multiply by 20, $(20x)^2 - y(20x) - 1980y^2$.
Put z for $20x$, $z^2 - yz - 1980y^2$.
Resolve into factors, $(z + 44y)(z - 45y)$.
Put $20x$ for z, $(20x + 44y)(20x - 45y)$.
Divide by 4×5, $(5x + 11y)(4x - 9y)$.

$$\therefore 20x^2 - xy - 99y^2 = (5x + 11y)(4x - 9y).$$

Exercise 46. Page 101

1. Resolve into factors $6x^2 - 19x + 15$.

Divide by 6, $x^2 - \frac{19}{6}x + \frac{5}{2}$.
Complete the square, $(x^2 - \frac{19}{6}x + \frac{361}{144}) + (\frac{5}{2} - \frac{361}{144})$.
Simplify, $(x - \frac{19}{12})^2 - (\frac{1}{12})^2$.
Resolve into factors, $(x - \frac{19}{12} + \frac{1}{12})(x - \frac{19}{12} - \frac{1}{12})$.
Simplify, $(x - \frac{3}{2})(x - \frac{5}{3})$.
Multiply by 2×3, $(2x - 3)(3x - 5)$.

$$\therefore 6x^2 - 19x + 15 = (2x - 3)(3x - 5).$$

2. Resolve into factors $6x^2 + 17x + 12$.

Divide by 6, $x^2 + \frac{17}{6}x + 2$.
Complete the square, $(x^2 + \frac{17}{6}x + \frac{289}{144}) - (\frac{289}{144} - 2)$.
Simplify, $(x + \frac{17}{12})^2 - (\frac{1}{12})^2$.
Resolve into factors, $(x + \frac{17}{12} + \frac{1}{12})(x + \frac{17}{12} - \frac{1}{12})$.
Simplify, $(x + \frac{3}{2})(x + \frac{4}{3})$.
Multiply by 2×3, $(2x + 3)(3x + 4)$.

$$\therefore 6x^2 + 17x + 12 = (2x + 3)(3x + 4).$$

3. Resolve into factors $12x^2 + 7x - 12$.

Divide by 12, $x^2 + \frac{7}{12}x - 1$.
Complete the square, $(x^2 + \frac{7}{12}x + \frac{49}{576}) - (\frac{49}{576} + 1)$.
Simplify, $(x + \frac{7}{24})^2 - (\frac{25}{24})^2$.
Resolve into factors, $(x + \frac{7}{24} + \frac{25}{24})(x + \frac{7}{24} - \frac{25}{24})$.
Simplify, $(x + \frac{4}{3})(x - \frac{3}{4})$.
Multiply by 3×4, $(3x + 4)(4x - 3)$.

$$\therefore 12x^2 + 7x - 12 = (3x + 4)(4x - 3).$$

4. Resolve into factors $12x^2 + 8x - 15$.

Divide by 12, $x^2 + \frac{2}{3}x - \frac{5}{4}$.

Complete the square, $(x^2 + \frac{2}{3}x + \frac{1}{9}) - (\frac{1}{9} + \frac{5}{4})$.

Simplify, $(x + \frac{1}{3})^2 - (\frac{7}{6})^2$.

Resolve into factors, $(x + \frac{1}{3} + \frac{7}{6})(x + \frac{1}{3} - \frac{7}{6})$.

Simplify, $(x + \frac{3}{2})(x - \frac{5}{6})$.

Multiply by 2×6, $(2x + 3)(6x - 5)$.

$$\therefore 12x^2 + 8x - 15 = (2x + 3)(6x - 5).$$

5. Resolve into factors $20x^2 - 41x + 20$.

Divide by 20, $x^2 - \frac{41}{20}x + 1$.

Complete the square, $(x^2 - \frac{41}{20}x + \frac{1681}{1600}) + (1 - \frac{1681}{1600})$.

Simplify, $(x - \frac{41}{40})^2 - (\frac{9}{40})^2$.

Resolve into factors, $(x - \frac{41}{40} + \frac{9}{40})(x - \frac{41}{40} - \frac{9}{40})$.

Simplify, $(x - \frac{4}{5})(x - \frac{5}{4})$.

Multiply by 5×4, $(5x - 4)(4x - 5)$.

$$\therefore 20x^2 - 41x + 20 = (5x - 4)(4x - 5).$$

6. Resolve into factors $6x^2 + 13x + 6$.

Divide by 6, $x^2 + \frac{13}{6}x + 1$.

Complete the square, $(x^2 + \frac{13}{6}x + \frac{169}{144}) + (1 - \frac{169}{144})$.

Simplify, $(x + \frac{13}{12})^2 - (\frac{5}{12})^2$.

Resolve into factors, $(x + \frac{13}{12} + \frac{5}{12})(x + \frac{13}{12} - \frac{5}{12})$.

Simplify, $(x + \frac{3}{2})(x + \frac{2}{3})$.

Multiply by 2×3, $(2x + 3)(3x + 2)$.

$$\therefore 6x^2 + 13x + 6 = (2x + 3)(3x + 2).$$

7. Resolve into factors $8x^2 + 2x - 15$.

Divide by 8, $x^2 + \frac{1}{4}x - \frac{15}{8}$.

Complete the square, $(x^2 + \frac{1}{4}x + \frac{1}{64}) - (\frac{15}{8} + \frac{1}{64})$.

Simplify, $(x + \frac{1}{8})^2 - (\frac{11}{8})^2$.

Resolve into factors, $(x + \frac{1}{8} + \frac{11}{8})(x + \frac{1}{8} - \frac{11}{8})$.

Simplify, $(x + \frac{3}{2})(x - \frac{5}{4})$.

Multiply by 2×4, $(2x + 3)(4x - 5)$.

$$\therefore 8x^2 + 2x - 15 = (2x + 3)(4x - 5).$$

8. Resolve into factors $6x^2 + 11xy - 35y^2$.

Divide by 6, $x^2 + \frac{11}{6}xy - \frac{35}{6}y^2$.
Complete the square, $(x + \frac{11}{6}xy + \frac{121}{144}y^2) - (\frac{35}{6}y^2 + \frac{121}{144}y^2)$.
Simplify, $(x + \frac{11}{12}y)^2 - (\frac{31}{12}y)^2$.
Resolve into factors, $(x + \frac{11}{12}y + \frac{31}{12}y)(x + \frac{11}{12}y - \frac{31}{12}y)$.
Simplify, $(x + \frac{7}{2}y)(x - \frac{5}{3}y)$.
Multiply by 2 × 3, $(2x + 7y)(3x - 5y)$.

$$\therefore\ 6x^2 + 11xy - 35y^2 = (2x + 7y)(3x - 5y).$$

9. Resolve into factors $30x^2 + 43xy + 15y^2$.

Divide by 30, $x^2 + \frac{43}{30}xy + \frac{1}{2}y^2$.
Complete the square, $(x^2 + \frac{43}{30}xy + \frac{1849}{3600}y^2) + (\frac{1}{2}y^2 - \frac{1849}{3600}y^2)$.
Simplify, $(x + \frac{43}{60}y)^2 - (\frac{7}{60}y)^2$.
Resolve into factors, $(x + \frac{43}{60}y + \frac{7}{60}y)(x + \frac{43}{60}y - \frac{7}{60}y)$.
Simplify, $(x + \frac{5}{6}y)(x + \frac{3}{5}y)$.
Multiply by 6 × 5, $(6x + 5y)(5x + 3y)$.

$$\therefore\ 30x^2 + 43xy + 15y^2 = (6x + 5y)(5x + 3y).$$

10. Resolve into factors $10x^2 + 19xy - 15y^2$.

Divide by 10, $x^2 + \frac{19}{10}xy - \frac{3}{2}y^2$.
Complete the square, $(x^2 + \frac{19}{10}xy + \frac{361}{400}y^2) - (\frac{3}{2}y^2 + \frac{361}{400}y^2)$.
Simplify, $(x + \frac{19}{20}y)^2 - (\frac{31}{20}y)^2$.
Resolve into factors, $(x + \frac{19}{20}y + \frac{31}{20}y)(x + \frac{19}{20}y - \frac{31}{20}y)$.
Simplify, $(x + \frac{5}{2}y)(x - \frac{3}{5}y)$.
Multiply by 2 × 5, $(2x + 5y)(5x - 3y)$.

$$\therefore\ 10x^2 + 19xy - 15y^2 = (2x + 5y)(5x - 3y).$$

11. Resolve into factors $12c^2 - 11cd - 15d^2$.

Divide by 12, $c^2 - \frac{11}{12}cd - \frac{5}{4}d^2$.
Complete the square, $(c^2 - \frac{11}{12}cd + \frac{121}{576}d^2) - (\frac{5}{4}d^2 + \frac{121}{576}d^2)$.
Simplify, $(c - \frac{11}{24}d)^2 - (\frac{29}{24}d)^2$.
Resolve into factors, $(c - \frac{11}{24}d + \frac{29}{24}d)(c - \frac{11}{24}d - \frac{29}{24}d)$.
Simplify, $(c + \frac{3}{4}d)(c - \frac{5}{3}d)$.
Multiply by 4 × 3, $(4c + 3d)(3c - 5d)$.

$$\therefore\ 12c^2 - 11cd - 15d^2 = (4c + 3d)(3c - 5d).$$

12. Resolve into factors $12m^2 + mn - 20n^2$.

Divide by 12, $m^2 + \frac{1}{12}mn - \frac{5}{3}n^2$.
Complete the square, $(m^2 + \frac{1}{12}mn + \frac{1}{576}n^2) - (\frac{5}{3}n^2 + \frac{1}{576}n^2)$.
Simplify, $(m + \frac{1}{24}n)^2 - (\frac{31}{24}n)^2$.
Resolve into factors, $(m + \frac{1}{24}n + \frac{31}{24}n)(m + \frac{1}{24}n - \frac{31}{24}n)$.
Simplify, $(m + \frac{4}{3}n)(m - \frac{5}{4}n)$.
Multiply by 3 × 4, $(3m + 4n)(4m - 5n)$.

$$\therefore\ 12m^2 + mn - 20n^2 = (3m + 4n)(4m - 5n).$$

13. Resolve into factors $12y^2 + 25yz + 12z^2$.

Divide by 12, $y^2 + \frac{25}{12}yz + z^2$.
Complete the square, $(y^2 + \frac{25}{12}yz + \frac{625}{576}z^2) + (z^2 - \frac{625}{576}z^2)$.
Simplify, $(y + \frac{25}{24}z)^2 - (\frac{7}{24}z)^2$.
Resolve into factors, $(y + \frac{25}{24}z + \frac{7}{24}z)(y + \frac{25}{24}z - \frac{7}{24}z)$.
Simplify, $(y + \frac{4}{3}z)(y + \frac{3}{4}z)$.
Multiply by 3 × 4, $(3y + 4z)(4y + 3z)$.

$$\therefore\ 12y^2 + 25yz + 12z^2 = (3y + 4z)(4y + 3z).$$

14. Resolve into factors $15a^2 - 16ab + 4b^2$.

Divide by 15, $a^2 - \frac{16}{15}ab + \frac{4}{15}b^2$.
Complete the square, $(a^2 - \frac{16}{15}ab + \frac{64}{225}b^2) + (\frac{4}{15}b^2 - \frac{64}{225}b^2)$.
Simplify, $(a - \frac{8}{15}b) - (\frac{2}{15}b)^2$.
Resolve into factors, $(a - \frac{8}{15}b + \frac{2}{15}b)(a - \frac{8}{15}b - \frac{2}{15}b)$.
Simplify, $(a - \frac{2}{5}b)(a - \frac{2}{3}b)$.
Multiply by 5 × 3, $(5a - 2b)(3a - 2b)$.

$$\therefore\ 15a^2 - 16ab + 4b^2 = (5a - 2b)(3a - 2b).$$

15. Resolve into factors $30c^2 - cd - 20d^2$.

Divide by 30, $c^2 - \frac{1}{30}cd - \frac{2}{3}d^2$.
Complete the square, $(c^2 - \frac{1}{30}cd + \frac{1}{3600}d^2) - (\frac{2}{3}d^2 + \frac{1}{3600}d^2)$.
Simplify, $(c - \frac{1}{60}d)^2 - (\frac{49}{60}d)^2$.
Resolve into factors, $(c - \frac{1}{60}d + \frac{49}{60}d)(c - \frac{1}{60}d - \frac{49}{60}d)$.
Simplify, $(c + \frac{4}{5}d)(c - \frac{5}{6}d)$.
Multiply by 5 × 6, $(5c + 4d)(6c - 5d)$.

$$\therefore\ 30c^2 - cd - 20d^2 = (5c + 4d)(6c - 5d).$$

16. Resolve into factors $30c^2 - 11cd - 30d^2$.

Divide by 30, $c^2 - \frac{11}{30}cd - d^2$.

Complete the square, $(c^2 - \frac{11}{30}cd + \frac{121}{3600}d^2) - (d^2 + \frac{121}{3600}d^2)$.

Simplify, $(c - \frac{11}{60}d)^2 - (\frac{61}{60}d)^2$.

Resolve into factors, $(c - \frac{11}{60}d + \frac{61}{60}d)(c - \frac{11}{60}d - \frac{61}{60}d)$.

Simplify, $(c + \frac{5}{6}d)(c - \frac{6}{5}d)$.

Multiply by 6 × 5, $(6c + 5d)(5c - 6d)$.

$$\therefore\ 30c^2 - 11cd - 30d^2 = (6c + 5d)(5c - 6d).$$

17. Resolve into factors $10a^2 - 51ab + 56b^2$.

Divide by 10, $a^2 - \frac{51}{10}ab + \frac{28}{5}b^2$.

Complete the square, $(a^2 - \frac{51}{10}ab + \frac{2601}{400}b^2) + (\frac{28}{5}b^2 - \frac{2601}{400}b^2)$.

Simplify, $(a - \frac{51}{20}b) - (\frac{19}{20}b)^2$.

Resolve into factors, $(a - \frac{51}{20}b + \frac{19}{20}b)(a - \frac{51}{20}b - \frac{19}{20}b)$.

Simplify, $(a - \frac{8}{5}b)(a - \frac{7}{2}b)$.

Multiply by 5 × 2, $(5a - 8b)(2a - 7b)$.

$$\therefore\ 10a^2 - 51ab + 56b^2 = (5a - 8b)(2a - 7b).$$

18. Resolve into factors $15x^2 + 44xy - 20y^2$.

Divide by 15, $x^2 + \frac{44}{15}xy - \frac{4}{3}y^2$.

Complete the square, $(x^2 + \frac{44}{15}xy + \frac{484}{225}y^2) - (\frac{4}{3}y^2 + \frac{484}{225}y^2)$.

Simplify, $(x + \frac{22}{15}y)^2 - (\frac{28}{15}y)^2$.

Resolve into factors, $(x + \frac{22}{15}y + \frac{28}{15}y)(x + \frac{22}{15}y - \frac{28}{15}y)$.

Simplify, $(x + \frac{10}{3}y)(x - \frac{2}{5}y)$.

Multiply by 3 × 5, $(3x + 10y)(5x - 2y)$.

$$\therefore\ 15x^2 + 44xy - 20y^2 = (3x + 10y)(5x - 2y).$$

Exercise 47. Page 102

Resolve into factors:

1. $$27x^3 + 343y^3 = (3x + 7y)[(3x)^2 - (3x)(7y) + (7y)^2]$$
$$= (3x + 7y)(9x^2 - 21xy + 49y^2).$$

2. $$216a^3 - 1331b^3 = (6a - 11b)[(6a)^2 + (6a)(11b) + (11b)^2]$$
$$= (6a - 11b)(36a^2 + 66ab + 121b^2).$$

3. $$125a^3b^3 + 216c^3d^3 = (5ab + 6cd)[(5ab)^2 - (5ab)(6cd) + (6cd)^2]$$
$$= (5ab + 6cd)(25a^2b^2 - 30abcd + 36c^2d^2).$$

4. $$\begin{aligned} 729x^3y^3 + 512a^3b^3 &= (9xy + 8ab)[(9xy)^2 - (9xy)(8ab) + (8ab)^2] \\ &= (9xy + 8ab)(81x^2y^2 - 72abxy + 64a^2b^2). \end{aligned}$$

5. $$\begin{aligned} 125x^3 - 343y^3z^3 &= (5x - 7yz)[(5x)^2 + (5x)(7yz) + (7yz)^2] \\ &= (5x - 7yz)(25x^2 + 35xyz + 49y^2z^2). \end{aligned}$$

6. $$\begin{aligned} 729a^3 - 343b^3 &= (9a - 7b)[(9a)^2 + (9a)(7b) + (7b)^2] \\ &= (9a - 7b)(81a^2 + 63ab + 49b^2). \end{aligned}$$

7. $$\begin{aligned} 8x^6 + 27y^9 &= (2x^2 + 3y^3)[(2x^2)^2 - (2x^2)(3y^3) + (3y^3)^2] \\ &= (2x^2 + 3y^3)(4x^4 - 6x^2y^3 + 9y^6). \end{aligned}$$

8. $$\begin{aligned} 27x^6 - 64y^6z^9 &= (3x^2 - 4y^2z^3)[(3x^2)^2 + (3x^2)(4y^2z^3) + (4y^2z^3)^2] \\ &= (3x^2 - 4y^2z^3)(9x^4 + 12x^2y^2z^3 + 16y^4z^6). \end{aligned}$$

9. $$\begin{aligned} 1000x^3 - 1331y^3 &= (10x - 11y)[(10x)^2 + (10x)(11y) + (11y)^2] \\ &= (10x - 11y)(100x^2 + 110xy + 121y^2). \end{aligned}$$

10. $$\begin{aligned} 1331m^3 + 1728n^3 &= (11m + 12n)[(11m)^2 - (11m)(12n) + (12n)^2] \\ &= (11m + 12n)(121m^2 - 132mn + 144n^2). \end{aligned}$$

11. $$27 + a^3b^6c^9 = (3 + ab^2c^3)(9 - 3ab^2c^3 + a^2b^4c^6).$$

12. $$\begin{aligned} 216x^3y^3z^3 - 125 &= (6xyz - 5)[(6xyz)^2 + (6xyz)(5) + 5^2] \\ &= (6xyz - 5)(36x^2y^2z^2 + 30xyz + 25). \end{aligned}$$

13. $$\begin{aligned} (a + 2b)^3 - 1 &= (a + 2b - 1)[(a + 2b)^2 + (a + 2b) + 1] \\ &= (a + 2b - 1)(a^2 + 4ab + 4b^2 + a + 2b + 1). \end{aligned}$$

14. $$\begin{aligned} (a - 2b)^3 + 1 &= (a - 2b + 1)[(a - 2b)^2 - (a - 2b) + 1] \\ &= (a - 2b + 1)(a^2 - 4ab + 4b^2 - a + 2b + 1). \end{aligned}$$

15. $$\begin{aligned} 1 - (x - y)^3 &= [1 - (x - y)][1 + (x - y) + (x - y)^2] \\ &= (1 - x + y)(1 + x - y + x^2 - 2xy + y^2). \end{aligned}$$

16. $$\begin{aligned} x^6 + y^6 &= (x^2 + y^2)[(x^2)^2 - (x^2)(y^2) + (y^2)^2] \\ &= (x^2 + y^2)(x^4 - x^2y^2 + y^4). \end{aligned}$$

17. $$\begin{aligned} a^9 - y^9 &= (a^3 - y^3)[(a^3)^2 + (a^3)(y^3) + (y^3)^2] \\ &= (a^3 - y^3)(a^6 + a^3y^3 + y^6) \\ &= (a - y)(a^2 + ay + y^2)(a^6 + a^3y^3 + y^6). \end{aligned}$$

18. $$\begin{aligned} 27a^3 - (a - b)^3 &= [3a - (a - b)][(3a)^2 + 3a(a - b) + (a - b)^2] \\ &= (3a - a + b)(9a^2 + 3a^2 - 3ab + a^2 - 2ab + b^2) \\ &= (2a + b)(13a^2 - 5ab + b^2). \end{aligned}$$

19.
$$\begin{aligned}64a^3 + (a+b)^3 &= [4a + (a+b)][(4a)^2 - (4a)(a+b) + (a+b)^2]\\ &= (4a + a + b)(16a^2 - 4a^2 - 4ab + a^2 + 2ab + b^2)\\ &= (5a + b)(13a^2 - 2ab + b^2).\end{aligned}$$

20.
$$\begin{aligned}(x^2 - 5)^3 - y^6 &= [(x^2 - 5) - y^2][(x^2 - 5)^2 + (x^2 - 5)(y^2) + (y^2)^2]\\ &= (x^2 - 5 - y^2)(x^4 - 10x^2 + 25 + x^2y^2 - 5y^2 + y^4).\end{aligned}$$

21.
$$\begin{aligned}(x^3 + 4)^3 + y^3 &= [(x^3 + 4) + y][(x^3 + 4)^2 - (x^3 + 4)(y) + y^2]\\ &= (x^3 + 4 + y)(x^6 + 8x^3 + 16 - x^3y - 4y + y^2).\end{aligned}$$

22.
$$\begin{aligned}&(a + b)^3 - (a - b)^3\\ &= [(a + b) - (a - b)][(a + b)^2 + (a + b)(a - b) + (a - b)^2]\\ &= (a + b - a + b)(a^2 + 2ab + b^2 + a^2 - b^2 + a^2 - 2ab + b^2)\\ &= 2b(3a^2 + b^2).\end{aligned}$$

23.
$$\begin{aligned}&64(x + y)^3 + 27(x - y)^3\\ &= [4(x + y) + 3(x - y)][\{4(x + y)\}^2 - \{4(x + y)\}\{3(x - y)\}\\ &\quad + \{3(x - y)\}^2]\\ &= (4x + 4y + 3x - 3y)(16x^2 + 32xy + 16y^2 - 12x^2 + 12y^2\\ &\quad + 9x^2 - 18xy + 9y^2)\\ &= (7x + y)(13x^2 + 14xy + 37y^2).\end{aligned}$$

24.
$$\begin{aligned}&(5a - 2b)^3 + 8c^3\\ &= [(5a - 2b) + 2c][(5a - 2b)^2 - (5a - 2b)(2c) + (2c)^2]\\ &= (5a - 2b + 2c)(25a^2 - 20ab + 4b^2 - 10ac + 4bc + 4c^2).\end{aligned}$$

25.
$$\begin{aligned}&x^{21}y^{18} + a^{15}b^{12}\\ &= (x^7y^6 + a^5b^4)[(x^7y^6)^2 - (x^7y^6)(a^5b^4) + (a^5b^4)^2]\\ &= (x^7y^6 + a^5b^4)(x^{14}y^{12} - a^5b^4x^7y^6 + a^{10}b^8).\end{aligned}$$

26.
$$\begin{aligned}&(2a - 3b)^3 - (x - 4y)^3\\ &= [(2a - 3b) - (x - 4y)][(2a - 3b)^2 + (2a - 3b)(x - 4y)\\ &\quad + (x - 4y)^2]\\ &= (2a - 3b - x + 4y)(4a^2 - 12ab + 9b^2 + 2ax - 8ay - 3bx\\ &\quad + 12by + x^2 - 8xy + 16y^2).\end{aligned}$$

27.
$$\begin{aligned}x^3 - (y - 2z)^3 &= [x - (y - 2z)][x^2 + x(y - 2z) + (y - 2z)^2]\\ &= (x - y + 2z)(x^2 + xy - 2xz + y^2 - 4yz + 4z^2).\end{aligned}$$

28.
$$\begin{aligned}&(x + y)^3 + (x - y)^3\\ &= [(x + y) + (x - y)][(x + y)^2 - (x + y)(x - y) + (x - y)^2]\\ &= (x + y + x - y)(x^2 + 2xy + y^2 - x^2 + y^2 + x^2 - 2xy + y^2)\\ &= 2x(x^2 + 3y^2).\end{aligned}$$

Exercise 48. Page 104

1. Resolve into factors $x^3 + 4x^2 + x - 6$.

If $x = +1$, $x^3 + 4x^2 + x - 6 = 1 + 4 + 1 - 6 = 0$.

Hence, $x - 1$ is a factor.

$\therefore x^3 + 4x^2 + x - 6 = (x - 1)(x^2 + 5x + 6) = (x - 1)(x + 2)(x + 3)$.

2. Resolve into factors $a^3 - 7a^2 + 16a - 12$.

If $a = +1$, $a^3 - 7a^2 + 16a - 12 = 1 - 7 + 16 - 12 = -2$.
If $a = -1$, $a^3 - 7a^2 + 16a - 12 = -1 - 7 - 16 - 12 = -36$.
If $a = +2$, $a^3 - 7a^2 + 16a - 12 = 8 - 28 + 32 - 12 = 0$.

Hence, $a - 2$ is a factor.

$\therefore a^3 - 7a^2 + 16a - 12 = (a - 2)(a^2 - 5a + 6) = (a - 2)(a - 2)(a - 3)$.

3. Resolve into factors $c^3 - 8c + 3$.

If $c = +1$, $c^3 - 8c + 3 = 1 - 8 + 3 = -4$.
If $c = -1$, $c^3 - 8c + 3 = -1 + 8 + 3 = +10$.
If $c = +3$, $c^3 - 8c + 3 = 27 - 24 + 3 = +6$.
If $c = -3$, $c^3 - 8c + 3 = -27 + 24 + 3 = 0$.

Hence, $c + 3$ is a factor.

$\therefore c^3 - 8c + 3 = (c + 3)(c^2 - 3c + 1)$.

4. Resolve into factors $2x^3 - x^2 - 13x - 6$.

If $x = +1$, $2x^3 - x^2 - 13x - 6 = 2 - 1 - 13 - 6 = -18$.
If $x = -1$, $2x^3 - x^2 - 13x - 6 = -2 - 1 + 13 - 6 = +4$.
If $x = +2$, $2x^3 - x^2 - 13x - 6 = 16 - 4 - 26 - 6 = -20$.
If $x = -2$, $2x^3 - x^2 - 13x - 6 = -16 - 4 + 26 - 6 = 0$.

Hence, $x + 2$ is a factor.

$\therefore 2x^3 - x^2 - 13x - 6 = (x + 2)(2x^2 - 5x - 3) = (x + 2)(x - 3)(2x + 1)$

5. Resolve into factors $x^3 - 19x + 12$.

If $x = +1$, $x^3 - 19x + 12 = 1 - 19 + 12 = -6$.
If $x = -1$, $x^3 - 19x + 12 = -1 + 19 + 12 = +30$.
If $x = +2$, $x^3 - 19x + 12 = 8 - 38 + 12 = -18$.
If $x = -2$, $x^3 - 19x + 12 = -8 + 38 + 12 = +42$.
If $x = +3$, $x^3 - 19x + 12 = 27 - 57 + 12 = -18$.
If $x = -3$, $x^3 - 19x + 12 = -27 + 57 + 12 = +42$.
If $x = +4$, $x^3 - 19x + 12 = 64 - 76 + 12 = 0$.

Hence, $x - 4$ is a factor.

$\therefore x^3 - 19x + 12 = (x - 4)(x^2 + 4x - 3)$.

6. Resolve into factors $6x^3 - 11x^2 + 6x - 1$.

If $x = +1$, $\quad 6x^3 - 11x^2 + 6x - 1 = 6 - 11 + 6 - 1 = 0$.

Hence, $x - 1$ is a factor.

$\therefore\ 6x^3 - 11x^2 + 6x - 1 = (x-1)(6x^2 - 5x + 1) = (x-1)(2x-1)(3x-1)$.

7. Resolve into factors $x^3 - 9x^2 + 26x - 24$.

If $x = +1$, $\quad x^3 - 9x^2 + 26x - 24 = 1 - 9 + 26 - 24 = -6$.
If $x = -1$, $\quad x^3 - 9x^2 + 26x - 24 = -1 - 9 - 26 - 24 = -60$.
If $x = +2$, $\quad x^3 - 9x^2 + 26x - 24 = 8 - 36 + 52 - 24 = 0$.

Hence, $x - 2$ is a factor.

$\therefore\ x^3 - 9x^2 + 26x - 24 = (x-2)(x^2 - 7x + 12) = (x-2)(x-3)(x-4)$.

8. Resolve into factors $2c^3 + 9c^2 + 12c + 4$.

If $c = +1$, $\quad 2c^3 + 9c^2 + 12c + 4 = 2 + 9 + 12 + 4 = +27$.
If $c = -1$, $\quad 2c^3 + 9c^2 + 12c + 4 = -2 + 9 - 12 + 4 = -1$.
If $c = +2$, $\quad 2c^3 + 9c^2 + 12c + 4 = 16 + 36 + 24 + 4 = +80$.
If $c = -2$, $\quad 2c^3 + 9c^2 + 12c + 4 = -16 + 36 - 24 + 4 = 0$.

Hence, $c + 2$ is a factor.

$\therefore\ 2c^3 + 9c^2 + 12c + 4 = (c+2)(2c^2 + 5c + 2) = (c+2)(c+2)(2c+1)$.

9. Resolve into factors $a^3 - 5a^2 - 2a + 24$.

If $a = +1$, $\quad a^3 - 5a^2 - 2a + 24 = 1 - 5 - 2 + 24 = +18$.
If $a = -1$, $\quad a^3 - 5a^2 - 2a + 24 = -1 - 5 + 2 + 24 = +20$.
If $a = +2$, $\quad a^3 - 5a^2 - 2a + 24 = 8 - 20 - 4 + 24 = +8$.
If $a = -2$, $\quad a^3 - 5a^2 - 2a + 24 = -8 - 20 + 4 + 24 = 0$.

Hence, $a + 2$ is a factor.

$\therefore\ a^3 - 5a^2 - 2a + 24 = (a+2)(a^2 - 7a + 12) = (a+2)(a-3)(a-4)$.

10. Resolve into factors $x^3 - 2x^2 - 15x + 36$.

If $x = +1$, $\quad x^3 - 2x^2 - 15x + 36 = 1 - 2 - 15 + 36 = +20$.
If $x = -1$, $\quad x^3 - 2x^2 - 15x + 36 = -1 - 2 + 15 + 36 = +48$.
If $x = +2$, $\quad x^3 - 2x^2 - 15x + 36 = 8 - 8 - 30 + 36 = +6$.
If $x = -2$, $\quad x^3 - 2x^2 - 15x + 36 = -8 - 8 + 30 + 36 = +50$.
If $x = +3$, $\quad x^3 - 2x^2 - 15x + 36 = 27 - 18 - 45 + 36 = 0$.

Hence, $x - 3$ is a factor.

$\therefore\ x^3 - 2x^2 - 15x + 36 = (x-3)(x^2 + x - 12) = (x-3)(x-3)(x+4)$.

11. Resolve into factors $m^3 + m^2 - 22m - 40$.

If $m = +1$, $m^3 + m^2 - 22m - 40 = 1 + 1 - 22 - 40 = -60$.
If $m = -1$, $m^3 + m^2 - 22m - 40 = -1 + 1 + 22 - 40 = -18$.
If $m = +2$, $m^3 + m^2 - 22m - 40 = 8 + 4 - 44 - 40 = -72$.
If $m = -2$, $m^3 + m^2 - 22m - 40 = -8 + 4 + 44 - 40 = 0$.

Hence, $m + 2$ is a factor.

$\therefore m^3 + m^2 - 22m - 40 = (m+2)(m^2 - m - 20) = (m+2)(m+4)(m-5)$.

12. Resolve into factors $n^3 - n^2 - 8n + 12$.

If $n = +1$, $n^3 - n^2 - 8n + 12 = 1 - 1 - 8 + 12 = +4$.
If $n = -1$, $n^3 - n^2 - 8n + 12 = -1 - 1 + 8 + 12 = +18$.
If $n = +2$, $n^3 - n^2 - 8n + 12 = 8 - 4 - 16 + 12 = 0$.

Hence, $n - 2$ is a factor.

$\therefore n^3 - n^2 - 8n + 12 = (n-2)(n^2 + n - 6) = (n-2)(n-2)(n+3)$.

13. Resolve into factors $6x^5 + 19x^4 - 20x^3 - 65x^2 + 24x + 36$.

If $x = +1$,
$6x^5 + 19x^4 - 20x^3 - 65x^2 + 24x + 36 = 6 + 19 - 20 - 65 + 24 + 36 = 0$.

Hence, $x - 1$ is a factor.

$$\therefore 6x^5 + 19x^4 - 20x^3 - 65x^2 + 24x + 36$$
$$= (x-1)(6x^4 + 25x^3 + 5x^2 - 60x - 36).$$

If $x = +1$,
$6x^4 + 25x^3 + 5x^2 - 60x - 36 = 6 + 25 + 5 - 60 - 36 = -60$.
If $x = -1$,
$6x^4 + 25x^3 + 5x^2 - 60x - 36 = 6 - 25 + 5 + 60 - 36 = +10$.
If $x = +2$,
$6x^4 + 25x^3 + 5x^2 - 60x - 36 = 96 + 200 + 20 - 120 - 36 = +160$.
If $x = -2$,
$6x^4 + 25x^3 + 5x^2 - 60x - 36 = 96 - 200 + 20 + 120 - 36 = 0$.

Hence, $x + 2$ is a factor.

$\therefore 6x^4 + 25x^3 + 5x^2 - 60x - 36 = (x+2)(6x^3 + 13x^2 - 21x - 18)$.

If $x = -2$, $6x^3 + 13x^2 - 21x - 18 = -48 + 52 + 42 - 18 = +28$.
If $x = +3$, $6x^3 + 13x^2 - 21x - 18 = 162 + 117 - 63 - 18 = +198$.
If $x = -3$, $6x^3 + 13x^2 - 21x - 18 = -162 + 117 + 63 - 18 = 0$.

Hence, $x + 3$ is a factor.

$$\therefore 6x^3 + 13x^2 - 21x - 18 = (x+3)(6x^2 - 5x - 6).$$

To factor $6x^2 - 5x - 6$.

Multiply by 6, $(6x)^2 - 5(6x) - 36$.
Put z for $6x$, $z^2 - 5z - 36$.
Resolve into factors, $(z + 4)(z - 9)$.
Put $6x$ for z, $(6x + 4)(6x - 9)$.
Divide by 2×3, $(3x + 2)(2x - 3)$.

$$\therefore\ 6x^2 - 5x - 6 = (3x + 2)(2x - 3).$$

$$\therefore\ 6x^5 + 19x^4 - 20x^3 - 65x^2 + 24x + 36$$
$$= (x - 1)(x + 2)(x + 3)(3x + 2)(2x - 3).$$

14. Resolve into factors $6x^5 + x^4 - 111x^3 + 66x^2 + 448x - 480$.

If $x = +1$, $6x^5 + x^4 - 111x^3 + 66x^2 + 448x - 480$
$= 6 + 1 - 111 + 66 + 448 - 480 = -70$.
If $x = -1$, $6x^5 + x^4 - 111x^3 + 66x^2 + 448x - 480$
$= -6 + 1 + 111 + 66 - 448 - 480 = -756$.
If $x = +2$, $6x^5 + x^4 - 111x^3 + 66x^2 + 448x - 480$
$= 192 + 16 - 888 + 264 + 896 - 480 = 0$.

Hence, $x - 2$ is a factor.

$$\therefore\ 6x^5 + x^4 - 111x^3 + 66x^2 + 448x - 480$$
$$= (x - 2)(6x^4 + 13x^3 - 85x^2 - 104x + 240).$$

If $x = +2$, $6x^4 + 13x^3 - 85x^2 - 104x + 240$
$= 96 + 104 - 340 - 208 + 240 = -108$.
If $x = -2$, $6x^4 + 13x^3 - 85x^2 - 104x + 240$
$= 96 - 104 - 340 + 208 + 240 = +100$.
If $x = +3$, $6x^4 + 13x^3 - 85x^2 - 104x + 240$
$= 486 + 351 - 765 - 312 + 240 = 0$.

Hence, $x - 3$ is a factor.

$$\therefore\ 6x^4 + 13x^3 - 85x^2 - 104x + 240 = (x - 3)(6x^3 + 31x^2 + 8x - 80).$$

If $x = +4$, $6x^3 + 31x^2 + 8x - 80 = 384 + 496 + 32 - 80 = +832$.
If $x = -4$, $6x^3 + 31x^2 + 8x - 80 = -384 + 496 - 32 - 80 = 0$.

Hence, $x + 4$ is a factor.

$$\therefore\ 6x^3 + 31x^2 + 8x - 80 = (x + 4)(6x^2 + 7x - 20).$$

To factor $6x^2 + 7x - 20$.

Multiply by 6, $(6x)^2 + 7(6x) - 120$.
Put z for $6x$, $z^2 + 7z - 120$.
Resolve into factors, $(z + 15)(z - 8)$.

Put $6x$ for z, $(6x + 15)(6x - 8)$.
Divide by 3×2, $(2x + 5)(3x - 4)$.

$$\therefore\ 6x^2 + 7x - 20 = (2x + 5)(3x - 4).$$
$$\therefore\ 6x^5 + x^4 - 111x^3 + 66x^2 + 448x - 480$$
$$= (x - 2)(x - 3)(x + 4)(2x + 5)(3x - 4).$$

15. Resolve into factors $4x^5 - 22x^4 + 17x^3 + 83x^2 - 152x + 60$.

If $x = +1$, $4x^5 - 22x^4 + 17x^3 + 83x^2 - 152x + 60$
$= 4 - 22 + 17 + 83 - 152 + 60 = -10$.

If $x = -1$, $4x^5 - 22x^4 + 17x^3 + 83x^2 - 152x + 60$
$= -4 - 22 - 17 + 83 + 152 + 60 = +252$.

If $x = +2$, $4x^5 - 22x^4 + 17x^3 + 83x^2 - 152x + 60$
$= 128 - 352 + 136 + 332 - 304 + 60 = 0$.

Hence, $x - 2$ is a factor.

$$\therefore\ 4x^5 - 22x^4 + 17x^3 + 83x^2 - 152x + 60$$
$$= (x - 2)(4x^4 - 14x^3 - 11x^2 + 61x - 30).$$

If $x = +2$,
$4x^4 - 14x^3 - 11x^2 + 61x - 30 = 64 - 112 - 44 + 122 - 30 = 0$.

Hence, $x - 2$ is again a factor.

$$\therefore\ 4x^4 - 14x^3 - 11x^2 + 61x - 30 = (x - 2)(4x^3 - 6x^2 - 23x + 15).$$

If $x = +3$, $4x^3 - 6x^2 - 23x + 15 = 108 - 54 - 69 + 15 = 0$.

Hence, $x - 3$ is a factor.

$$\therefore\ 4x^3 - 6x^2 - 23x + 15 = (x - 3)(4x^2 + 6x - 5).$$

$4x^2 + 6x - 5$ cannot be resolved into rational factors.

$$\therefore\ 4x^5 - 22x^4 + 17x^3 + 83x^2 - 152x + 60 = (x - 2)^2(x - 3)(4x^2 + 6x - 5).$$

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Resolve into factors:

1. $a^3bc^2 - a^2bc^2x - a^2b^2c^2 = a^2bc^2(a - x - b)$.

2. $6ax - 15by + 9bx - 10ay = (6ax + 9bx) - (10ay + 15by)$
$= 3x(2a + 3b) - 5y(2a + 3b)$
$= (3x - 5y)(2a + 3b)$.

3. $9a^2 - 24ac + 16c^2 = (3a - 4c)(3a - 4c)$.

4. $49x^2 - 81y^2 = (7x + 9y)(7x - 9y)$.

5. $$\begin{aligned} a^2 - 25c^2 + 9b^2 + 6ab &= (a^2 + 6ab + 9b^2) - (25c^2) \\ &= (a + 3b)^2 - (5c)^2 \\ &= (a + 3b + 5c)(a + 3b - 5c). \end{aligned}$$

6. $$\begin{aligned} 16a^4 + 25b^4 + 24a^2b^2 &= (16a^4 + 40a^2b^2 + 25b^4) - (16a^2b^2) \\ &= (4a^2 + 5b^2)^2 - (4ab)^2 \\ &= (4a^2 + 5b^2 + 4ab)(4a^2 + 5b^2 - 4ab) \\ &= (4a^2 + 4ab + 5b^2)(4a^2 - 4ab + 5b^2). \end{aligned}$$

7. $$x^2 + 2xy - 99y^2 = (x - 9y)(x + 11y).$$

8. $24x^2 + 2xy - 35y^2.$

Multiply by 24, $(24x)^2 + 2y(24x) - 840y^2.$

Put z for $24x$, $z^2 + 2yz - 840y^2.$

Resolve into factors, $(z - 28y)(z + 30y).$

Put $24x$ for z, $(24x - 28y)(24x + 30y).$

Divide by 4×6, $(6x - 7y)(4x + 5y).$

$$\therefore\ 24x^2 + 2xy - 35y^2 = (6x - 7y)(4x + 5y).$$

9. $$x^6 + 8y^3 = (x^2 + 2y)(x^4 - 2x^2y + 4y^2).$$

10. $x^3 + 8x^2 + 20x + 16.$

If $x = +1$, $x^3 + 8x^2 + 20x + 16 = 1 + 8 + 20 + 16 = 45.$

If $x = -1$, $x^3 + 8x^2 + 20x + 16 = -1 + 8 - 20 + 16 = 3.$

If $x = +2$, $x^3 + 8x^2 + 20x + 16 = 8 + 32 + 40 + 16 = 96.$

If $x = -2$, $x^3 + 8x^2 + 20x + 16 = -8 + 32 - 40 + 16 = 0.$

Hence, $x + 2$ is a factor.

$$\therefore\ x^3 + 8x^2 + 20x + 16 = (x + 2)(x^2 + 6x + 8) = (x + 2)(x + 2)(x + 4)$$

11. $15x^2 - 76xy + 77y^2.$

Multiply by 15, $(15x)^2 - 76y(15x) + 1155y^2.$

Put z for $15x$, $z^2 - 76yz + 1155y^2.$

Resolve into factors, $(z - 21y)(z - 55y).$

Put $15x$ for z, $(15x - 21y)(15x - 55y).$

Divide by 3×5, $(5x - 7y)(3x - 11y).$

$$\therefore\ 15x^2 - 76xy + 77y^2 = (5x - 7y)(3x - 11y).$$

12. $$a^4y^2 + a^3xy^2 + a^2y^4 = a^2y^2(a^2 + ax + y^2).$$

13. $$4x^2 + 28xy + 49y^2 = (2x + 7y)(2x + 7y).$$

14. $$\begin{aligned} 36x^4 + 49y^4 - 100x^2y^2 &= (36x^4 - 84x^2y^2 + 49y^4) - (16x^2y^2) \\ &= (6x^2 - 7y^2)^2 - (4xy)^2 \\ &= (6x^2 - 7y^2 + 4xy)(6x^2 - 7y^2 - 4xy) \\ &= (6x^2 + 4xy - 7y^2)(6x^2 - 4xy - 7y^2). \end{aligned}$$

15. $$\begin{aligned} 4x^2+25y^2-16z^2-20xy &= (4x^2-20xy+25y^2)-(16z^2) \\ &= (2x-5y)^2-(4z)^2 \\ &= (2x-5y+4z)(2x-5y-4z). \end{aligned}$$

16. $$36a^2-121b^2=(6a+11b)(6a-11b).$$

17. $$x^2-20x+51=(x-3)(x-17).$$

18. $6x^2-13xy-63y^2$.

Multiply by 6, $(6x)^2-13y(6x)-378y^2$.

Put z for $6x$, $z^2-13yz-378y^2$.

Resolve into factors, $(z+14y)(z-27y)$.

Put $6x$ for z, $(6x+14y)(6x-27y)$.

Divide by 2×3, $(3x+7y)(2x-9y)$.

$$\therefore\ 6x^2-13xy-63y^2=(3x+7y)(2x-9y).$$

19. $$\begin{aligned} 289a^4+151a^2b^2+64b^4 &= (289a^4+272a^2b^2+64b^4)-(121a^2b^2) \\ &= (17a^2+8b^2)^2-(11ab)^2 \\ &= (17a^2+8b^2+11ab)(17a^2+8b^2-11ab) \\ &= (17a^2+11ab+8b^2)(17a^2-11ab+8b^2). \end{aligned}$$

20. $$x^2+19xy+70y^2=(x+5y)(x+14y).$$

21. $$25a^2-40abc+16b^2c^2=(5a-4bc)(5a-4bc).$$

22. $$\begin{aligned} 64x^4y^4-49y^4z^4 &= y^4(64x^4-49z^4) \\ &= y^4(8x^2+7z^2)(8x^2-7z^2). \end{aligned}$$

23. $$\begin{aligned} 81a^4+64b^4-225a^2b^2 &= (81a^4-144a^2b^2+64b^4)-(81a^2b^2) \\ &= (9a^2-8b^2)^2-(9ab)^2 \\ &= (9a^2-8b^2+9ab)(9a^2-8b^2-9ab) \\ &= (9a^2+9ab-8b^2)(9a^2-9ab-8b^2). \end{aligned}$$

24. $$729x^3-1000y^3=(9x-10y)(81x^2+90xy+100y^2).$$

25. $$\begin{aligned} 25m^2+16n^2-64c^2-40mn &= (25m^2-40mn+16n^2)-(64c^2) \\ &= (5m-4n)^2-(8c)^2 \\ &= (5m-4n+8c)(5m-4n-8c). \end{aligned}$$

26. $$\begin{aligned} &144m^4+68m^2n^4+49n^8 \\ &= (144m^4+168m^2n^4+49n^8)-(100m^2n^4) \\ &= (12m^2+7n^4)^2-(10mn^2)^2 \\ &= (12m^2+7n^4+10mn^2)(12m^2+7n^4-10mn^2) \\ &= (12m^2+10mn^2+7n^4)(12m^2-10mn^2+7n^4). \end{aligned}$$

27. $$\begin{aligned} a^4-5a^3-84a^2 &= a^2(a^2-5a-84) \\ &= a^2(a+7)(a-12). \end{aligned}$$

28. $$\begin{aligned} 4x^2+10ab-5bx-8ax &= (4x^2-5bx)-(8ax-10ab) \\ &= x(4x-5b)-2a(4x-5b) \\ &= (x-2a)(4x-5b). \end{aligned}$$

29. $32a^2+4ab-15b^2.$

Multiply by 32, $(32a)^2+4b(32a)-480b^2.$

Put z for $32a$, $z^2+4bz-480b^2.$

Resolve into factors, $(z+24b)(z-20b).$

Put $32a$ for z, $(32a+24b)(32a-20b).$

Divide by 8×4, $(4a+3b)(8a-5b).$

$$\therefore\ 32a^2+4ab-15b^2=(4a+3b)(8a-5b).$$

30. $$144m^2n^2-121p^2=(12mn+11p)(12mn-11p).$$

31. $$\begin{aligned} c^3+23c^2+102c &= c(c^2+23c+102) \\ &= c(c+6)(c+17). \end{aligned}$$

32. $36m^2-31mn-56n^2.$

Multiply by 36, $(36m)^2-31n(36m)-2016n^2.$

Put z for $36m$, $z^2-31nz-2016n^2.$

Resolve into factors, $(z+32n)(z-63n).$

Put $36m$ for z, $(36m+32n)(36m-63n).$

Divide by 4×9, $(9m+8n)(4m-7n).$

$$\therefore\ 36m^2-31mn-56n^2=(9m+8n)(4m-7n).$$

33. $$b^2x^5-b^2x^3+2b^2x^2=b^2x^2(x^3-x+2).$$

34. $$\begin{aligned} 121c^4+144d^4+183c^2d^2 &= (121c^4+264c^2d^2+144d^4)-(81c^2d^2) \\ &= (11c^2+12d^2)^2-(9cd)^2 \\ &= (11c^2+12d^2+9cd)(11c^2+12d^2-9cd) \\ &= (11c^2+9cd+12d^2)(11c^2-9cd+12d^2). \end{aligned}$$

35. $$\begin{aligned} 49x^4-48yz-9y^2-64z^2 &= 49x^4-(9y^2+48yz+64z^2) \\ &= (7x^2)^2-(3y+8z)^2 \\ &= (7x^2+3y+8z)(7x^2-3y-8z). \end{aligned}$$

36. $$\begin{aligned} 81a^4-256b^4 &= (9a^2+16b^2)(9a^2-16b^2) \\ &= (9a^2+16b^2)(3a+4b)(3a-4b). \end{aligned}$$

37. $$m^2-16mn+60n^2=(m-6n)(m-10n).$$

38. $$\begin{aligned} 2y^2+3dy-14cy-21cd &= (2y^2+3dy)-(14cy+21cd) \\ &= y(2y+3d)-7c(2y+3d) \\ &= (y-7c)(2y+3d). \end{aligned}$$

39. $$27a^3 - 1331y^3 = (3a - 11y)(9a^2 + 33ay + 121y^2).$$

40. $$\begin{aligned} 49x^4 - 263x^2y^2 + 169y^4 &= (49x^4 - 182x^2y^2 + 169y^4) - (81x^2y^2) \\ &= (7x^2 - 13y^2)^2 - (9xy)^2 \\ &= (7x^2 - 13y^2 + 9xy)(7x^2 - 13y^2 - 9xy) \\ &= (7x^2 + 9xy - 13y^2)(7x^2 - 9xy - 13y^2). \end{aligned}$$

41. $$y^2 - 9yz - 112z^2 = (y + 7z)(y - 16z).$$

42. $$36x^2 - 84xyz + 49y^2z^2 = (6x - 7yz)(6x - 7yz).$$

43. $$\begin{aligned} 49a^4 + 3a^2b^2 + 4b^4 &= (49a^4 + 28a^2b^2 + 4b^4) - (25a^2b^2) \\ &= (7a^2 + 2b^2)^2 - (5ab)^2 \\ &= (7a^2 + 2b^2 + 5ab)(7a^2 + 2b^2 - 5ab) \\ &= (7a^2 + 5ab + 2b^2)(7a^2 - 5ab + 2b^2). \end{aligned}$$

44. $$9a^{4n} - 25b^{2n} = (3a^{2n} + 5b^n)(3a^{2n} - 5b^n).$$

45. $$343x^3 + 125y^3 = (7x + 5y)(49x^2 - 35xy + 25y^2).$$

46. $$\begin{aligned} y^5 + 5y^4 - 36y^3 &= y^3(y^2 + 5y - 36) \\ &= y^3(y + 9)(y - 4). \end{aligned}$$

47. $$x^4y^2 + x^3y^3 - x^3y^2z = x^3y^2(x + y - z).$$

48. $$81a^2b^2 + 90abc + 25c^2 = (9ab + 5c)(9ab + 5c).$$

49. $$\begin{aligned} 64b^4 - 176b^2y^2 + 49y^4 &= (64b^4 - 112b^2y^2 + 49y^4) - (64b^2y^2) \\ &= (8b^2 - 7y^2)^2 - (8by)^2 \\ &= (8b^2 - 7y^2 + 8by)(8b^2 - 7y^2 - 8by) \\ &= (8b^2 + 8by - 7y^2)(8b^2 - 8by - 7y^2) \end{aligned}$$

50. $$c^2 + cd - 110d^2 = (c + 11d)(c - 10d).$$

51. $$\begin{aligned} 500a^3 + 864b^3 &= 4(125a^3 + 216b^3) \\ &= 4(5a + 6b)(25a^2 - 30ab + 36b^2). \end{aligned}$$

52. $$\begin{aligned} d^6 + 21d^5 + 90d^4 &= d^4(d^2 + 21d + 90) \\ &= d^4(d + 6)(d + 15). \end{aligned}$$

53. $22c^2 + 37cd - 45d^2$.

Multiply by 22, $(22c)^2 + 37d(22c) - 990d^2$.

Put z for $22c$, $z^2 + 37dz - 990d^2$.

Resolve into factors, $(z - 18d)(z + 55d)$.

Put $22c$ for z, $(22c - 18d)(22c + 55d)$.

Divide by 2×11, $(11c - 9d)(2c + 5d)$.

$$\therefore\ 22c^2 + 37cd - 45d^2 = (11c - 9d)(2c + 5d).$$

54. $$x^2 - 17xy + 60y^2 = (x - 5y)(x - 12y).$$

55. $$\begin{aligned} c^2 + 6abdx - 2acx - 3bcd &= (c^2 - 2acx) - (3bcd - 6abdx) \\ &= c(c - 2ax) - 3bd(c - 2ax) \\ &= (c - 3bd)(c - 2ax). \end{aligned}$$

56. $$\begin{aligned} 36a^2 + 42bc - 9b^2 - 49c^2 &= 36a^2 - (9b^2 - 42bc + 49c^2) \\ &= (6a)^2 - (3b - 7c)^2 \\ &= (6a + 3b - 7c)(6a - 3b + 7c). \end{aligned}$$

57. $$25x^4 - 81y^2 = (5x^2 + 9y)(5x^2 - 9y).$$

58. $$64b^2c^2 + 144bcx^2 + 81x^4 = (8bc + 9x^2)(8bc + 9x^2).$$

59. $$\begin{aligned} 256x^4 - 705x^2z^2 + 225z^4 &= (256x^4 - 480x^2z^2 + 225z^4) - (225x^2z^2) \\ &= (16x^2 - 15z^2)^2 - (15xz)^2 \\ &= (16x^2 - 15z^2 + 15xz)(16x^2 - 15z^2 - 15xz) \\ &= (16x^2 + 15xz - 15z^2)(16x^2 - 15xz - 15z^2). \end{aligned}$$

60. $$x^2 - 7xy - 44y^2 = (x + 4y)(x - 11y).$$

61. $$1728a^3 - 125x^3 = (12a - 5x)(144a^2 + 60ax + 25x^2).$$

62. $$\begin{aligned} 36a^4b^4 - 121b^4c^4 &= b^4(36a^4 - 121c^4) \\ &= b^4(6a^2 + 11c^2)(6a^2 - 11c^2). \end{aligned}$$

63. $$a^2 + 13ab + 30b^2 = (a + 3b)(a + 10b).$$

64. $$\begin{aligned} 4m^4 + 8m^2n^2 + 121n^4 &= (4m^4 + 44m^2n^2 + 121n^4) - (36m^2n^2) \\ &= (2m^2 + 11n^2)^2 - (6mn)^2 \\ &= (2m^2 + 11n^2 + 6mn)(2m^2 + 11n^2 - 6mn) \\ &= (2m^2 + 6mn + 11n^2)(2m^2 - 6mn + 11n^2). \end{aligned}$$

65. $14x^2 + 61xy + 42y^2$.

Multiply by 14, $(14x)^2 + 61y(14x) + 588y^2.$
Put z for $14x$, $z^2 + 61yz + 588y^2.$
Resolve into factors, $(z + 12y)(z + 49y).$
Put $14x$ for z, $(14x + 12y)(14x + 49y).$
Divide by 2×7, $(7x + 6y)(2x + 7y).$

$$\therefore\ 14x^2 + 61xy + 42y^2 = (7x + 6y)(2x + 7y).$$

66. $$144x^2 - 264xy^2 + 121y^4 = (12x - 11y^2)(12x - 11y^2).$$

67. $20a^2 - 49ac + 30c^2.$

Multiply by 20, $(20a)^2 - 49c(20a) + 600c^2.$
Put z for $20a$, $z^2 - 49cz + 600c^2.$
Resolve into factors, $(z - 24c)(z - 25c).$
Put $20a$ for z, $(20a - 24c)(20a - 25c).$
Divide by 4×5, $(5a - 6c)(4a - 5c).$

$$\therefore\ 20a^2 - 49ac + 30c^2 = (5a - 6c)(4a - 5c).$$

68. $$a^5b - 3a^4b^2 + 3a^3b^3 = a^3b(a^2 - 3ab + 3b^2).$$

69. $$\begin{aligned} 25a^4 - 211a^2y^2 + 81y^4 &= (25a^4 - 90a^2y^2 + 81y^4) - (121a^2y^2) \\ &= (5a^2 - 9y^2)^2 - (11ay)^2 \\ &= (5a^2 - 9y^2 + 11ay)(5a^2 - 9y^2 - 11ay) \\ &= (5a^2 + 11ay - 9y^2)(5a^2 - 11ay - 9y^2). \end{aligned}$$

70. $$144a^2b^2 - 25c^2d^2 = (12ab + 5cd)(12ab - 5cd).$$

71. $$\begin{aligned} x^9 - 512y^9 &= (x^3)^3 - (8y^3)^3 \\ &= (x^3 - 8y^3)(x^6 + 8x^3y^3 + 64y^6) \\ &= (x - 2y)(x^2 + 2xy + 4y^2)(x^6 + 8x^3y^3 + 64y^6). \end{aligned}$$

72. $$\begin{aligned} 2mx + 4nx - 10an - 5am &= (2mx - 5am) + (4nx - 10an) \\ &= m(2x - 5a) + 2n(2x - 5a) \\ &= (m + 2n)(2x - 5a). \end{aligned}$$

73. $$x^2 - 5xy - 36y^2 = (x + 4y)(x - 9y).$$

74. $$49x^2 - 169y^2z^2 = (7x + 13yz)(7x - 13yz).$$

75. $99x^2 - 17xy - 12y^2.$

Multiply by 99, $(99x)^2 - 17y(99x) - 1188y^2.$
Put z for $99x$, $z^2 - 17yz - 1188y^2.$
Resolve into factors, $(z + 27y)(z - 44y).$
Put $99x$ for z, $(99x + 27y)(99x - 44y).$
Divide by 9×11, $(11x + 3y)(9x - 4y).$

$$\therefore\ 99x^2 - 17xy - 12y^2 = (11x + 3y)(9x - 4y).$$

76. $$\begin{aligned} 81x^2 - 121a^2 + 176ab - 64b^2 &= 81x^2 - (121a^2 - 176ab + 64b^2) \\ &= (9x)^2 - (11a - 8b)^2 \\ &= (9x + 11a - 8b)(9x - 11a + 8b). \end{aligned}$$

77. $$64x^6 + 125y^9 = (4x^2 + 5y^3)(16x^4 - 20x^2y^3 + 25y^6).$$

78. $$\begin{aligned} 81x^2y^2 - 180xy^2z + 100y^2z^2 &= y^2(81x^2 - 180xz + 100z^2) \\ &= y^2(9x - 10z)(9x - 10z). \end{aligned}$$

79. $$x^2 - 12xy + 35y^2 = (x - 5y)(x - 7y).$$

80. $$\begin{aligned} 225a^3b - 256ab^3 &= ab(225a^2 - 256b^2) \\ &= ab(15a + 16b)(15a - 16b). \end{aligned}$$

81. $$\begin{aligned} 64x^4 - 73x^2y^2 + 9y^4 &= (64x^4 - 48x^2y^2 + 9y^4) - (25x^2y^2) \\ &= (8x^2 - 3y^2)^2 - (5xy)^2 \\ &= (8x^2 - 3y^2 + 5xy)(8x^2 - 3y^2 - 5xy) \\ &= (8x^2 + 5xy - 3y^2)(8x^2 - 5xy - 3y^2). \end{aligned}$$

To factor $8x^2 + 5xy - 3y^2$.

Multiply by 8, $(8x)^2 + 5y(8x) - 24y^2.$
Put z for $8x$, $z^2 + 5yz - 24y^2.$
Resolve into factors, $(z + 8y)(z - 3y).$
Put $8x$ for z, $(8x + 8y)(8x - 3y).$
Divide by 8, $(x + y)(8x - 3y).$

$$\therefore\ 8x^2 + 5xy - 3y^2 = (x + y)(8x - 3y).$$

To factor $8x^2 - 5xy - 3y^2$.

Multiply by 8, $(8x)^2 - 5y(8x) - 24y^2.$
Put z for $8x$, $z^2 - 5yz - 24y^2.$
Resolve into factors, $(z + 3y)(z - 8y).$
Put $8x$ for z, $(8x + 3y)(8x - 8y).$
Divide by 8, $(8x + 3y)(x - y).$

$$\therefore\ 8x^2 - 5xy - 3y^2 = (8x + 3y)(x - y).$$
$$\therefore\ 64x^4 - 73x^2y^2 + 9y^4 = (x + y)(x - y)(8x + 3y)(8x - 3y).$$

82. $$\begin{aligned} 6x^3 - 8ax + 20ab - 15bx^2 &= (6x^3 - 15bx^2) - (8ax - 20ab) \\ &= 3x^2(2x - 5b) - 4a(2x - 5b) \\ &= (3x^2 - 4a)(2x - 5b). \end{aligned}$$

83. $$24x^2 - 86xy + 42y^2 = 2(12x^2 - 43xy + 21y^2).$$

To factor $12x^2 - 43xy + 21y^2$.

Multiply by 12, $(12x)^2 - 43y(12x) + 252y^2.$
Put z for $12x$, $z^2 - 43yz + 252y^2.$
Resolve into factors, $(z - 7y)(z - 36y).$
Put $12x$ for z, $(12x - 7y)(12x - 36y).$
Divide by 1 × 12, $(12x - 7y)(x - 3y).$

$$\therefore\ 12x^2 - 43xy + 21y^2 = (x - 3y)(12x - 7y).$$
$$\therefore\ 24x^2 - 86xy + 42y^2 = 2(x - 3y)(12x - 7y).$$

84.
$$\begin{aligned}100a^4+36a^2b^2+81b^4&=(100a^4+180a^2b^2+81b^4)-(144a^2b^2)\\&=(10a^2+9b^2)^2-(12ab)^2\\&=(10a^2+9b^2+12ab)(10a^2+9b^2-12ab)\\&=(10a^2+12ab+9b^2)(10a^2-12ab+9b^2).\end{aligned}$$

85.
$$a^2+8ab-33b^2=(a-3b)(a+11b).$$

86. $6a^2+25ab-91b^2.$

Multiply by 6, $(6a)^2+25b(6a)-546b^2.$

Put z for $6a$, $z^2+25bz-546b^2.$

Resolve into factors, $(z+39b)(z-14b).$

Put $6a$ for z, $(6a+39b)(6a-14b).$

Divide by 3×2, $(2a+13b)(3a-7b).$

$$\therefore 6a^2+25ab-91b^2=(2a+13b)(3a-7b).$$

87.
$$\begin{aligned}16x^6y^2-40x^4y^4+25x^2y^6&=x^2y^2(16x^4-40x^2y^2+25y^4)\\&=x^2y^2(4x^2-5y^2)(4x^2-5y^2).\end{aligned}$$

88. $343a^3+(3x+5y)^3$
$$\begin{aligned}&=[7a+(3x+5y)][49a^2-7a(3x+5y)+(3x+5y)^2]\\&=(7a+3x+5y)(49a^2-21ax-35ay+9x^2+30xy+25y^2).\end{aligned}$$

89.
$$\begin{aligned}196x^4y^2-225x^2y^4&=x^2y^2(196x^2-225y^2)\\&=x^2y^2(14x+15y)(14x-15y).\end{aligned}$$

90.
$$a^4b^2y+a^3b^3x+ab^4z=ab^2(a^3y+a^2bx+b^2z).$$

91.
$$\begin{aligned}81x^4-18x^2y^2+49y^4&=(81x^4+126x^2y^2+49y^4)-(144x^2y^2)\\&=(9x^2+7y^2)^2-(12xy)^2\\&=(9x^2+7y^2+12xy)(9x^2+7y^2-12xy)\\&=(9x^2+12xy+7y^2)(9x^2-12xy+7y^2).\end{aligned}$$

92.
$$\begin{aligned}49x^2+56xy+16y^2-16a^2&=(49x^2+56xy+16y^2)-(16a^2)\\&=(7x+4y)^2-(4a)^2\\&=(7x+4y+4a)(7x+4y-4a).\end{aligned}$$

93.
$$\begin{aligned}121a^2mn-169mny^2&=mn(121a^2-169y^2)\\&=mn(11a+13y)(11a-13y).\end{aligned}$$

94.
$$x^2+16xy+28y^2=(x+2y)(x+14y).$$

95.
$$\begin{aligned}49a^5b-154a^3b^3+121ab^5&=ab(49a^4-154a^2b^2+121b^4)\\&=ab(7a^2-11b^2)(7a^2-11b^2).\end{aligned}$$

96. $$8a^6 - 6a^4b^2 - 35a^2b^4 = a^2(8a^4 - 6a^2b^2 - 35b^4).$$

To factor $8a^4 - 6a^2b^2 - 35b^4$.

Multiply by 8, $(8a^2)^2 - 6b^2(8a^2) - 280b^4.$

Put z for $8a^2$, $z^2 - 6b^2z^2 - 280b^4.$

Resolve into factors, $(z + 14b^2)(z - 20b^2).$

Put $8a^2$ for z, $(8a^2 + 14b^2)(8a^2 - 20b^2).$

Divide by 2×4, $(4a^2 + 7b^2)(2a^2 - 5b^2).$

$$\therefore\ 8a^4 - 6a^2b^2 - 35b^4 = (4a^2 + 7b^2)(2a^2 - 5b^2).$$
$$\therefore\ 8a^6 - 6a^4b^2 - 35a^2b^4 = a^2(4a^2 + 7b^2)(2a^2 - 5b^2).$$

97. $$\begin{aligned} 20ax + 6by - 15ay - 8bx &= (20ax - 15ay) - (8bx - 6by) \\ &= 5a(4x - 3y) - 2b(4x - 3y) \\ &= (5a - 2b)(4x - 3y). \end{aligned}$$

98. $$\begin{aligned} 169a^8 + 9a^4b^2 + 81b^4 &= (169a^8 + 234a^4b^2 + 81b^4) - (225a^4b^2) \\ &= (13a^4 + 9b^2)^2 - (15a^2b)^2 \\ &= (13a^4 + 9b^2 + 15a^2b)(13a^4 + 9b^2 - 15a^2b) \\ &= (13a^4 + 15a^2b + 9b^2)(13a^4 - 15a^2b + 9b^2). \end{aligned}$$

99. $$\begin{aligned} 16a^4c^2 + 88a^3c^3 + 121a^2c^4 &= a^2c^2(16a^2 + 88ac + 121c^2) \\ &= a^2c^2(4a + 11c)(4a + 11c). \end{aligned}$$

100. $$x^2 + xy - 30y^2 = (x - 5y)(x + 6y).$$

101. $$\begin{aligned} 100a^3x - 484ax^3 &= 4ax(25a^2 - 121x^2) \\ &= 4ax(5a + 11x)(5a - 11x). \end{aligned}$$

102. $$a^4x^2 - 3a^3x^3 + 6a^2x^4 = a^2x^2(a^2 - 3ax + 6x^2).$$

103. $$\begin{aligned} a^{2m+2} - a^2c^{2n} &= a^2(a^{2m} - c^{2n}) \\ &= a^2(a^m + c^n)(a^m - c^n). \end{aligned}$$

104. $12x^4 - x^2y^2 - 35y^4$.

Multiply by 12, $(12x^2)^2 - y^2(12x^2) - 420y^4.$

Put z for $12x^2$, $z^2 - y^2z - 420y^4.$

Resolve into factors, $(z + 20y^2)(z - 21y^2).$

Put $12x^2$ for z, $(12x^2 + 20y^2)(12x^2 - 21y^2).$

Divide by 4×3, $(3x^2 + 5y^2)(4x^2 - 7y^2).$

$$\therefore\ 12x^4 - x^2y^2 - 35y^4 = (3x^2 + 5y^2)(4x^2 - 7y^2).$$

105. $$x^4 - 15x^2y + 56y^2 = (x^2 - 7y)(x^2 - 8y).$$

106. $$\begin{aligned} &729 - 8(4y - 7z)^3 \\ &= [9 - 2(4y - 7z)][81 + 18(4y - 7z) + 4(4y - 7z)^2] \\ &= (9 - 8y + 14z)(81 + 72y - 126z + 64y^2 - 224yz + 196z^2) \end{aligned}$$

107.
$$\begin{aligned} a^7b - ab^5 &= ab(a^6 - b^4)\\ &= ab(a^3 + b^2)(a^3 - b^2). \end{aligned}$$

108.
$$\begin{aligned} x^3y - 2x^2y^2 - 99xy^3 &= xy(x^2 - 2xy - 99y^2)\\ &= xy(x + 9y)(x - 11y). \end{aligned}$$

109.
$$x^2 - 18xy + 45y^2 = (x - 3y)(x - 15y).$$

110. $24x^2 + 118xy + 45y^2$.

Multiply by 24, $(24x)^2 + 118y(24x) + 1080y^2$.

Put z for $24x$, $z^2 + 118yz + 1080y^2$.

Resolve into factors, $(z + 10y)(z + 108y)$.

Put $24x$ for z, $(24x + 10y)(24x + 108y)$.

Divide by 2×12, $(12x + 5y)(2x + 9y)$.

$$\therefore 24x^2 + 118xy + 45y^2 = (12x + 5y)(2x + 9y).$$

111.
$$b^4c^2 - 7b^3c^3 + 4b^2c^4 = b^2c^2(b^2 - 7bc + 4c^2).$$

112.
$$\begin{aligned} 144x^3y - 1089xy^3 &= 9xy(16x^2 - 121y^2)\\ &= 9xy(4x + 11y)(4x - 11y). \end{aligned}$$

113.
$$\begin{aligned} 25x^4y^2z^2 + 110x^2yz^3 + 121z^4 &= z^2(25x^4y^2 + 110x^2yz + 121z^2)\\ &= z^2(5x^2y + 11z)(5x^2y + 11z). \end{aligned}$$

114.
$$\begin{aligned} 196x^4 - 221x^2y^2 + 25y^4 &= (196x^4 - 140x^2y^2 + 25y^4) - (81x^2y^2)\\ &= (14x^2 - 5y^2)^2 - (9xy)^2\\ &= (14x^2 - 5y^2 + 9xy)(14x^2 - 5y^2 - 9xy)\\ &= (14x^2 + 9xy - 5y^2)(14x^2 - 9xy - 5y^2). \end{aligned}$$

To factor $14x^2 + 9xy - 5y^2$.

Multiply by 14, $(14x)^2 + 9y(14x) - 70y^2$.

Put z for $14x$, $z^2 + 9yz - 70y^2$.

Resolve into factors, $(z - 5y)(z + 14y)$.

Put $14x$ for z, $(14x - 5y)(14x + 14y)$.

Divide by 1×14, $(14x - 5y)(x + y)$.

$$\therefore 14x^2 + 9xy - 5y^2 = (14x - 5y)(x + y).$$

To factor $14x^2 - 9xy - 5y^2$.

Multiply by 14, $(14x)^2 - 9y(14x) - 70y^2$.

Put z for $14x$, $z^2 - 9yz - 70y^2$.

Resolve into factors, $(z + 5y)(z - 14y)$.

Put $14x$ for z, $(14x + 5y)(14x - 14y)$.

Divide by 1×14, $(14x + 5y)(x - y)$.

$$\therefore 14x^2 - 9xy - 5y^2 = (14x + 5y)(x - y).$$

$$\therefore 196x^4 - 221x^2y^2 + 25y^4 = (x + y)(x - y)(14x + 5y)(14x - 5y).$$

115. $$\begin{aligned}10ax + 45ay - 4bx - 18by &= (10ax + 45ay) - (4bx + 18by)\\ &= 5a(2x + 9y) - 2b(2x + 9y)\\ &= (5a - 2b)(2x + 9y).\end{aligned}$$

116. $$16x^2 + 72a^2b^2x + 81a^4b^4 = (4x + 9a^2b^2)(4x + 9a^2b^2).$$

117. $18m^4 - 53m^2n^2 - 35n^4$.

Multiply by 18, $(18m^2)^2 - 53n^2(18m^2) - 630n^4$.

Put z for $18m^2$, $z^2 - 53n^2z - 630n^4$.

Resolve into factors, $(z + 10n^2)(z - 63n^2)$.

Put $18m^2$ for z, $(18m^2 + 10n^2)(18m^2 - 63n^2)$.

Divide by 2×9, $(9m^2 + 5n^2)(2m^2 - 7n^2)$.

$$\therefore 18m^4 - 53m^2n^2 - 35n^4 = (9m^2 + 5n^2)(2m^2 - 7n^2).$$

118. $$\begin{aligned}&(2x + 5y)^3 - (3y - 7z)^3\\ &= [(2x + 5y) - (3y - 7z)][(2x + 5y)^2\\ &\quad + (2x + 5y)(3y - 7z) + (3y - 7z)^2]\\ &= (2x + 5y - 3y + 7z)(4x^2 + 20xy + 25y^2 + 6xy + 15y^2\\ &\quad - 14xz - 35yz + 9y^2 - 42yz + 49z^2)\\ &= (2x + 2y + 7z)(4x^2 + 49y^2 + 49z^2 + 26xy - 14xz - 77yz).\end{aligned}$$

119. $$\begin{aligned}&625a^4 - 1300a^2b^2 + 324b^4\\ &= (625a^4 - 900a^2b^2 + 324b^4) - (400a^2b^2)\\ &= (25a^2 - 18b^2)^2 - (20ab)^2\\ &= (25a^2 - 18b^2 + 20ab)(25a^2 - 18b^2 - 20ab)\\ &= (25a^2 + 20ab - 18b^2)(25a^2 - 20ab - 18b^2).\end{aligned}$$

120. $$\begin{aligned}36x^6 - 156x^4y^2 + 169x^2y^4 &= x^2(36x^4 - 156x^2y^2 + 169y^4)\\ &= x^2(6x^2 - 13y^2)(6x^2 - 13y^2).\end{aligned}$$

121. $$x^4 + 21x^2y + 38y^2 = (x^2 + 2y)(x^2 + 19y).$$

122. $$\begin{aligned}144x^2 - 144ac - 81a^2 - 64c^2 &= 144x^2 - (81a^2 + 144ac + 64c^2)\\ &= (12x)^2 - (9a + 8c)^2\\ &= (12x + 9a + 8c)(12x - 9a - 8c).\end{aligned}$$

123. $$\begin{aligned}&21ax - 28ay + 24by - 30bz + 35az - 18bx\\ &= (21ax - 18bx) - (28ay - 24by) + (35az - 30bz)\\ &= 3x(7a - 6b) - 4y(7a - 6b) + 5z(7a - 6b)\\ &= (7a - 6b)(3x - 4y + 5z).\end{aligned}$$

124. $484x^8 - 664x^4y^2 + 225y^4$
$= (484x^8 - 660x^4y^2 + 225y^4) - (4x^4y^2)$
$= (22x^4 - 15y^2)^2 - (2\,{}^2xy)^2$
$= (22x^4 - 15y^2 + 2x^2y)(22x^4 - 15y^2 - 2x^2y)$
$= (22x^4 + 2x^2y - 15y^2)(22x^4 - 2x^2y - 15y^2).$

125. $x^4 + 4x^3y - 13x^2y^2 - 40xy^3 + 48y^4.$

If $x = +y$, $x^4 + 4x^3y - 13x^2y^2 - 40xy^3 + 48y^4$
$= y^4 + 4y^4 - 13y^4 - 40y^4 + 48y^4 = 0.$

Hence, $x - y$ is a factor.

$\therefore x^4 + 4x^3y - 13x^2y^2 - 40xy^3 + 48y^4$
$= (x - y)(x^3 + 5x^2y - 8xy^2 - 48y^3).$

If $x = +y$, $x^3 + 5x^2y - 8xy^2 - 48y^3$
$= y^3 + 5y^3 - 8y^3 - 48y^3 = -50y^3.$
If $x = -y$, $x^3 + 5x^2y - 8xy^2 - 48y^3$
$= -y^3 + 5y^3 + 8y^3 - 48y^3 = -36y^3.$
If $x = +2y$, $x^3 + 5x^2y - 8xy^2 - 48y^3$
$= 8y^3 + 20y^3 - 16y^3 - 48y^3 = -36y^3.$
If $x = -2y$, $x^3 + 5x^2y - 8xy^2 - 48y^3$
$= -8y^3 + 20y^3 + 16y^3 - 48y^3 = -20y^3.$
If $x = +3y$, $x^3 + 5x^2y - 8xy^2 - 48y^3$
$= 27y^3 + 45y^3 - 24y^3 - 48y^3 = 0.$

Hence, $x - 3y$ is a factor.

$\therefore x^3 + 5x^2y - 8xy^2 - 48y^3 = (x - 3y)(x^2 + 8xy + 16y^2)$
$= (x - 3y)(x + 4y)(x + 4y).$
$\therefore x^4 + 4x^3y - 13x^2y^2 - 40xy^3 + 48y^4$
$= (x - y)(x - 3y)(x + 4y)(x + 4y).$

126. $x^2 - 4n^2 + 4mn - 6xy - m^2 + 9y^2$
$= (x^2 - 6xy + 9y^2) - (m^2 - 4mn + 4n^2)$
$= (x - 3y)^2 - (m - 2n)^2$
$= (x - 3y + m - 2n)(x - 3y - m + 2n).$

127. $(3a - 4b)^3 + (2x + 3y)^3$
$= [(3a - 4b) + (2x + 3y)][(3a - 4b)^2 - (3a - 4b)(2x + 3y)$
$+ (2x + 3y)^2]$
$= (3a - 4b + 2x + 3y)(9a^2 - 24ab + 16b^2 - 6ax - 9ay + 8bx$
$+ 12by + 4x^2 + 12xy + 9y^2).$

128. $12a^4b^2 + a^3b^3 - 35a^2b^4 = a^2b^2(12a^2 + ab - 35b^2).$

To factor $12a^2 + ab - 35b^2.$

Multiply by 12, $(12a)^2 + b(12a) - 420b^2.$

Put z for $12a$, $z^2 + bz - 420b^2.$

Resolve into factors, $(z - 20b)(z + 21b).$

Put $12a$ for z, $(12a - 20b)(12a + 21b).$

Divide by 4×3, $(3a - 5b)(4a + 7b).$

$$\therefore 12a^2 + ab - 35b^2 = (3a - 5b)(4a + 7b).$$

$$\therefore 12a^4b^2 + a^3b^3 - 35a^2b^4 = a^2b^2(3a - 5b)(4a + 7b).$$

129. $6x^4 + 49x^3y + 146x^2y^2 + 189xy^3 + 90y^4.$

If $x = +y$, $6x^4 + 49x^3y + 146x^2y^2 + 189xy^3 + 90y^4$
$= 6y^4 + 49y^4 + 146y^4 + 189y^4 + 90y^4 = +480y^4.$

If $x = -y$, $6x^4 + 49x^3y + 146x^2y^2 + 189xy^3 + 90y^4$
$= 6y^4 - 49y^4 + 146y^4 - 189y^4 + 90y^4 = +4y^4.$

If $x = +2y$, $6x^4 + 49x^3y + 146x^2y^2 + 189xy^3 + 90y^4$
$= 96y^4 + 392y^4 + 584y^4 + 378y^4 + 90y^4 = +1540y^4.$

If $x = -2y$, $6x^4 + 49x^3y + 146x^2y^2 + 189xy^3 + 90y^4$
$= 96y^4 - 392y^4 + 584y^4 - 378y^4 + 90y^4 = 0.$

Hence, $x + 2y$ is a factor.

$\therefore 6x^4 + 49x^3y + 146x^2y^2 + 189xy^3 + 90y^4$
$= (x + 2y)(6x^3 + 37x^2y + 72xy^2 + 45y^3).$

If $x = -2y$, $6x^3 + 37x^2y + 72xy^2 + 45y^3$
$= -48y^3 + 148y^3 - 144y^3 + 45y^3 = +y^3.$

If $x = +3y$, $6x^3 + 37x^2y + 72xy^2 + 45y^3$
$= 162y^3 + 333y^3 + 216y^3 + 45y^3 = +756y^3.$

If $x = -3y$, $6x^3 + 37x^2y + 72xy^2 + 45y^3$
$= -162y^3 + 333y^3 - 216y^3 + 45y^3 = 0.$

Hence, $x + 3y$ is a factor.

$\therefore 6x^3 + 37x^2y + 72xy^2 + 45y^3 = (x + 3y)(6x^2 + 19xy + 15y^2).$

To factor $6x^2 + 19xy + 15y^2.$

Multiply by 6, $(6x)^2 + 19y(6x) + 90y^2.$

Put z for $6x$, $z^2 + 19yz + 90y^2.$

Resolve into factors, $(z + 9y)(z + 10y).$

Put $6x$ for z, $(6x + 9y)(6x + 10y).$

Divide by 3×2, $(2x + 3y)(3x + 5y).$

$$\therefore 6x^2 + 19xy + 15y^2 = (2x + 3y)(3x + 5y).$$

$\therefore 6x^4 + 49x^3y + 146x^2y^2 + 189xy^3 + 90y^4$
$= (x + 2y)(x + 3y)(2x + 3y)(3x + 5y).$

130. $9a^4x^3 - 35a^2b^2xy^2 - 21a^4xy^2 + 15a^2b^2x^3$
$= a^2x(9a^2x^2 - 35b^2y^2 - 21a^2y^2 + 15b^2x^2)$
$= a^2x[(9a^2x^2 - 21a^2y^2) + (15b^2x^2 - 35b^2y^2)]$
$= a^2x[3a^2(3x^2 - 7y^2) + 5b^2(3x^2 - 7y^2)]$
$= a^2x(3a^2 + 5b^2)(3x^2 - 7y^2).$

131. $9x^2 - 9a^2 + 64y^2 - 64b^2 + 48xy + 48ab$
$= (9x^2 + 48xy + 64y^2) - (9a^2 - 48ab + 64b^2)$
$= (3x + 8y)^2 - (3a - 8b)^2$
$= (3x + 8y + 3a - 8b)(3x + 8y - 3a + 8b).$

132. $15a^6 - 89a^5 + 152a^4 - 44a^3 - 48a^2$
$= a^2(15a^4 - 89a^3 + 152a^2 - 44a - 48).$

If $a = +1$, $15a^4 - 89a^3 + 152a^2 - 44a - 48$
$= 15 - 89 + 152 - 44 - 48 = -14.$
If $a = -1$, $15a^4 - 89a^3 + 152a^2 - 44a - 48$
$= 15 + 89 + 152 + 44 - 48 = +252.$
If $a = +2$, $15a^4 - 89a^3 + 152a^2 - 44a - 48$
$= 240 - 712 + 608 - 88 - 48 = 0.$

Hence, $a - 2$ is a factor.

$\therefore 15a^4 - 89a^3 + 152a^2 - 44a - 48 = (a-2)(15a^3 - 59a^2 + 34a + 24).$

If $a = +2$, $15a^3 - 59a^2 + 34a + 24 = 120 - 236 + 68 + 24 = -24.$
If $a = -2$, $15a^3 - 59a^2 + 34a + 24 = -120 - 236 - 68 + 24 = -400.$
If $a = +3$, $15a^3 - 59a^2 + 34a + 24 = 405 - 531 + 102 + 24 = 0.$

Hence, $a - 3$ is a factor.

$\therefore 15a^3 - 59a^2 + 34a + 24 = (a - 3)(15a^2 - 14a - 8).$

To factor $15a^2 - 14a - 8$.

Multiply by 15, $(15a)^2 - 14(15a) - 120.$
Put z for $15a$, $z^2 - 14z - 120.$
Resolve into factors, $(z - 20)(z + 6).$
Put $15a$ for z, $(15a - 20)(15a + 6).$
Divide by 5×3, $(3a - 4)(5a + 2).$

$\therefore 15a^2 - 14a - 8 = (3a - 4)(5a + 2).$

$\therefore 15a^6 - 89a^5 + 152a^4 - 44a^3 - 48a^2 = a^2(a-2)(a-3)(3a-4)(5a+2).$

133. $16a^2 - 34ab + 15b^2.$

Multiply by 16, $(16a)^2 - 34b(16a) + 240b^2.$
Put z for $16a$, $z^2 - 34bz + 240b^2.$
Resolve into factors, $(z - 10b)(z - 24b).$
Put $16a$ for z, $(16a - 10b)(16a - 24b).$
Divide by 2×8, $(8a - 5b)(2a - 3b).$

$$\therefore 16a^2 - 34ab + 15b^2 = (8a - 5b)(2a - 3b).$$

134. $(5a - 6b)^3 - (3x - 2y)^3$

$$= [(5a - 6b) - (3x - 2y)][(5a - 6b)^2 + (5a - 6b)(3x - 2y) + (3x - 2y)^2]$$
$$= (5a - 6b - 3x + 2y)(25a^2 - 60ab + 36b^2 + 15ax - 10ay - 18bx + 12by + 9x^2 - 12xy + 4y^2).$$

135. $16x^4 - 81b^2 - 25a^4 + 56x^2y - 90a^2b + 49y^2$

$$= (16x^4 + 56x^2y + 49y^2) - (25a^4 + 90a^2b + 81b^2)$$
$$= (4x^2 + 7y)^2 - (5a^2 + 9b)^2$$
$$= (4x^2 + 7y + 5a^2 + 9b)(4x^2 + 7y - 5a^2 - 9b).$$

136. $x^4 - 10x^3 + 18x^2 + x - 10.$

If $x = +1$, $x^4 - 10x^3 + 18x^2 + x - 10 = 1 - 10 + 18 + 1 - 10 = 0.$

Hence, $x - 1$ is a factor.

$$\therefore x^4 - 10x^3 + 18x^2 + x - 10 = (x - 1)(x^3 - 9x^2 + 9x + 10).$$

If $x = +1$, $x^3 - 9x^2 + 9x + 10 = 1 - 9 + 9 + 10 = +11.$
If $x = -1$, $x^3 - 9x^2 + 9x + 10 = -1 - 9 - 9 + 10 = -9.$
If $x = +2$, $x^3 - 9x^2 + 9x + 10 = 8 - 36 + 18 + 10 = 0.$

Hence, $x - 2$ is a factor.

$$\therefore x^3 - 9x^2 + 9x + 10 = (x - 2)(x^2 - 7x - 5).$$

If $x = +5$, $x^2 - 7x - 5 = 25 - 35 - 5 = -15.$
If $x = -5$, $x^2 - 7x - 5 = 25 + 35 - 5 = +55.$

Hence, $x^2 - 7x - 5$ has no rational integral factors.

$$\therefore x^4 - 10x^3 + 18x^2 + x - 10 = (x - 1)(x - 2)(x^2 - 7x - 5).$$

137. $6a^2 - 9ay - 15by - 20bx - 12ax + 10ab$

$$= (6a^2 + 10ab) - (12ax + 20bx) - (9ay + 15by)$$
$$= 2a(3a + 5b) - 4x(3a + 5b) - 3y(3a + 5b)$$
$$= (3a + 5b)(2a - 4x - 3y).$$

138. $x^5 - 6x^4 - 3x^3 + 88x^2 - 204x + 144.$

If $x = +1$, $x^5 - 6x^4 - 3x^3 + 88x^2 - 204x + 144$
$= 1 - 6 - 3 + 88 - 204 + 144 = +20.$

If $x = -1$, $x^5 - 6x^4 - 3x^3 + 88x^2 - 204x + 144$
$= -1 - 6 + 3 + 88 + 204 + 144 = +432.$

If $x = +2$, $x^5 - 6x^4 - 3x^3 + 88x^2 - 204x + 144$
$= 32 - 96 - 24 + 352 - 408 + 144 = 0.$

Hence, $x - 2$ is a factor.

$\therefore x^5 - 6x^4 - 3x^3 + 88x^2 - 204x + 144$
$= (x - 2)(x^4 - 4x^3 - 11x^2 + 66x - 72).$

If $x = +2$, $x^4 - 4x^3 - 11x^2 + 66x - 72$
$= 16 - 32 - 44 + 132 - 72 = 0.$

Hence, $x - 2$ is again a factor.

$\therefore x^4 - 4x^3 - 11x^2 + 66x - 72 = (x - 2)(x^3 - 2x^2 - 15x + 36).$

If $x = +2$, $x^3 - 2x^2 - 15x + 36 = 8 - 8 - 30 + 36 = +6.$
If $x = -2$, $x^3 - 2x^2 - 15x + 36 = -8 - 8 + 30 + 36 = +50.$
If $x = +3$, $x^3 - 2x^2 - 15x + 36 = 27 - 18 - 45 + 36 = 0.$

Hence, $x - 3$ is a factor.

$$\therefore x^3 - 2x^2 - 15x + 36 = (x - 3)(x^2 + x - 12)$$
$$= (x - 3)(x - 3)(x + 4).$$
$$\therefore x^5 - 6x^4 - 3x^3 + 88x^2 - 204x + 144$$
$$= (x - 2)(x - 2)(x - 3)(x - 3)(x + 4).$$

139. $25a^2 - 36x^2 + 121b^2 - 49y^2 - 110ab - 84xy$
$= (25a^2 - 110ab + 121b^2) - (36x^2 + 84xy + 49y^2)$
$= (5a - 11b)^2 - (6x + 7y)^2$
$= (5a - 11b + 6x + 7y)(5a - 11b - 6x - 7y).$

Exercise 50. Page 110

1. Find the H.C.F. of $27x^3$ and $45x^4$.

$$27x^3 = 3^3 \times x^3.$$
$$45x^4 = 3^2 \times 5 \times x^4.$$
$$\therefore \text{the H.C.F.} = 3^2 \times x^3 = 9x^3.$$

2. Find the H.C.F. of $50a^2b^3$ and $75a^3b^2$.

$$50a^2b^3 = 2 \times 5^2 \times a^2b^3.$$
$$75a^3b^2 = 3 \times 5^2 \times a^3b^2.$$
$$\therefore \text{the H.C.F.} = 5^2 \times a^2b^2 = 25a^2b^2.$$

3. Find the H.C.F. of $24a^3x^4$ and $36ax^3$.

$$24a^3x^4 = 2^3 \times 3 \times a^3x^4.$$
$$36ax^3 = 2^2 \times 3^2 \times ax^3.$$
$$\therefore \text{ the H.C.F.} = 2^2 \times 3 \times ax^3 = 12ax^3.$$

4. Find the H.C.F. of $108a^2c^4$ and $84ac^3$.

$$108a^2c^4 = 2^2 \times 3^3 \times a^2c^4.$$
$$84ac^3 = 2^2 \times 3 \times 7 \times ac^3.$$
$$\therefore \text{ the H.C.F.} = 2^2 \times 3 \times ac^3 = 12ac^3.$$

5. Find the H.C.F. of $99x^4y^4$ and $66x^3y^2$.

$$99x^4y^4 = 3^2 \times 11 \times x^4y^4.$$
$$66x^3y^2 = 2 \times 3 \times 11 \times x^3y^2.$$
$$\therefore \text{ the H.C.F.} = 3 \times 11 \times x^3y^2 = 33x^3y^2.$$

6. Find the H.C.F. of $144x^5$ and $108x^7$.

$$144x^5 = 2^4 \times 3^2 \times x^5.$$
$$108x^7 = 2^2 \times 3^3 \times x^7.$$
$$\therefore \text{ the H.C.F.} = 2^2 \times 3^2 \times x^5 = 36x^5.$$

7. Find the H.C.F. of $x^2 - y^2$ and $x + y$.

$$x^2 - y^2 = (x + y)(x - y).$$
$$x + y = x + y.$$
$$\therefore \text{ the H.C.F.} = x + y.$$

8. Find the H.C.F. of $6(x + y)^2$ and $9(x + y)^3$.

$$6(x + y)^2 = 2 \times 3 \times (x + y)^2.$$
$$9(x + y)^3 = 3^2 \times (x + y)^3.$$
$$\therefore \text{ the H.C.F.} = 3(x + y)^2.$$

9. Find the H.C.F. of $a + b$ and $a^3 + b^3$.

$$a + b = a + b.$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$
$$\therefore \text{ the H.C.F.} = a + b.$$

10. Find the H.C.F. of $a^3 - 8x^3$ and $a - 2x$.

$$a^3 - 8x^3 = (a - 2x)(a^2 + 2ax + 4x^2).$$
$$a - 2x = a - 2x.$$
$$\therefore \text{ the H.C.F.} = a - 2x.$$

11. Find the H.C.F. of $x^2 - 9$ and $x^2 - 4x + 3$.

$$x^2 - 9 = (x + 3)(x - 3).$$
$$x^2 - 4x + 3 = (x - 1)(x - 3).$$
$$\therefore \text{ the H.C.F.} = x - 3.$$

12. Find the H.C.F. of $a^2 - 16$ and $a^2 - 2a - 8$.

$$a^2 - 16 = (a + 4)(a - 4).$$
$$a^2 - 2a - 8 = (a + 2)(a - 4).$$
$$\therefore \text{ the H.C.F.} = a - 4.$$

13. Find the H.C.F. of $x^2 - 7x + 12$ and $x^2 - 5x + 6$.

$$x^2 - 7x + 12 = (x - 3)(x - 4).$$ 2
$$x^2 - 5x + 6 = (x - 2)(x - 3).$$
$$\therefore \text{ the H.C.F.} = x - 3.$$

14. Find the H.C.F. of $a^2 + a - 6$ and $a^2 + 8a + 15$.

$$a^2 + a - 6 = (a + 3)(a - 2).$$
$$a^2 + 8a + 15 = (a + 3)(a + 5).$$
$$\therefore \text{ the H.C.F.} = a + 3.$$

15. Find the H.C.F. of $a^2 + ab - 12b^2$ and $a^2 + 6ab + 8b^2$.

$$a^2 + ab - 12b^2 = (a + 4b)(a - 3b).$$
$$a^2 + 6ab + 8b^2 = (a + 2b)(a + 4b).$$
$$\therefore \text{ the H.C.F.} = a + 4b.$$

16. Find the H.C.F. of $x^3 + ax^2 - 6a^2x$ and $x^4 + 5ax^3 + 6a^2x^2$.

$$x^3 + ax^2 - 6a^2x = x(x^2 + ax - 6a^2) = x(x + 3a)(x - 2a).$$
$$x^4 + 5ax^3 + 6a^2x^2 = x^2(x^2 + 5ax + 6a^2) = x^2(x + 2a)(x + 3a).$$
$$\therefore \text{ the H.C.F.} = x(x + 3a).$$

17. Find the H.C.F. of $8x^2 - 2xy - 3y^2$ and $12x^2 - xy - 6y^2$.

$$8x^2 - 2xy - 3y^2 = (2x + y)(4x - 3y).$$ 1
$$12x^2 - xy - 6y^2 = (3x + 2y)(4x - 3y).$$
$$\therefore \text{ the H.C.F.} = 4x - 3y.$$

18. Find the H.C.F. of $6a^2 - ab - 2b^2$ and $4a^2 - 4ab - 3b^2$.

$$6a^2 - ab - 2b^2 = (2a + b)(3a - 2b).$$
$$4a^2 - 4ab - 3b^2 = (2a + b)(2a - 3b).$$
$$\therefore \text{ the H.C.F.} = 2a + b.$$

19. Find the H.C.F. of $12a^2 - 23ax + 10x^2$ and $8a^2 - 22ax + 15x^2$.

$$12a^2 - 23ax + 10x^2 = (3a - 2x)(4a - 5x).$$
$$8a^2 - 22ax + 15x^2 = (2a - 3x)(4a - 5x).$$
$$\therefore \text{ the H.C.F.} = 4a - 5x.$$

20. Find the H.C.F. of $12x^2 - 34x + 14$ and $42a - 4ax - 6ax^2$.

$$12x^2 - 34x + 14 = 2(6x^2 - 17x + 7) = 2(2x - 1)(3x - 7).$$
$$42a - 4ax - 6ax^2 = 2a(21 - 2x - 3x^2) = 2a(7 - 3x)(3 + x).$$
$$\therefore \text{ the H.C.F.} = 2(3x - 7).$$

21. Find the H.C.F. of $x^4 - a^4$; $x^3 + a^3$; $x^2 - a^2$.

$$\begin{aligned}
x^4 - a^4 &= (x + a)(x - a)(x^2 + a^2).\\
x^3 + a^3 &= (x + a)(x^2 - ax + a^2).\\
x^2 - a^2 &= (x + a)(x - a).\\
\therefore \text{ the H.C.F.} &= x + a.
\end{aligned}$$

22. Find the H.C.F. of $x^4 + a^3x - ax^3 - a^4$ and $x^3 - a^3$.

$$\begin{aligned}
x^4 + a^3x - ax^3 - a^4 &= (x - a)(x^3 + a^3)\\
&= (x - a)(x + a)(x^2 - ax + a^2).\\
x^3 - a^3 &= (x - a)(x^2 + ax + a^2).\\
\therefore \text{ the H.C.F.} &= x - a.
\end{aligned}$$

23. Find the H.C.F. of $2x^4 - 11x^2y^2 + 12y^4$ and $3x^6 - 48x^2y^4$.

$$\begin{aligned}
2x^4 - 11x^2y^2 + 12y^4 &= (x^2 - 4y^2)(2x^2 - 3y^2)\\
&= (x + 2y)(x - 2y)(2x^2 - 3y^2).\\
3x^6 - 48x^2y^4 &= 3x^2(x^4 - 16y^4)\\
&= 3x^2(x + 2y)(x - 2y)(x^2 + 4y^2).\\
\therefore \text{ the H.C.F.} &= (x + 2y)(x - 2y).
\end{aligned}$$

24. Find the H.C.F. of $36x^6 - 18x^5 - 27x^4 + 9x^3$
and $27x^2y^2 - 18x^4y^2 - 9x^3y^2$.

$$\begin{aligned}
36x^6 - 18x^5 - 27x^4 + 9x^3 &= 9x^3(4x^3 - 2x^2 - 3x + 1)\\
&= 9x^3(x - 1)(4x^2 + 2x - 1).\\
27x^2y^2 - 18x^4y^2 - 9x^3y^2 &= 9x^2y^2(3 - 2x^2 - x)\\
&= 9x^2y^2(1 - x)(3 + 2x).\\
\therefore \text{ the H.C.F.} &= 9x^2(x - 1).
\end{aligned}$$

25. Find the H.C.F. of $x^5 + y^5$; $x^4 - y^4$; $x^3 + y^3$.

$$\begin{aligned}
x^5 + y^5 &= (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).\\
x^4 - y^4 &= (x + y)(x - y)(x^2 + y^2).\\
x^3 + y^3 &= (x + y)(x^2 - xy + y^2).\\
\therefore \text{ the H.C.F.} &= x + y.
\end{aligned}$$

26. Find the H.C.F. of $x^2 - 2a^2 - ax$; $x^2 - 6a^2 + ax$; $x^2 - 8a^2 + 2ax$.

$$\begin{aligned}
x^2 - 2a^2 - ax &= (x + a)(x - 2a).\\
x^2 - 6a^2 + ax &= (x - 2a)(x + 3a).\\
x^2 - 8a^2 + 2ax &= (x - 2a)(x + 4a).\\
\therefore \text{ the H.C.F.} &= x - 2a.
\end{aligned}$$

27. Find the H.C.F. of $30a^2b^4 - 25a^3b^3 + 5a^4b^2$ and $9ab^3 - 9a^2b^2 + 2a^3b$.

$$\begin{aligned} 30a^2b^4 - 25a^3b^3 + 5a^4b^2 &= 5a^2b^2(6b^2 - 5ab + a^2) \\ &= 5a^2b^2(2b - a)(3b - a). \\ 9ab^3 - 9a^2b^2 + 2a^3b &= ab(9b^2 - 9ab + 2a^2) \\ &= ab(3b - a)(3b - 2a). \\ \therefore \text{ the H.C.F. } &= ab(3b - a). \end{aligned}$$

28. Find the H.C.F. of $x^3 + 3x^2y + 3xy^2 + y^3$ and $x^2 + 2xy + y^2$.

$$\begin{aligned} x^3 + 3x^2y + 3xy^2 + y^3 &= (x + y)^3. \\ x^2 + 2xy + y^2 &= (x + y)^2. \\ \therefore \text{ the H.C.F. } &= (x + y)^2. \end{aligned}$$

29. Find the H.C.F. of $6x^4y - 13x^3y^2 + 6x^2y^3$ and $8x^3y^2 - 14x^2y^3 + 3xy^4$.

$$\begin{aligned} 6x^4y - 13x^3y^2 + 6x^2y^3 &= x^2y(6x^2 - 13xy + 6y^2) \\ &= x^2y(3x - 2y)(2x - 3y). \\ 8x^3y^2 - 14x^2y^3 + 3xy^4 &= xy^2(8x^2 - 14xy + 3y^2) \\ &= xy^2(4x - y)(2x - 3y). \\ \therefore \text{ the H.C.F. } &= xy(2x - 3y). \end{aligned}$$

Exercise 51. Page 112

1. Find the L.C.M. of a^3b^2 and a^2b^4.

$$\begin{aligned} a^3b^2 &= a^3 \times b^2. \\ a^2b^4 &= a^2 \times b^4. \\ \therefore \text{ the L.C.M. } &= a^3b^4. \end{aligned}$$

2. Find the L.C.M. of $5xy^2$ and $3x^2y$.

$$\begin{aligned} 5xy^2 &= 5 \times x \times y^2. \\ 3x^2y &= 3 \times x^2 \times y. \\ \therefore \text{ the L.C.M. } &= 3 \times 5 \times x^2 \times y^2 \\ &= 15x^2y^2. \end{aligned}$$

3. Find the L.C.M. of $6a^2c^3$ and $9a^3c^4$.

$$\begin{aligned} 6a^2c^3 &= 2 \times 3 \times a^2 \times c^3. \\ 9a^3c^4 &= 3^2 \times a^3 \times c^4. \\ \therefore \text{ the L.C.M. } &= 2 \times 3^2 \times a^3 \times c^4 \\ &= 18a^3c^4. \end{aligned}$$

4. Find the L.C.M. of $28x^2y^3$ and $35xy^4$.

$$\begin{aligned} 28x^2y^3 &= 2^2 \times 7 \times x^2 \times y^3. \\ 35xy^4 &= 5 \times 7 \times x \times y^4. \\ \therefore \text{ the L.C.M. } &= 2^2 \times 5 \times 7 \times x^2 \times y^4 \\ &= 140x^2y^4. \end{aligned}$$

5. Find the L.C.M. of $49a^4b^3$ and $56a^2b^2$.

$$\begin{aligned} 49a^4b^3 &= 7^2 \times a^4 \times b^3. \\ 56a^2b^2 &= 2^3 \times 7 \times a^2 \times b^2. \\ \therefore \text{the L.C.M.} &= 2^3 \times 7^2 \times a^4 \times b^3 \\ &= 392a^4b^3. \end{aligned}$$

6. Find the L.C.M. of $25a^6b^9$ and $60a^3b^5$.

$$\begin{aligned} 25a^6b^9 &= 5^2 \times a^6 \times b^9. \\ 60a^3b^5 &= 2^2 \times 3 \times 5 \times a^3 \times b^5. \\ \therefore \text{the L.C.M.} &= 2^2 \times 3 \times 5^2 \times a^6 \times b^9 \\ &= 300a^6b^9. \end{aligned}$$

7. Find the L.C.M. of $x-1$ and $x+1$.

$$\begin{aligned} x-1 &= x-1. \\ x+1 &= x+1. \\ \therefore \text{the L.C.M.} &= (x-1)(x+1) \\ &= x^2-1. \end{aligned}$$

8. Find the L.C.M. of a^2+2b and a^2-2b.

$$\begin{aligned} a^2+2b &= a^2+2b. \\ a^2-2b &= a^2-2b. \\ \therefore \text{the L.C.M.} &= (a^2+2b)(a^2-2b) \\ &= a^4-4b^2. \end{aligned}$$

9. Find the L.C.M. of a^3 and a^3+a^2.

$$\begin{aligned} a^3 &= a^3. \\ a^3+a^2 &= a^2(a+1). \\ \therefore \text{the L.C.M.} &= a^3(a+1). \end{aligned}$$

10. Find the L.C.M. of x^2+xy and $xy+y^2$.

$$\begin{aligned} x^2+xy &= x(x+y). \\ xy+y^2 &= y(x+y). \\ \therefore \text{the L.C.M.} &= xy(x+y). \end{aligned}$$

11. Find the L.C.M. of $1-y^2$ and $1+y^3$.

$$\begin{aligned} 1-y^2 &= (1+y)(1-y). \\ 1+y^3 &= (1+y)(1-y+y^2). \\ \therefore \text{the L.C.M.} &= (1+y)(1-y)(1-y+y^2). \end{aligned}$$

12. Find the L.C.M. of a^3-b^3 and a^2-b^2.

$$\begin{aligned} a^3-b^3 &= (a-b)(a^2+ab+b^2). \\ a^2-b^2 &= (a-b)(a+b). \\ \therefore \text{the L.C.M.} &= (a-b)(a+b)(a^2+ab+b^2). \end{aligned}$$

13. Find the L.C.M. of $x^2-7x+12$ and x^2-5x+6.

$$\begin{aligned} x^2-7x+12 &= (x-3)(x-4). \\ x^2-5x+6 &= (x-3)(x-2). \\ \therefore \text{the L.C.M.} &= (x-2)(x-3)(x-4). \end{aligned}$$

14. Find the L.C.M. of x^2-x-6 and x^2+6x+8.

$$\begin{aligned} x^2-x-6 &= (x+2)(x-3). \\ x^2+6x+8 &= (x+2)(x+4). \\ \therefore \text{the L.C.M.} &= (x+2)(x-3)(x+4). \end{aligned}$$

15. Find the L.C.M. of $x^2 + 15x + 36$ and $x^2 - 6x - 27$.

$$\begin{aligned} x^2 + 15x + 36 &= (x + 3)(x + 12).\\ x^2 - 6x - 27 &= (x + 3)(x - 9).\\ \therefore \text{ the L.C.M.} &= (x + 3)(x - 9)(x + 12). \end{aligned}$$

16. Find the L.C.M. of $x^4y - 2x^3y^2 - 35x^2y^3$ and $x^3y^2 - 5x^2y^3 - 14xy^4$.

$$\begin{aligned} x^4y - 2x^3y^2 - 35x^2y^3 &= x^2y(x^2 - 2xy - 35y^2)\\ &= x^2y(x + 5y)(x - 7y).\\ x^3y^2 - 5x^2y^3 - 14xy^4 &= xy^2(x^2 - 5xy - 14y^2)\\ &= xy^2(x + 2y)(x - 7y).\\ \therefore \text{ the L.C.M.} &= x^2y^2(x + 2y)(x + 5y)(x - 7y). \end{aligned}$$

17. Find the L.C.M. of $x^2 + ax - 6a^2$ and $x^2 + 2ax - 8a^2$.

$$\begin{aligned} x^2 + ax - 6a^2 &= (x - 2a)(x + 3a).\\ x^2 + 2ax - 8a^2 &= (x - 2a)(x + 4a).\\ \therefore \text{ the L.C.M.} &= (x - 2a)(x + 3a)(x + 4a). \end{aligned}$$

18. Find the L.C.M. of $6x^2 + xy - 12y^2$ and $6x^2 - 17xy + 12y^2$.

$$\begin{aligned} 6x^2 + xy - 12y^2 &= (3x - 4y)(2x + 3y).\\ 6x^2 - 17xy + 12y^2 &= (3x - 4y)(2x - 3y).\\ \therefore \text{ the L.C.M.} &= (3x - 4y)(2x + 3y)(2x - 3y). \end{aligned}$$

19. Find the L.C.M. of $15a^2 + 2ab - 24b^2$ and $10a^2 + 13ab - 30b^2$.

$$\begin{aligned} 15a^2 + 2ab - 24b^2 &= (3a + 4b)(5a - 6b).\\ 10a^2 + 13ab - 30b^2 &= (2a + 5b)(5a - 6b).\\ \therefore \text{ the L.C.M.} &= (2a + 5b)(3a + 4b)(5a - 6b). \end{aligned}$$

20. Find the L.C.M. of

$36a^3b + 39a^2b^2 - 42ab^3$ and $48a^2b + 156ab^2 + 126b^3$.

$$\begin{aligned} 36a^3b + 39a^2b^2 - 42ab^3 &= 3ab(12a^2 + 13ab - 14b^2)\\ &= 3ab(3a - 2b)(4a + 7b).\\ 48a^2b + 156ab^2 + 126b^3 &= 6b(8a^2 + 26ab + 21b^2)\\ &= 6b(2a + 3b)(4a + 7b).\\ \therefore \text{ the L.C.M.} &= 6ab(2a + 3b)(3a - 2b)(4a + 7b) \end{aligned}$$

21. Find the L.C.M. of $a^3 - x^3$; $a - x$; $a^2 - 2ax + x^2$; $a^2 - x^2$.

$$\begin{aligned} a^3 - x^3 &= (a - x)(a^2 + ax + x^2).\\ a - x &= (a - x).\\ a^2 - 2ax + x^2 &= (a - x)^2.\\ a^2 - x^2 &= (a - x)(a + x).\\ \therefore \text{ the L.C.M.} &= (a - x)^2(a + x)(a^2 + ax + x^2). \end{aligned}$$

22. Find the L.C.M. of $(x^2+2ax)^2$; x^3-4a^2x; $(x-a)^3$; $(x^2+ax-2a^2)^2$.

$$\begin{aligned}
(x^2+2ax)^2 &= x^2(x+2a)^2.\\
x^3-4a^2x &= x(x^2-4a^2)\\
&= x(x+2a)(x-2a).\\
(x-a)^3 &= (x-a)^3.\\
(x^2+ax-2a^2)^2 &= (x-a)^2(x+2a)^2.\\
\therefore \text{ the L.C.M.} &= x^2(x-a)^3(x+2a)^2(x-2a).
\end{aligned}$$

23. Find the L.C.M. of $x(xy-y^2)$; $x^4y^2-x^2y^4$; $y(x^2-xy)^2$.

$$\begin{aligned}
x(xy-y^2) &= xy(x-y).\\
x^4y^2-x^2y^4 &= x^2y^2(x^2-y^2)\\
&= x^2y^2(x+y)(x-y).\\
y(x^2-xy)^2 &= x^2y(x-y)^2.\\
\therefore \text{ the L.C.M.} &= x^2y^2(x+y)(x-y)^2.
\end{aligned}$$

24. Find the L.C.M. of

$$x^4+ax^3-a^2x^2+2a^3x \text{ and } 2x^3+6ax^2+2a^2x-4a^3.$$

$$\begin{aligned}
x^4+ax^3-a^2x^2+2a^3x &= x(x^3+ax^2-a^2x+2a^3)\\
&= x(x+2a)(x^2-ax+a^2).\\
2x^3+6ax^2+2a^2x-4a^3 &= 2(x^3+3ax^2+a^2x-2a^3)\\
&= 2(x+2a)(x^2+ax-a^2).\\
\therefore \text{ the L.C.M.} &= 2x(x+2a)(x^2-ax+a^2)(x^2+ax-a^2).
\end{aligned}$$

25. Find the L.C.M. of

$$25x^3y-20x^2y+4xy;\ 10x^2-29x+10;\ 125x^3-8.$$

$$\begin{aligned}
25x^3y-20x^2y+4xy &= xy(25x^2-20x+4)\\
&= xy(5x-2)^2.\\
10x^2-29x+10 &= (2x-5)(5x-2).\\
125x^3-8 &= (5x-2)(25x^2+10x+4).\\
\therefore \text{ the L.C.M.} &= xy(2x-5)(5x-2)^2(25x^2+10x+4).
\end{aligned}$$

26. Find the L.C.M. of

$$x^3-8y^3;\ x^2-4y^2;\ x^2+3xy+2y^2;\ x^2-xy-2y^2.$$

$$\begin{aligned}
x^3-8y^3 &= (x-2y)(x^2+2xy+4y^2).\\
x^2-4y^2 &= (x-2y)(x+2y).\\
x^2+3xy+2y^2 &= (x+y)(x+2y).\\
x^2-xy-2y^2 &= (x+y)(x-2y).\\
\therefore \text{ the L.C.M.} &= (x+y)(x+2y)(x-2y)(x^2+2xy+4y^2).
\end{aligned}$$

27. Find the L.C.M. of

$$12x^3-16x^2-16x;\ 12x^3+6x^2-36x;\ 12x^3+24x^2+4x+8.$$

$$\begin{aligned}
12x^3-16x^2-16x &= 4x(3x^2-4x-4)\\
&= 4x(x-2)(3x+2).
\end{aligned}$$

$$\begin{aligned}
12x^3 + 6x^2 - 36x &= 6x(2x^2 + x - 6)\\
&= 6x(x+2)(2x-3).\\
12x^3 + 24x^2 + 4x + 8 &= 4(3x^3 + 6x^2 + x + 2)\\
&= 4[3x^2(x+2) + (x+2)]\\
&= 4(x+2)(3x^2+1).\\
\therefore \text{ the L.C.M.} &= 12x(x+2)(x-2)(2x-3)(3x+2)(3x^2+1).
\end{aligned}$$

Exercise 52. Page 119

1. Find the H.C.F. and the L.C.M. of

$$6x^4 - 13x^3 + 6x^2;\ 8x^4 - 36x^3 + 54x^2 - 27x.$$

$$\begin{array}{r|r|l}
x^2)\,\underline{6x^4 - 13x^3 + 6x^2} & x)\,\underline{8x^4 - 36x^3 + 54x^2 - 27x} & \\
6x^2 - 13x + 6 & 8x^3 - 36x^2 + 54x - 27 & \\
\underline{6x^2 - 9x\phantom{{}+6}} & \underline{3} & \\
-4x + 6 & 24x^3 - 108x^2 + 162x - 81 & 4x \\
\underline{-4x + 6} & \underline{24x^3 - 52x^2 + 24x\phantom{{}-81}} & \\
 & -56x^2 + 138x - 81 & \\
 & \underline{3} & \\
 & -168x^2 + 414x - 243 & -28 \\
 & \underline{-168x^2 + 364x - 168} & \\
 & 25)\,\underline{50x - 75} & \\
 & 2x - 3 & 3x - 2
\end{array}$$

∴ the H.C.F. $= x(2x-3)$.

$$\begin{aligned}
6x^4 - 13x^3 + 6x^2 &= x^2(2x-3)(3x-2).\\
8x^4 - 36x^3 + 54x^2 - 27x &= x(2x-3)(4x^2 - 12x + 9)\\
&= x(2x-3)(2x-3)^2\\
&= x(2x-3)^3.
\end{aligned}$$

∴ the L.C.M. $= x^2(2x-3)^3(3x-2)$.

2. Find the H.C.F. and the L.C.M. of

$$6x^3 - 7x^2 - 16x + 12;\ 4x^3 - 8x^2 - 9x + 18.$$

$$\begin{array}{r|r|l}
4x^3 - 8x^2 - 9x + 18 & 6x^3 - 7x^2 - 16x + 12 & \\
\underline{4x^3 - 2x^2 - 12x\phantom{{}+18}} & \underline{2} & \\
-6x^2 + 3x + 18 & 12x^3 - 14x^2 - 32x + 24 & 3 \\
\underline{-6x^2 + 3x + 18} & \underline{12x^3 - 24x^2 - 27x + 54} & \\
 & 5)\,\underline{10x^2 - 5x - 30} & \\
 & 2x^2 - x - 6 & 2x - 3
\end{array}$$

∴ the H.C.F. $= 2x^2 - x - 6 = (x-2)(2x+3)$.

$$\begin{aligned}6x^3-7x^2-16x+12 &= (2x^2-x-6)(3x-2)\\ &= (x-2)(2x+3)(3x-2).\\ 4x^3-8x^2-9x+18 &= (2x^2-x-6)(2x-3)\\ &= (x-2)(2x+3)(2x-3).\end{aligned}$$

$\therefore$ the L.C.M. $=(x-2)(2x+3)(2x-3)(3x-2)$.

3. Find the H.C.F. and the L.C.M. of

$$x^3-6x^2+11x-6;\ x^3-9x^2+26x-24.$$

$$\begin{array}{r|r|l}
x^3-6x^2+11x-6 & x^3-9x^2+26x-24 & 1\\
\underline{x^3-5x^2+6x} & \underline{x^3-6x^2+11x-6} & \\
-x^2+5x-6 & -3\overline{)\underline{-3x^2+15x-18}} & \\
\underline{-x^2+5x-6} & x^2-5x+6 & x-1
\end{array}$$

$\therefore$ the H.C.F. $=x^2-5x+6=(x-2)(x-3)$.

$$\begin{aligned}x^3-6x^2+11x-6 &= (x^2-5x+6)(x-1)\\ &= (x-1)(x-2)(x-3).\\ x^3-9x^2+26x-24 &= (x^2-5x+6)(x-4)\\ &= (x-2)(x-3)(x-4).\end{aligned}$$

$\therefore$ the L.C.M. $=(x-1)(x-2)(x-3)(x-4)$.

4. Find the H.C.F. and the L.C.M. of

$$4x^4+7x^2+16;\ 4x^4+12x^3+9x^2-16.$$

$$\begin{array}{r|r|l}
4x^4+7x^2+16 & 4x^4+12x^3+9x^2-16 & 1\\
\underline{3} & \underline{4x^4+7x^2+16} & \\
12x^4+21x^2+48 & 2\overline{)\underline{12x^3+2x^2-32}} & \\
\underline{12x^4+2x^3-32x} & 6x^3+x^2-16 & 2x-1\\
-2x^3+21x^2+32x+48 & \underline{6x^3+9x^2+12x} & 3x-4\\
\underline{3} & -8x^2-12x-16 & \\
-6x^3+63x^2+96x+144 & \underline{-8x^2-12x-16} & \\
\underline{-6x^3-x^2+16} & & \\
32\overline{)\underline{64x^2+96x+128}} & & \\
2x^2+3x+4 & &
\end{array}$$

$\therefore$ the H.C.F. $=2x^2+3x+4$.

$$\begin{aligned}4x^4+7x^2+16 &= (2x^2+3x+4)(2x^2-3x+4).\\ 4x^4+12x^3+9x^2-16 &= (2x^2+3x+4)(2x^2+3x-4).\end{aligned}$$

$\therefore$ the L.C.M. $=(2x^2+3x+4)(2x^2-3x+4)(2x^2+3x-4)$.

5. Find the H.C.F. and the L.C.M. of

$$2x^3+x^2-4x-3;\ 2x^3+x^2-9.$$

$$\begin{array}{r|r|l}
2x^3+x^2-4x-3 & 2x^3+x^2\qquad -9 & 1 \\
\underline{2x^3-3x^2\qquad\qquad} & \underline{2x^3+x^2-4x-3} & \\
4x^2-4x\quad & 2\underline{)4x-6} & \\
\underline{4x^2-6x\quad} & 2x-3 & x^2+2x+1 \\
2x-3 & & \\
\underline{2x-3} & &
\end{array}$$

$\therefore$ the H.C.F. $=2x-3$.

$$\begin{aligned}2x^3+x^2-4x-3&=(2x-3)(x^2+2x+1)\\&=(2x-3)(x+1)^2.\\2x^3+x^2-9&=(2x-3)(x^2+2x+3).\end{aligned}$$

$\therefore$ the L.C.M. $=(2x-3)(x+1)^2(x^2+2x+3)$.

6. Find the H.C.F. and the L.C.M. of

$$n^3+4n^2+5n+2;\ n^3+2n^2-n-2.$$

$$\begin{array}{r|r|l}
n^3+4n^2+5n+2 & n^3+2n^2-\quad n-2 & 1 \\
\underline{n^3+3n^2+2n\qquad} & \underline{n^3+4n^2+5n+2} & \\
n^2+3n+2 & -2\underline{)-2n^2-6n-4} & \\
\underline{n^2+3n+2} & n^2+3n+2 & n+1
\end{array}$$

$\therefore$ the H.C.F. $=n^2+3n+2=(n+1)(n+2)$.

$$\begin{aligned}n^3+4n^2+5n+2&=(n^2+3n+2)(n+1)\\&=(n+1)^2(n+2).\\n^3+2n^2-n-2&=(n^2+3n+2)(n-1)\\&=(n-1)(n+1)(n+2).\end{aligned}$$

$\therefore$ the L.C.M. $=(n+1)^2(n-1)(n+2)$.

7. Find the H.C.F. and the L.C.M. of

$$6x^3-13x^2+19x-7;\ 9x^3-27x^2+41x-28.$$

$$\begin{array}{r|r|l}
6x^3-13x^2+19x-7 & 9x^3-27x^2+41x-28 & \\
\underline{6x^3-10x^2+14x\qquad} & \underline{2\qquad\qquad\qquad\qquad\qquad} & \\
-\quad 3x^2+\quad 5x-7 & 18x^3-54x^2+82x-56 & 3 \\
\underline{-\quad 3x^2+\quad 5x-7} & \underline{18x^3-39x^2+57x-21} & \\
 & -5\underline{)-15x^2+25x-35} & \\
 & 3x^2-\quad 5x+\quad 7 & 2x-1
\end{array}$$

$\therefore$ the H.C.F. $=3x^2-5x+7$.

$$6x^3 - 13x^2 + 19x - 7 = (3x^2 - 5x + 7)(2x - 1).$$
$$9x^3 - 27x^2 + 41x - 28 = (3x^2 - 5x + 7)(3x - 4).$$

$\therefore$ the L.C.M. $= (2x - 1)(3x - 4)(3x^2 - 5x + 7)$.

8. Find the H.C.F. and the L.C.M. of

$$2x^3 - 9x^2 + 11x - 3;\ 4x^3 - 4x^2 - 5x + 3.$$

$$\begin{array}{r|r|l}
2x^3 - 9x^2 + 11x - 3 & 4x^3 - 4x^2 - 5x + 3 & 2 \\
\underline{7 \qquad\qquad\qquad\qquad} & \underline{4x^3 - 18x^2 + 22x - 6} & \\
14x^3 - 63x^2 + 77x - 21 & 14x^2 - 27x + 9 & x - 18 \\
\underline{14x^3 - 27x^2 + 9x \qquad} & \underline{14x^2 - 21x \qquad} & 7x - 3 \\
- 36x^2 + 68x - 21 & - 6x + 9 & \\
\underline{7 \qquad\qquad\qquad} & \underline{- 6x + 9} & \\
- 252x^2 + 476x - 147 & & \\
\underline{- 252x^2 + 486x - 162} & & \\
-5 \overline{)\, - 10x + 15} & & \\
2x - 3 & &
\end{array}$$

$\therefore$ the H.C.F. $= 2x - 3$.

$$2x^3 - 9x^2 + 11x - 3 = (2x - 3)(x^2 - 3x + 1).$$
$$4x^3 - 4x^2 - 5x + 3 = (2x - 3)(2x^2 + x - 1)$$
$$= (2x - 3)(x + 1)(2x - 1).$$

$\therefore$ the L.C.M. $= (x + 1)(2x - 1)(2x - 3)(x^2 - 3x + 1)$.

9. Find the H.C.F. and the L.C.M. of

$$x^3 - 3x^2y + 3xy^2 - 2y^3;\ x^3 + 3xy^2 - 2x^2y - 6y^3.$$

$$\begin{array}{r|r|l}
x^3 - 3x^2y + 3xy^2 - 2y^3 & x^3 - 2x^2y + 3xy^2 - 6y^3 & 1 \\
\underline{x^3 \qquad\qquad - 4xy^2 \qquad} & \underline{x^3 - 3x^2y + 3xy^2 - 2y^3} & \\
- 3x^2y + 7xy^2 - 2y^3 & y \overline{)\, x^2y \qquad\qquad - 4y^3} & \\
\underline{- 3x^2y \qquad\qquad + 12y^3} & x^2 \qquad\qquad - 4y^2 & x - 3y \\
7y^2 \overline{)\, 7xy^2 - 14y^3} & \underline{x^2 - 2xy \qquad\qquad} & x + 2y \\
x - 2y & 2xy - 4y^2 & \\
 & \underline{2xy - 4y^2} &
\end{array}$$

$\therefore$ the H.C.F. $= x - 2y$.

$$x^3 - 3x^2y + 3xy^2 - 2y^3 = (x - 2y)(x^2 - xy + y^2).$$
$$x^3 + 3xy^2 - 2x^2y - 6y^3 = (x - 2y)(x^2 + 3y^2).$$

$\therefore$ the L.C.M. $= (x - 2y)(x^2 - xy + y^2)(x^2 + 3y^2)$.

10. **Find the H.C.F. and the L.C.M. of**

$$7x^4 - 10x^3 + 3x^2 - 4x + 4;\ 8x^4 - 13x^3 + 5x^2 - 3x + 3.$$

$$\begin{array}{r|r|l}
7 - 10 + 3 - 4 + 4 & 8 - 13 + 5 - 3 + 3 & 1 \\
\underline{7 - 21 + 14 + 7 - 7} & \underline{7 - 10 + 3 - 4 + 4} & \\
11\,\underline{)11 - 11 - 11 + 11} & 1 - 3 + 2 + 1 - 1 & 7 \\
1 - 1 - 1 + 1 & \underline{1 - 1 - 1 + 1} & 1 - 2 \\
\underline{1 - 2 + 1} & - 2 + 3 + 0 - 1 & \\
1 - 2 + 1 & \underline{- 2 + 2 + 2 - 2} & \\
\underline{1 - 2 + 1} & 1 - 2 + 1 & 1 + 1
\end{array}$$

$\therefore$ the H.C.F. $= x^2 - 2x + 1 = (x - 1)^2$.

$$\begin{aligned}
7x^4 - 10x^3 + 3x^2 - 4x + 4 &= (x^2 - 2x + 1)(7x^2 + 4x + 4) \\
&= (x - 1)^2(7x^2 + 4x + 4). \\
8x^4 - 13x^3 + 5x^2 - 3x + 3 &= (x^2 - 2x + 1)(8x^2 + 3x + 3) \\
&= (x - 1)^2(8x^2 + 3x + 3).
\end{aligned}$$

$\therefore$ the L.C.M. $= (x - 1)^2(7x^2 + 4x - 4)(8x^2 + 3x + 3)$.

Factor Method

If $x = 1$, $7x^4 - 10x^3 + 3x^2 - 4x + 4 = 7 - 10 + 3 - 4 + 4 = 0$,

and $8x^4 - 13x^3 + 5x^2 - 3x + 3 = 8 - 13 + 5 - 3 + 3 = 0$.

$\therefore x - 1$ is a factor of each expression.

$\therefore 7x^4 - 10x^3 + 3x^2 - 4x + 4 = (x - 1)(7x^3 - 3x^2 - 4)$,

and $8x^4 - 13x^3 + 5x^2 - 3x + 3 = (x - 1)(8x^3 - 5x^2 - 3)$.

If $x = 1$, $7x^3 - 3x^2 - 4 = 7 - 3 - 4 = 0$,

and $8x^3 - 5x^2 - 3 = 8 - 5 - 3 = 0$.

$\therefore x - 1$ is again a factor of each expression.

$\therefore 7x^3 - 3x^2 - 4 = (x - 1)(7x^2 + 4x + 4)$,

and $8x^3 - 5x^2 - 3 = (x - 1)(8x^2 + 3x + 3)$.

Now $7x^2 + 4x + 4$ and $8x^2 + 3x + 3$ cannot be resolved into rational integral factors.

$\therefore$ the H.C.F. $= (x - 1)^2$,
and the L.C.M. $= (x - 1)^2(7x^2 + 4x + 4)(8x^2 + 3x + 3)$.

11. Find the H.C.F. and the L.C.M. of

$$x^4 - x^3y - x^2y^2 + xy^3;\ x^5 - x^4y - xy^4 + y^5.$$

$$\begin{aligned}
x^4 - x^3y - x^2y^2 + xy^3 &= x(x^3 - x^2y - xy^2 + y^3)\\
&= x[x^2(x-y) - y^2(x-y)]\\
&= x(x^2-y^2)(x-y)\\
&= x(x+y)(x-y)^2.\\
x^5 - x^4y - xy^4 + y^5 &= x^4(x-y) - y^4(x-y)\\
&= (x^4-y^4)(x-y)\\
&= (x^2+y^2)(x^2-y^2)(x-y)\\
&= (x^2+y^2)(x+y)(x-y)^2.
\end{aligned}$$

$\therefore$ the H.C.F. $= (x+y)(x-y)^2$.

$\therefore$ the L.C.M. $= x(x+y)(x-y)^2(x^2+y^2)$.

12. Find the H.C.F. and the L.C.M. of x^4+2x^2+9; $x^4-4x^3+8x-21$.

$$\begin{array}{r|r|l}
x^4 + 2x^2 + 9 & x^4 - 4x^3 + 8x - 21 & 1\\
\underline{2} & \underline{x^4 + 2x^2 + 9} & \\
2x^4 + 4x^2 + 18 & -2\overline{)-4x^3 - 2x^2 + 8x - 30} & \\
\underline{2x^4 + x^3 - 4x^2 + 15x} & 2x^3 + x^2 - 4x + 15 & x-1\\
-x^3 + 8x^2 - 15x + 18 & \underline{2x^3 - 4x^2 + 6x} & 2x+5\\
\underline{2} & 5x^2 - 10x + 15 & \\
-2x^3 + 16x^2 - 30x + 36 & \underline{5x^2 - 10x + 15} & \\
\underline{-2x^3 - x^2 + 4x - 15} & & \\
17\overline{)17x^2 - 34x + 51} & & \\
x^2 - 2x + 3 & &
\end{array}$$

$\therefore$ the H.C.F. $= x^2 - 2x + 3$.

$$x^4 + 2x^2 + 9 = (x^2 - 2x + 3)(x^2 + 2x + 3).$$

$$x^4 - 4x^3 + 8x - 21 = (x^2 - 2x + 3)(x^2 - 2x - 7).$$

$\therefore$ the L.C.M. $= (x^2 - 2x + 3)(x^2 + 2x + 3)(x^2 - 2x - 7)$.

13. Find the H.C.F. and the L.C.M. of

$$2x^4 + x^3 - 9x^2 + 8x - 2;\ 2x^4 - 7x^3 + 11x^2 - 8x + 2.$$

$$\begin{array}{r|r|l}
2 + 1 - 9 + 8 - 2 & 2 - 7 + 11 - 8 + 2 & 1\\
\underline{2 - 5 + 4 - 1} & \underline{2 + 1 - 9 + 8 - 2} & \\
6 - 13 + 9 - 2 & -4\overline{)-8 + 20 - 16 + 4} & \\
\underline{6 - 15 + 12 - 3} & 2 - 5 + 4 - 1 & 1+3\\
2 - 3 + 1 & \underline{2 - 3 + 1} & 1-1\\
 & -2 + 3 - 1 & \\
 & \underline{-2 + 3 - 1} &
\end{array}$$

$\therefore$ the H.C.F. $= 2x^2 - 3x + 1 = (x-1)(2x-1)$.

$$\begin{aligned}2x^4 + x^3 - 9x^2 + 8x - 2 &= (2x^2 - 3x + 1)(x^2 + 2x - 2)\\ &= (x-1)(2x-1)(x^2+2x-2).\\ 2x^4 - 7x^3 + 11x^2 - 8x + 2 &= (2x^2 - 3x + 1)(x^2 - 2x + 2)\\ &= (x-1)(2x-1)(x^2-2x+2).\end{aligned}$$

∴ the L.C.M. $= (x-1)(2x-1)(x^2+2x-2)(x^2-2x+2)$.

14. Find the H.C.F. and the L.C.M. of

$6x^4 - 10x^3 + x^2 - 31x - 21$; $6x^4 - 4x^3 + 3x^2 - 68x - 42$.

$$\begin{array}{r|r|l}
6 - 10 + 1 - 31 - 21 & 6 - 4 + 3 - 68 - 42 & 1\\
\underline{6 + 2 - 37 - 21\phantom{{}-21}} & \underline{6 - 10 + 1 - 31 - 21} & \\
-12 + 38 - 10 - 21 & 6 + 2 - 37 - 21 & 1 - 2\\
\underline{-12 - 4 + 74 + 42} & \underline{6 - 12 - 9\phantom{{}-21}} & 3 + 7\\
21\overline{)\,42 - 84 - 63} & 14 - 28 - 21 & \\
2 - 4 - 3 & \underline{14 - 28 - 21} &
\end{array}$$

∴ the H.C.F. $= 2x^2 - 4x - 3$.

$$\begin{aligned}6x^4 - 10x^3 + x^2 - 31x - 21 &= (2x^2 - 4x - 3)(3x^2 + x + 7).\\ 6x^4 - 4x^3 + 3x^2 - 68x - 42 &= (2x^2 - 4x - 3)(3x^2 + 4x + 14).\end{aligned}$$

∴ the L.C.M. $= (2x^2 - 4x - 3)(3x^2 + x + 7)(3x^2 + 4x + 14)$.

15. Find the H.C.F. and the L.C.M. of

$6x^3 - 7ax^2 - 11a^2x + 12a^3$; $8x^3 - 14ax^2 - 7a^2x + 15a^3$.

$$\begin{array}{r|r|l}
6x^3 - 7ax^2 - 11a^2x + 12a^3 & 8x^3 - 14ax^2 - 7a^2x + 15a^3 & \\
\underline{7} & \underline{3} & \\
42x^3 - 49ax^2 - 77a^2x + 84a^3 & 24x^3 - 42ax^2 - 21a^2x + 45a^3 & 4\\
\underline{42x^3 - 69ax^2 + 9a^2x\phantom{{}+84a^3}} & \underline{24x^3 - 28ax^2 - 44a^2x + 48a^3} & \\
20ax^2 - 86a^2x + 84a^3 & -a\overline{)\,-14ax^2 + 23a^2x - 3a^3} & \\
\underline{7} & 14x^2 - 23ax + 3a^2 & 3x + 10a\\
140ax^2 - 602a^2x + 588a^3 & \underline{14x^2 - 21ax\phantom{{}+3a^2}} & 7x - a\\
\underline{140ax^2 - 230a^2x + 30a^3} & -2ax + 3a^2 & \\
-186a^2\overline{)\,-372a^2x + 558a^3} & \underline{-2ax + 3a^2} & \\
2x - 3a & &
\end{array}$$

∴ the H.C.F. $= 2x - 3a$.

$$\begin{aligned}6x^3 - 7ax^2 - 11a^2x + 12a^3 &= (2x - 3a)(3x^2 + ax - 4a^2)\\ &= (2x-3a)(x-a)(3x+4a).\\ 8x^3 - 14ax^2 - 7a^2x + 15a^3 &= (2x - 3a)(4x^2 - ax - 5a^2)\\ &= (2x-3a)(x+a)(4x-5a).\end{aligned}$$

∴ the L.C.M. $= (x+a)(x-a)(2x-3a)(3x+4a)(4x-5a)$.

16. Find the H.C.F. and the L.C.M. of

$$42x^4 + 41x^3 - 9x^2 - 9x - 1;\ 35x^3 + 47x^2 + 13x + 1.$$

$$\begin{array}{r|r|l}
42 + 41 - 9 - 9 - 1 & 35 + 47 + 13 + 1 & 6 - 11 \\
\underline{5} & \underline{35 + 40 + 5} & 5 + 1 \\
210 + 205 - 45 - 45 - 5 & 7 + 8 + 1 & \\
\underline{210 + 282 + 78 + 6} & \underline{7 + 8 + 1} & \\
- 77 - 123 - 51 - 5 & & \\
\underline{5} & & \\
- 385 - 615 - 255 - 25 & & \\
\underline{- 385 - 517 - 143 - 11} & & \\
-14)\underline{- 98 - 112 - 14} & & \\
7 + 8 + 1 & &
\end{array}$$

$\therefore$ the H.C.F. $= 7x^2 + 8x + 1 = (x+1)(7x+1)$.

$$\begin{aligned}
42x^4 + 41x^3 - 9x^2 - 9x - 1 &= (x+1)(7x+1)(6x^2 - x - 1) \\
&= (x+1)(7x+1)(2x-1)(3x+1). \\
35x^3 + 47x^2 + 13x + 1 &= (x+1)(7x+1)(5x+1).
\end{aligned}$$

$\therefore$ the L.C.M. $= (x+1)(7x+1)(2x-1)(3x+1)(5x+1)$.

17. Find the H.C.F. and the L.C.M. of

$$c^5 - 2c^4 - 6c^3 + 4c^2 + 13c + 6;\ c^5 + 3c^4 - 8c^2 - 9c - 3.$$

$$\begin{array}{r|r|l}
1 - 2 - 6 + 4 + 13 + 6 & 1 + 3 + 0 - 8 - 9 - 3 & 1 \\
\underline{5} & \underline{1 - 2 - 6 + 4 + 13 + 6} & \\
5 - 10 - 30 + 20 + 65 + 30 & 5 + 6 - 12 - 22 - 9 & 1 + 8 \\
\underline{5 + 6 - 12 - 22 - 9} & \underline{5 + 15 + 15 + 5} & 5 - 9 \\
-2)\underline{- 16 - 18 + 42 + 74 + 30} & - 9 - 27 - 27 - 9 & \\
8 + 9 - 21 - 37 - 15 & \underline{- 9 - 27 - 27 - 9} & \\
\underline{5} & & \\
40 + 45 - 105 - 185 - 75 & & \\
\underline{40 + 48 - 96 - 176 - 72} & & \\
-3)\underline{- 3 - 9 - 9 - 3} & & \\
1 + 3 + 3 + 1 & &
\end{array}$$

$\therefore$ the H.C.F. $= c^3 + 3c^2 + 3c + 1 = (c+1)^3$.

$$\begin{aligned}
c^5 - 2c^4 - 6c^3 + 4c^2 + 13c + 6 &= (c+1)^3(c^2 - 5c + 6) \\
&= (c+1)^3(c-2)(c-3). \\
c^5 + 3c^4 - 8c^2 - 9c - 3 &= (c+1)^3(c^2 - 3).
\end{aligned}$$

$\therefore$ the L.C.M. $= (c+1)^3(c-2)(c-3)(c^2-3)$.

18. Find the H.C.F. and the L.C.M. of

$$x^3+8x^2+19x+12;\ x^3+9x^2+26x+24.$$

$$\begin{array}{r|r|l}
1+8+19+12 & 1+9+26+24 & 1 \\
\underline{1+7+12} & \underline{1+8+19+12} & \\
1+7+12 & 1+7+12 & 1+1 \\
\underline{1+7+12} & &
\end{array}$$

$\therefore$ the H.C.F. $= x^2+7x+12=(x+3)(x+4)$.

$$x^3+8x^2+19x+12=(x+3)(x+4)(x+1).$$
$$x^3+9x^2+26x+24=(x+3)(x+4)(x+2).$$

$\therefore$ the L.C.M. $=(x+1)(x+2)(x+3)(x+4)$.

19. Find the H.C.F. and the L.C.M. of

$$6x^4-x^3-x^2+17x-6;\ 9x^4-30x^3+46x^2-41x+10.$$

$$\begin{array}{r|r|l}
6-1-1+17-6 & 9-30+46-41+10 & \\
\underline{6-10+14-4} & \underline{2} & \\
9-15+21-6 & 18-60+92-82+20 & 3 \\
\underline{9-15+21-6} & \underline{18-3-3+51-18} & \\
 & -19\,\underline{)-57+95-133+38} & \\
 & 3-5+7-2 & 2+3
\end{array}$$

$\therefore$ the H.C.F. $=3x^3-5x^2+7x-2$.

$$6x^4-x^3-x^2+17x-6=(3x^3-5x^2+7x-2)(2x+3).$$
$$9x^4-30x^3+46x^2-41x+10=(3x^3-5x^2+7x-2)(3x-5).$$

$\therefore$ the L.C.M. $=(2x+3)(3x-5)(3x^3-5x^2+7x-2)$.

20. Find the H.C.F. and the L.C.M. of

$$4x^4+2x^3-18x^2+3x-5;\ 6x^5-4x^4-11x^3-3x^2-3x-1.$$

$$\begin{array}{r|r|l}
4+2-18+3-5 & 6-4-11-3-3-1 & \\
\underline{4-8+2-2} & \underline{2} & \\
10-20+5-5 & 12-8-22-6-6-2 & 3-7 \\
\underline{10-20+5-5} & \underline{12+6-54+9-15} & \\
 & -14+32-15+9-2 & \\
 & \underline{2} & \\
 & -28+64-30+18-4 & \\
 & \underline{-28-14+126-21+35} & \\
 & 39\,\underline{)78-156+39-39} & \\
 & 2-4+1-1 & 2+5
\end{array}$$

$\therefore$ the H.C.F. $=2x^3-4x^2+x-1$.

$$4x^4+2x^3-18x^2+3x-5=(2x^3-4x^2+x-1)(2x+5).$$
$$\begin{aligned}6x^5-4x^4-11x^3-3x^2-3x-1&=(2x^3-4x^2+x-1)(3x^2+4x+1)\\&=(2x^3-4x^2+x-1)(x+1)(3x+1).\end{aligned}$$

$\therefore$ the L.C.M. $=(x+1)(3x+1)(2x+5)(2x^3-4x^2+x-1)$.

21. Find the H.C.F. and the L.C.M. of

$$2x^4+3x^3+5x^2+9x-3;\ 3x^4-2x^3+10x^2-6x+3.$$

$$\begin{array}{r|r|l}
2x^4+3x^3+5x^2+9x-3 & 3x^4-2x^3+10x^2-6x+3 & \\
\underline{2x^4\qquad+6x^2\qquad\qquad} & \underline{2\qquad\qquad\qquad\qquad\qquad\qquad} & \\
3x^3-x^2+9x-3 & 6x^4-4x^3+20x^2-12x+6 & 3\\
\underline{3x^3\qquad+9x\quad} & \underline{6x^4+9x^3+15x^2+27x-9} & \\
-x^2\qquad-3 & -1\,)\underline{-13x^3+5x^2-39x+15} & \\
\underline{-x^2\qquad-3} & 13x^3-5x^2+39x-15 & \\
 & 13x-5\,)\underline{=(x^2+3)(13x-5)} & \\
 & x^2+3\qquad & 2x^2+3x-1
\end{array}$$

$\therefore$ the H.C.F. $=x^2+3$.

$$2x^4+3x^3+5x^2+9x-3=(x^2+3)(2x^2+3x-1).$$
$$3x^4-2x^3+10x^2-6x+3=(x^2+3)(3x^2-2x+1).$$

$\therefore$ the L.C.M. $=(x^2+3)(2x^2+3x-1)(3x^2-2x+1)$.

22. Find the H.C.F. and the L.C.M. of

$$x^4-6x^3+13x^2-12x+4;\ x^5-4x^4+8x^3-16x^2+16x.$$

$$\begin{array}{r|r|l}
x^4-6x^3+13x^2-12x+4 & x\,)\underline{x^5-4x^4+8x^3-16x^2+16x} & \\
\underline{2\qquad\qquad\qquad\qquad\qquad\qquad} & x^4-4x^3+8x^2-16x+16 & 1\\
2x^4-12x^3+26x^2-24x+8 & \underline{x^4-6x^3+13x^2-12x+4} & \\
\underline{2x^4-5x^3-4x^2+12x\qquad} & 2x^3-5x^2-4x+12 & x-7\\
-7x^3+30x^2-36x+8 & \underline{2x^3-8x^2+8x\qquad} & 2x+3\\
\underline{2\qquad\qquad\qquad\qquad\qquad} & 3x^2-12x+12 & \\
-14x^3+60x^2-72x+16 & \underline{3x^2-12x+12} & \\
\underline{-14x^3+35x^2+28x-84} & & \\
25\,)\underline{25x^2-100x+100} & & \\
x^2-4x+4 & &
\end{array}$$

$\therefore$ the H.C.F. $=x^2-4x+4=(x-2)^2$.

$$\begin{aligned} x^4 - 6x^3 + 13x^2 - 12x + 4 &= (x-2)^2(x^2-2x+1) \\ &= (x-2)^2(x-1)^2. \\ x^5 - 4x^4 + 8x^3 - 16x^2 + 16x &= x(x^4 - 4x^3 + 8x^2 - 16x + 16) \\ &= x(x-2)^2(x^2+4). \end{aligned}$$

$\therefore$ the L.C.M. $= x(x-1)^2(x-2)^2(x^2+4)$.

23. Find the H.C.F. and the L.C.M. of

$$6x^5 - 9x^4y + 19x^3y^2 - 12x^2y^3 + 19xy^4 - 15y^5;$$
$$4x^4 - 2x^3y + 10x^2y^2 + xy^3 + 15y^4.$$

$$\begin{array}{r|r|l}
4 - 2 + 10 + 1 + 15 & 6 - 9 + 19 - 12 + 19 - 15 & \\
\underline{4 + 6 - 8 + 30} & \underline{2 \qquad\qquad\qquad\qquad\qquad} & \\
-8 + 18 - 29 + 15 & 12 - 18 + 38 - 24 + 38 - 30 & 3 - 3 \\
\underline{-8 - 12 + 16 - 60} & \underline{12 - 6 + 30 + 3 + 45 \quad} & \\
15\,)\underline{30 - 45 + 75} & -12 + 8 - 27 - 7 - 30 & \\
2 - 3 + 5 & \underline{-12 + 6 - 30 - 3 - 45} & \\
 & 2 + 3 - 4 + 15 & 2 - 4 \\
 & \underline{2 - 3 + 5 \quad} & 1 + 3 \\
 & 6 - 9 + 15 & \\
 & \underline{6 - 9 + 15} &
\end{array}$$

$\therefore$ the H.C.F. $= 2x^2 - 3xy + 5y^2$.

$$\begin{aligned} &6x^5 - 9x^4y + 19x^3y^2 - 12x^2y^3 + 19xy^4 - 15y^5 \\ &\quad = (2x^2 - 3xy + 5y^2)(3x^3 + 2xy^2 - 3y^3). \\ &4x^4 - 2x^3y + 10x^2y^2 + xy^3 + 15y^4 \\ &\quad = (2x^2 - 3xy + 5y^2)(2x^2 + 2xy + 3y^2). \end{aligned}$$

$\therefore$ the L.C.M. $= (2x^2 - 3xy + 5y^2)(3x^3 + 2xy^2 - 3y^3)(2x^2 + 2xy + 3y^2)$.

24. Find the H.C.F. and the L.C.M. of $a^6-3a^5+6a^4-7a^3+6a^2-3a+1$; $a^6-a^5+2a^4-a^3+2a^2-a+1$.

$$\begin{array}{r|r|l}
a^6-a^5+2a^4-a^3+2a^2-a+1 & a^6-3a^5+6a^4-7a^3+6a^2-3a+1 & 1 \\
\underline{a^6-2a^5+3a^4-2a^3+a^2} & \underline{a^6-a^5+2a^4-a^3+2a^2-a+1} & \\
a^5-a^4+a^3+a^2-a & -2a)\underline{-2a^5+4a^4-6a^3+4a^2-2a} & \\
\underline{a^5-2a^4+3a^3-2a^2+a} & a^4-2a^3+3a^2-2a+1 & a^2+a+1 \\
a^4-2a^3+3a^2-2a+1 & & \\
\underline{a^4-2a^3+3a^2-2a+1} & &
\end{array}$$

$\therefore$ the H.C.F. $=a^4-2a^3+3a^2-2a+1=(a^2-a+1)^2.$

$$\begin{aligned}
a^6-a^5+2a^4-a^3+2a^2-a+1&=(a^4-2a^3+3a^2-2a+1)(a^2+a+1)\\
&=(a^2-a+1)^2(a^2+a+1).\\
a^6-3a^5+6a^4-7a^3+6a^2-3a+1&=(a^4-2a^3+3a^2-2a+1)(a^2-a+1)\\
&=(a^2-a+1)^2(a^2-a+1)\\
&=(a^2-a+1)^3.
\end{aligned}$$

$\therefore$ the L.C.M. $=(a^2+a+1)(a^2-a+1)^3.$

25. Find the H.C.F. and the L.C.M. of $15x^5+10x^4y+4x^3y^2+6x^2y^3-3xy^4$; $12x^3y^2+38x^2y^3+16xy^4-10y^5$.

$$\begin{array}{r|r|l}
2y^2\,\underline{)\,12x^3y^2+38x^2y^3+16xy^4-10y^5} & x\,\underline{)\,15x^5+10x^4y+4x^3y^2+6x^2y^3-3xy^4} & \\
6x^3+19x^2y+8xy^2-5y^3 & 15x^4+10x^3y+4x^2y^2+6xy^3-3y^4 & 5x \\
\underline{6x^3+4x^2y-2xy^2} & \underline{2} & \\
+15x^2y+10xy^2-5y^3 & 30x^4+20x^3y+8x^2y^2+12xy^3-6y^4 & \\
\underline{+15x^2y+10xy^2-5y^3} & \underline{30x^4+95x^3y+40x^2y^2-25xy^3} & \\
 & -75x^3y-32x^2y^2+37xy^3-6y^4 & -25y \\
 & \underline{2} & \\
 & -150x^3y-64x^2y^2+74xy^3-12y^4 & \\
 & \underline{-150x^3y-475x^2y^2-200xy^3+125y^4} & \\
 & 137y^2\,\underline{)\,411x^2y^2+274xy^3-137y^4} & \\
 & 3x^2+2xy-y^2 & 2x+5y
\end{array}$$

$\therefore$ the H.C.F. $=3x^2+2xy-y^2=(x+y)(3x-y)$.

$$\begin{aligned}
15x^5+10x^4y+4x^3y^2+6x^2y^3-3xy^4 &= x(15x^4+10x^3y+4x^2y^2+6xy^3-3y^4)\\
&= x(3x^2+2xy-y^2)(5x^2+3y^2)\\
&= x(x+y)(3x-y)(5x^2+3y^2).\\
12x^3y^2+38x^2y^3+16xy^4-10y^5 &= 2y^2(6x^3+19x^2y+8xy^2-5y^3)\\
&= 2y^2(3x^2+2xy-y^2)(2x+5y)\\
&= 2y^2(x+y)(3x-y)(2x+5y).
\end{aligned}$$

$\therefore$ the L.C.M. $=2xy^2(x+y)(3x-y)(2x+5y)(5x^2+3y^2)$.

Factor Method

$$12x^3y^2 + 38x^2y^3 + 16xy^4 - 10y^5 = 2y^2(6x^3 + 19x^2y + 8xy^2 - 5y^3).$$

If $x = -y$, $\quad 6x^3 + 19x^2y + 8xy^2 - 5y^3 = -6y^3 + 19y^3 - 8y^3 - 5y^3 = 0.$

$$\begin{aligned} \therefore\ 6x^3 + 19x^2y + 8xy^2 - 5y^3 &= (x + y)(6x^2 + 13xy - 5y^2) \\ &= (x + y)(3x - y)(2x + 5y). \end{aligned}$$

$$\therefore\ 12x^3y^2 + 38x^2y^3 + 16xy^4 - 10y^5 = 2y^2(x + y)(3x - y)(2x + 5y).$$

$$15x^5 + 10x^4y + 4x^3y^2 + 6x^2y^3 - 3xy^4 = x(15x^4 + 10x^3y + 4x^2y^2 + 6xy^3 - 3y^4).$$

If $x = -y$, $\quad 15x^4 + 10x^3y + 4x^2y^2 + 6xy^3 - 3y^4 = 15y^4 - 10y^4 + 4y^4 - 6y^4 - 3y^4 = 0.$

$$\begin{aligned} \therefore\ 15x^4 + 10x^3y + 4x^2y^2 + 6xy^3 - 3y^4 &= (x + y)(15x^3 - 5x^2y + 9xy^2 - 3y^3) \\ &= (x + y)[5x^2(3x - y) + 3y^2(3x - y)] \\ &= (x + y)(3x - y)(5x^2 + 3y^2). \end{aligned}$$

$$\therefore\ 15x^5 + 10x^4y + 4x^3y^2 + 6x^2y^3 - 3xy^4 = x(x + y)(3x - y)(5x^2 + 3y^2).$$

$\therefore$ the H.C.F. $= (x + y)(3x - y)$,
and the L.C.M. $= 2xy^2(x + y)(3x - y)(2x + 5y)(5x^2 + 3y^2)$.

26. Find the H.C.F. and the L.C.M. of $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$; $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$.

$$\begin{array}{rrrrrr|c|rrrrrr|l}
x^5 + & x^4y + & x^3y^2 + & x^2y^3 + & xy^4 + & y^5 & & x^5 - & x^4y + & x^3y^2 - & x^2y^3 + & xy^4 - & y^5 & 1 \\
x^5 & & + x^3y^2 & & + xy^4 & & & x^5 + & x^4y + & x^3y^2 + & x^2y^3 + & xy^4 + & y^5 & \\
\hline
 & + x^4y & & + x^2y^3 & & + y^5 & -2y) & & -2x^4y & & -2x^2y^3 & & -2y^5 & \\
 & + x^4y & & + x^2y^3 & & + y^5 & & & x^4 & + & x^2y^2 & + & y^4 & x + y \\
\hline
\end{array}$$

$\therefore$ the H.C.F. $= x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2)$.

$$\begin{aligned}
x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 &= (x^4 + x^2y^2 + y^4)(x + y) \\
&= (x^2 + xy + y^2)(x^2 - xy + y^2)(x + y). \\
x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5 &= (x^4 + x^2y^2 + y^4)(x - y) \\
&= (x^2 + xy + y^2)(x^2 - xy + y^2)(x - y).
\end{aligned}$$

$\therefore$ the L.C.M. $= (x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$.

Factor Method

$$\begin{aligned}
x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 &= x^4(x + y) + x^2y^2(x + y) + y^4(x + y) \\
&= (x + y)(x^4 + x^2y^2 + y^4) \\
&= (x + y)(x^2 + xy + y^2)(x^2 - xy + y^2). \\
x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5 &= x^4(x - y) + x^2y^2(x - y) + y^4(x - y) \\
&= (x - y)(x^4 + x^2y^2 + y^4) \\
&= (x - y)(x^2 + xy + y^2)(x^2 - xy + y^2).
\end{aligned}$$

$\therefore$ the H.C.F. $= (x^2 + xy + y^2)(x^2 - xy + y^2)$,
and the L.C.M. $= (x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$.

27. Find the H.C.F. and the L.C.M. of $4a^5b - 4a^3b^3 + 4a^2b^4 - ab^5$; $24a^5 + 16a^4b - 36a^3b^2 - 12a^2b^3 + 8ab^4$.

$$\begin{array}{r|l}
ab) & \underline{4a^5b - 4a^3b^3 + 4a^2b^4 - ab^5} \\
 & 4a^4 - 4a^2b^2 + 4ab^3 - b^4 \\
 & \underline{2} \\
 & 8a^4 - 8a^2b^2 + 8ab^3 - 2b^4 \\
 & \underline{8a^4 - 6a^3b - 18a^2b^2 + 7ab^3} \\
 & 6a^3b + 10a^2b^2 + ab^3 - 2b^4 \\
 & \underline{4} \\
 & 24a^3b + 40a^2b^2 + 4ab^3 - 8b^4 \\
 & \underline{24a^3b - 18a^2b^2 - 54ab^3 + 21b^4} \\
29b^2) & \underline{58a^2b^2 + 58ab^3 - 29b^4} \\
 & 2a^2 + 2ab - b^2
\end{array}
\qquad
\begin{array}{r|l|l}
4a) & \underline{24a^5 + 16a^4b - 36a^3b^2 - 12a^2b^3 + 8ab^4} & \\
 & 6a^4 + 4a^3b - 9a^2b^2 - 3ab^3 + 2b^4 & \\
 & \underline{2} & \\
 & 12a^4 + 8a^3b - 18a^2b^2 - 6ab^3 + 4b^4 & 3 \\
 & \underline{12a^4 - 12a^2b^2 + 12ab^3 - 3b^4} & \\
b) & \underline{8a^3b - 6a^2b^2 - 18ab^3 + 7b^4} & \\
 & 8a^3 - 6a^2b - 18ab^2 + 7b^3 & a + 3b \\
 & \underline{8a^3 + 8a^2b - 4ab^2} & 4a - 7b \\
 & -14a^2b - 14ab^2 + 7b^3 & \\
 & \underline{-14a^2b - 14ab^2 + 7b^3} &
\end{array}$$

$\therefore$ the H.C.F. $= a(2a^2 + 2ab - b^2)$.

$$\begin{aligned}
4a^5b - 4a^3b^3 + 4a^2b^4 - ab^5 &= ab(4a^4 - 4a^2b^2 + 4ab^3 - b^4) \\
&= ab(2a^2 + 2ab - b^2)(2a^2 - 2ab + b^2). \\
24a^5 + 16a^4b - 36a^3b^2 - 12a^2b^3 + 8ab^4 &= 4a(6a^4 + 4a^3b - 9a^2b^2 - 3ab^3 + 2b^4) \\
&= 4a(2a^2 + 2ab - b^2)(3a^2 - ab - 2b^2) \\
&= 4a(2a^2 + 2ab - b^2)(a - b)(3a + 2b).
\end{aligned}$$

$\therefore$ the L.C.M. $= 4ab(a - b)(3a + 2b)(2a^2 + 2ab - b^2)(2a^2 - 2ab + b^2)$.

28. Find the H.C.F. and the L.C.M. of

$$a^6 + 4a^5b - 3a^4b^2 - 16a^3b^3 + 11a^2b^4 + 12ab^5 - 9b^6;\ 6a^5 + 20a^4b - 12a^3b^2 - 48a^2b^3 + 22ab^4 + 12b^5.$$

$$\begin{array}{rrrrrrr}
2\,)\ 6a^5 + & 20a^4b - & 12a^3b^2 - & 48a^2b^3 + & 22ab^4 + & 12b^5 & \\ \hline
3a^5 + & 10a^4b - & 6a^3b^2 - & 24a^2b^3 + & 11ab^4 + & 6b^5 & \\
29 & & & & & & \\ \hline
87a^5 + & 290a^4b - & 174a^3b^2 - & 696a^2b^3 + & 319ab^4 + & 174b^5 & \\
87a^5 + & 180a^4b - & 342a^3b^2 - & 204a^2b^3 + & 279ab^4 & & \\ \hline
 & 110a^4b + & 168a^3b^2 - & 492a^2b^3 + & 40ab^4 + & 174b^5 & \\
 & 29 & & & & & \\ \hline
 & 3190a^4b + & 4872a^3b^2 - & 14268a^2b^3 + & 1160ab^4 + & 5046b^5 & \\
 & 3190a^4b + & 6600a^3b^2 - & 12540a^2b^3 - & 7480ab^4 + & 10230b^5 & \\ \hline
 & -1728b^2\,) & -1728a^3b^2 - & 1728a^2b^3 + & 8640ab^4 - & 5184b^5 & \\ \hline
 & & a^3 + & a^2b - & 5ab^2 + & 3b^3 &
\end{array}$$

$$\begin{array}{rrrrrrr|l}
a^6 + & 4a^5b - & 3a^4b^2 - & 16a^3b^3 + & 11a^2b^4 + & 12ab^5 - & 9b^6 & \\
3 & & & & & & & \\ \hline
3a^6 + & 12a^5b - & 9a^4b^2 - & 48a^3b^3 + & 33a^2b^4 + & 36ab^5 - & 27b^6 & a \\
3a^6 + & 10a^5b - & 6a^4b^2 - & 24a^3b^3 + & 11a^2b^4 + & 6ab^5 & & \\ \hline
 & 2a^5b - & 3a^4b^2 - & 24a^3b^3 + & 22a^2b^4 + & 30ab^5 - & 27b^6 & \\
 & 3 & & & & & & \\ \hline
 & 6a^5b - & 9a^4b^2 - & 72a^3b^3 + & 66a^2b^4 + & 90ab^5 - & 81b^6 & 2b \\
 & 6a^5b + & 20a^4b^2 - & 12a^3b^3 - & 48a^2b^4 + & 22ab^5 + & 12b^6 & \\ \hline
 & -b^2\,) & -29a^4b^2 - & 60a^3b^3 + & 114a^2b^4 + & 68ab^5 - & 93b^6 & \\ \hline
 & & 29a^4 & +60a^3b & -114a^2b^2 - & 68ab^3 + & 93b^4 & \\
 & & 29a^4 & +29a^3b & -145a^2b^2 + & 87ab^3 & & 3a+110b \\ \hline
 & & & 31a^3b + & 31a^2b^2 - & 155ab^3 + & 93b^4 & 29a+31b \\
 & & & 31a^3b + & 31a^2b^2 - & 155ab^3 + & 93b^4 & \\ \hline
\end{array}$$

$\therefore$ the H.C.F. $= a^3 + a^2b - 5ab^2 + 3b^3 = (a-b)(a^2 + 2ab - 3b^2) = (a-b)(a-b)(a+3b) = (a-b)^2(a+3b).$

$$\begin{aligned}
a^6 + 4a^5b - 3a^4b^2 - 16a^3b^3 + 11a^2b^4 + 12ab^5 - 9b^6 &= (a^3 + a^2b - 5ab^2 + 3b^3)(a^3 + 3a^2b - ab^2 - 3ab^3) \\
&= (a-b)^2(a+3b)[a^2(a+3b) - b^2(a+3b)] \\
&= (a-b)^2(a+3b)(a^2-b^2)(a+3b) \\
&= (a-b)^2(a+3b)(a+b)(a-b)(a+3b) \\
&= (a+b)(a-b)^3(a+3b)^2.
\end{aligned}$$

$$\begin{aligned}6a^5 + 20a^4b - 12a^3b^2 - 48a^2b^3 + 22ab^4 + 12b^5 &= 2(3a^5 + 10a^4b - 6a^3b^2 - 24a^2b^3 + 11ab^4 + 6b^5)\\ &= 2(a^3 + a^2b - 5ab^2 + 3b^3)(3a^2 + 7ab + 2b^2)\\ &= 2(a-b)^2(a+3b)(a+2b)(3a+b).\end{aligned}$$

$\therefore$ the L.C.M. $= 2(a+b)(a-b)^3(a+3b)^2(a+2b)(3a+b)$.

Factor Method

$$\begin{aligned}a^6 + 4a^5b - 3a^4b^2 - 16a^3b^3 + 11a^2b^4 + 12ab^5 - 9b^6 &= (a-b)(a^5 + 5a^4b + 2a^3b^2 - 14a^2b^3 - 3ab^4 + 9b^5)\\ &= (a-b)^2(a^4 + 6a^3b + 8a^2b^2 - 6ab^3 - 9b^4)\\ &= (a-b)^3(a^3 + 7a^2b + 15ab^2 + 9b^3)\\ &= (a-b)^3(a+b)(a^2 + 6ab + 9b^2)\\ &= (a-b)^3(a+b)(a+3b)^2.\\ 6a^5 + 20a^4b - 12a^3b^2 - 48a^2b^3 + 22ab^4 + 12b^5 &= 2(3a^5 + 10a^4b - 6a^3b^2 - 24a^2b^3 + 11ab^4 + 6b^5)\\ &= 2(a-b)(3a^4 + 13a^3b + 7a^2b^2 - 17ab^3 - 6b^4)\\ &= 2(a-b)^2(3a^3 + 16a^2b + 23ab^2 + 6b^3)\\ &= 2(a-b)^2(a+2b)(3a^2 + 10ab + 3b^2)\\ &= 2(a-b)^2(a+2b)(a+3b)(3a+b).\end{aligned}$$

$\therefore$ the H.C.F. $= (a-b)^2(a+3b)$,

and the L.C.M. $= 2(a-b)^3(a+b)(a+3b)^2(a+2b)(3a+b)$.

29. Find the H.C.F. and the L.C.M. of $2x^5 - 4x^4 + 8x^3 - 12x^2 + 6x$; $3x^5 - 3x^4 - 6x^3 + 9x^2 - 3x$.

$$\begin{array}{r|r|l}
2x\,\underline{)\,2x^5 - 4x^4 + 8x^3 - 12x^2 + 6x} & 3x\,\underline{)\,3x^5 - 3x^4 - 6x^3 + 9x^2 - 3x} & \\
x^4 - 2x^3 + 4x^2 - 6x + 3 & x^4 - x^3 - 2x^2 + 3x - 1 & 1 \\
\underline{x^4 - 6x^3 + 9x^2 - 4x } & \underline{x^4 - 2x^3 + 4x^2 - 6x + 3} & \\
4x^3 - 5x^2 - 2x + 3 & x^3 - 6x^2 + 9x - 4 & x + 4 \\
\underline{4x^3 - 24x^2 + 36x - 16} & \underline{x^3 - 2x^2 + x } & x - 4 \\
19\,\underline{)\,19x^2 - 38x + 19} & -4x^2 + 8x - 4 & \\
x^2 - 2x + 1 & \underline{-4x^2 + 8x - 4} &
\end{array}$$

$\therefore$ the H.C.F. $= x(x^2 - 2x + 1) = x(x - 1)^2$.

$$2x^5 - 4x^4 + 8x^3 - 12x^2 + 6x = 2x(x - 1)^2(x^2 + 3).$$
$$3x^5 - 3x^4 - 6x^3 + 9x^2 - 3x = 3x(x - 1)^2(x^2 + x - 1).$$

$\therefore$ the L.C.M. $= 6x(x - 1)^2(x^2 + x - 1)(x^2 + 3)$.

30. Find the H.C.F. and the L.C.M. of

$$4x^5 + 20x^4y + 13x^3y^2 - 16x^2y^3 - 17xy^4 - 4y^5;\ 6x^5 + 11x^4y - 54x^3y^2 - x^2y^3 + 30xy^4 + 8y^5.$$

$$\begin{array}{rrrrrr|rrrrrr|l}
4+ & 20+ & 13- & 16- & 17- & 4 & 6+ & 11- & 54- & 1+ & 30+ & 8 & \\
19 & & & & & & 2 & & & & & & \\
\hline
76+ & 380+ & 247- & 304- & 323- & 76 & 12+ & 22- & 108- & 2+ & 60+ & 16 & 3 \\
76+ & 294- & 92- & 222- & 56 & & 12+ & 60+ & 39- & 48- & 51- & 12 & \\
\hline
& 86+ & 339- & 82- & 267- & 76 & & -38- & 147+ & 46+ & 111+ & 28 & -2-43 \\
& 19 & & & & & & -38- & 133+ & 95+ & 76 & & -19-7 \\
\hline
& 1634+ & 6441- & 1558- & 5073- & 1444 & & & -14- & 49+ & 35+ & 28 & \\
& 1634+ & 6321- & 1978- & 4773- & 1204 & & & -14- & 49+ & 35+ & 28 & \\
\hline
& 60\,) & 120+ & 420- & 300- & 240 & & & & & & & \\
& & 2+ & 7- & 5- & 4 & & & & & & &
\end{array}$$

$\therefore$ the H.C.F. $= 2x^3 + 7x^2y - 5xy^2 - 4y^3 = (x - y)(2x^2 + 9xy + 4y^2) = (x - y)(x + 4y)(2x + y).$

$$\begin{aligned}
4x^5 + 20x^4y + 13x^3y^2 - 16x^2y^3 - 17xy^4 - 4y^5 &= (2x^3 + 7x^2y - 5xy^2 - 4y^3)(2x^2 + 3xy + y^2)\\
&= (x - y)(x + 4y)(2x + y)(x + y)(2x + y)\\
&= (x + y)(x - y)(x + 4y)(2x + y)^2.\\
6x^5 + 11x^4y - 54x^3y^2 - x^2y^3 + 30xy^4 + 8y^5 &= (2x^3 + 7x^2y - 5xy^2 - 4y^3)(3x^2 - 5xy - 2y^2)\\
&= (x - y)(x + 4y)(2x + y)(3x^2 - 5xy - 2y^2).
\end{aligned}$$

$\therefore$ the L.C.M. $= (x + y)(x - y)(x + 4y)(2x + y)^2(3x^2 - 5xy - 2y^2).$

31. Find the H.C.F. and the L.C.M. of

$$6x^5 + 2x^4y - 46x^3y^2 + 38x^2y^3 - 16xy^4 - 32y^5;\ 4x^5 - 34x^3y^2 + 28x^2y^3 - 14xy^4 - 24y^5.$$

$$\begin{array}{r|r|l}
2\,\underline{)6 + 2 - 46 + 38 - 16 - 32} & 2\,\underline{)4 + 0 - 34 + 28 - 14 - 24} & \\
3 + 1 - 23 + 19 - 8 - 16 & 2 + 0 - 17 + 14 - 7 - 12 & \\
\underline{2} & \underline{3} & \\
6 + 2 - 46 + 38 - 16 - 32 & 6 + 0 - 51 + 42 - 21 - 36 & 2 \\
\underline{6 + 15 - 12 + 15 + 12} & \underline{6 + 2 - 46 + 38 - 16 - 32} & \\
-13 - 34 + 23 - 28 - 32 & -2 - 5 + 4 - 5 - 4 & -3 + 13 \\
\underline{2} & \underline{-2 - 4 + 6 - 8} & -2 - 1 \\
-26 - 68 + 46 - 56 - 64 & -1 - 2 + 3 - 4 & \\
\underline{-26 - 65 + 52 - 65 - 52} & \underline{-1 - 2 + 3 - 4} & \\
-3\,\underline{)- 3 - 6 + 9 - 12} & & \\
1 + 2 - 3 + 4 & &
\end{array}$$

$\therefore$ the H.C.F. $= 2(x^3 + 2x^2y - 3xy^2 + 4y^3)$.

$$\begin{aligned}
6x^5 + 2x^4y - 46x^3y^2 + 38x^2y^3 - 16xy^4 - 32y^5 &= 2(3x^5 + x^4y - 23x^3y^2 + 19x^2y^3 - 8xy^4 - 16y^5)\\
&= 2(x^3 + 2x^2y - 3xy^2 + 4y^3)(3x^2 - 5xy - 4y^2).\\
4x^5 - 34x^3y^2 + 28x^2y^3 - 14xy^4 - 24y^5 &= 2(2x^5 - 17x^3y^2 + 14x^2y^3 - 7xy^4 - 12y^5)\\
&= 2(x^3 + 2x^2y - 3xy^2 + 4y^3)(2x^2 - 4xy - 3y^2).
\end{aligned}$$

$\therefore$ the L.C.M. $= 2(3x^2 - 5xy - 4y^2)(2x^2 - 4xy - 3y^2)(x^3 + 2x^2y - 3xy^2 + 4y^3)$.

Exercise 53. Page 122

1. Find the H.C.F. and the L.C.M. of

$$a^3 + a^2b - ab^2 - b^3;\ a^3 - 3ab^2 + 2b^3;\ a^3 - 2a^2b - ab^2 + 2b^3.$$

$$\begin{aligned}
a^3 + a^2b - ab^2 - b^3 &= a^2(a+b) - b^2(a+b)\\
&= (a^2 - b^2)(a+b)\\
&= (a-b)(a+b)^2.\\
a^3 - 3ab^2 + 2b^3 &= (a^3 - ab^2) - (2ab^2 - 2b^3)\\
&= a(a+b)(a-b) - 2b^2(a-b)\\
&= (a-b)(a^2 + ab - 2b^2)\\
&= (a-b)(a-b)(a+2b)\\
&= (a-b)^2(a+2b).\\
a^3 - 2a^2b - ab^2 + 2b^3 &= a^2(a-2b) - b^2(a-2b)\\
&= (a^2 - b^2)(a-2b)\\
&= (a+b)(a-b)(a-2b).
\end{aligned}$$

$\therefore$ the H.C.F. $= a - b$,

and the L.C.M. $= (a+b)^2(a-b)^2(a+2b)(a-2b)$.

2. Find the H.C.F. and the L.C.M. of

$$a^2 + 3ab + 2b^2;\ a^2 + 4ab + 3b^2;\ a^2 + 5ab + 6b^2.$$

$$\begin{aligned}
a^2 + 3ab + 2b^2 &= (a+b)(a+2b).\\
a^2 + 4ab + 3b^2 &= (a+b)(a+3b).\\
a^2 + 5ab + 6b^2 &= (a+2b)(a+3b).
\end{aligned}$$

$\therefore$ the H.C.F. $= 1$,

and the L.C.M. $= (a+b)(a+2b)(a+3b)$.

3. Find the H.C.F. and the L.C.M. of

$$a^2 + 5ab + 10b^2;\ a^3 + 3a^2b - 20b^3;\ a^3 - 15ab^2 - 50b^3.$$

$$\begin{array}{r|r|l}
1+3+\ 0-20 & 1+0-15-50 & 1\\
\underline{1+5+10} & \underline{1+3+\ 0-20} & \\
-2-10-20 & -3\,\overline{)}-3-15-30 & \\
\underline{-2-10-20} & 1+\ 5+10 & 1-2
\end{array}$$

$\therefore$ the H.C.F. of $a^3 + 3a^2b - 20b^3$ and $a^3 - 15ab^2 - 50b^3$
$= a^2 + 5ab + 10b^2$.

$$\begin{aligned}
a^2 + 5ab + 10b^2 &= (a^2 + 5ab + 10b^2).\\
a^3 + 3a^2b - 20b^3 &= (a^2 + 5ab + 10b^2)(a - 2b).\\
a^3 - 15ab^2 - 50b^3 &= (a^2 + 5ab + 10b^2)(a - 5b).
\end{aligned}$$

$\therefore$ the H.C.F. $= a^2 + 5ab + 10b^2$,

and the L.C.M. $= (a - 2b)(a - 5b)(a^2 + 5ab + 10b^2)$.

4. Find the H.C.F. and the L.C.M. of

$$x^2+2x-3;\ x^3+3x^2-x-3;\ x^3+4x^2+x-6.$$

$$\begin{aligned} x^2+2x-3 &= (x-1)(x+3).\\ x^3+3x^2-x-3 &= x^2(x+3)-(x+3)\\ &= (x^2-1)(x+3)\\ &= (x+1)(x-1)(x+3).\\ x^3+4x^2+x-6 &= (x-1)(x^2+5x+6)\\ &= (x-1)(x+2)(x+3). \end{aligned}$$

$\therefore$ the H.C.F. $=(x-1)(x+3)$,
and the L.C.M. $=(x+1)(x-1)(x+2)(x+3)$.

5. Find the H.C.F. and the L.C.M. of

$$6x^2-13x+6;\ 6x^2+5x-6;\ 9x^2-4.$$

$$\begin{aligned} 6x^2-13x+6 &= (2x-3)(3x-2).\\ 6x^2+5x-6 &= (2x+3)(3x-2).\\ 9x^2-4 &= (3x+2)(3x-2). \end{aligned}$$

$\therefore$ the H.C.F. $=3x-2$,
and the L.C.M. $=(3x-2)(3x+2)(2x-3)(2x+3)$.

6. Find the H.C.F. and the L.C.M. of

$$x^3-6x^2+11x-6;\ x^3-9x^2+26x-24;\ x^3-8x^2+19x-12.$$

$$\begin{array}{r|r|l} 1-6+11-6 & 1-8+19-12 & 1\\ \underline{1-4+\ 3} & \underline{1-6+11-\ 6} & \\ -2+\ 8-6 & -2\,\big)\,\underline{-2+\ 8-\ 6} & \\ \underline{-2+\ 8-6} & 1-\ 4+\ 3 & 1-2 \end{array}$$

$\therefore$ the H.C.F. of $x^3-6x^2+11x-6$ and $x^3-8x^2+19x-12$
$=x^2-4x+3=(x-1)(x-3)$.

$$\begin{aligned} x^3-6x^2+11x-6 &= (x^2-4x+3)(x-2)\\ &= (x-1)(x-3)(x-2).\\ x^3-9x^2+26x-24 &= (x-3)(x^2-6x+8)\\ &= (x-3)(x-2)(x-4).\\ x^3-8x^2+19x-12 &= (x^2-4x+3)(x-4)\\ &= (x-1)(x-3)(x-4). \end{aligned}$$

$\therefore$ the H.C.F. $=x-3$,
and the L.C.M. $=(x-1)(x-2)(x-3)(x-4)$.

7. Find the H.C.F. and the L.C.M. of

$$3a^2 + 14a - 5;\ 6a^2 + 39a + 45;\ 6a^3 + 7a^2 - 3a.$$

$$\begin{aligned} 3a^2 + 14a - 5 &= (a+5)(3a-1).\\ 6a^2 + 39a + 45 &= 3(2a^2 + 13a + 15)\\ &= 3(a+5)(2a+3).\\ 6a^3 + 7a^2 - 3a &= a(6a^2 + 7a - 3)\\ &= a(2a+3)(3a-1). \end{aligned}$$

$\therefore$ the H.C.F. $= 1$,
and the L.C.M. $= 3a(a+5)(3a-1)(2a+3)$.

8. Find the H.C.F. and the L.C.M. of

$$14x^2 + 5xy - y^2;\ 21x^2 - 17xy + 2y^2;\ 6x^2 - xy - 2y^2.$$

$$\begin{aligned} 14x^2 + 5xy - y^2 &= (2x+y)(7x-y).\\ 21x^2 - 17xy + 2y^2 &= (3x-2y)(7x-y).\\ 6x^2 - xy - 2y^2 &= (2x+y)(3x-2y). \end{aligned}$$

$\therefore$ the H.C.F. $= 1$,
and the L.C.M. $= (2x+y)(3x-2y)(7x-y)$.

9. Find the H.C.F. and the L.C.M. of $x^4 - 2x^3 - 2x^2 + 7x - 10$; $x^4 - 7x^3 + 18x^2 - 23x + 15$; $2x^4 - 11x^3 + 24x^2 - 28x + 15$.

$$\begin{array}{r|r|l} 1 - 2 - 2 + 7 - 10 & 1 - 7 + 18 - 23 + 15 & 1 \\ \underline{1 - 4 + 6 - 5} & \underline{1 - 2 - 2 + 7 - 10} & \\ 2 - 8 + 12 - 10 & -5\,\big)\,\underline{-5 + 20 - 30 + 25} & \\ \underline{2 - 8 + 12 - 10} & 1 - 4 + 6 - 5 & 1 + 2 \end{array}$$

$\therefore$ the H.C.F. of $x^4 - 2x^3 - 2x^2 + 7x - 10$ and $x^4 - 7x^3 + 18x^2 - 23x + 15$
$= x^3 - 4x^2 + 6x - 5$.

$$\begin{array}{r|r|l} 1 - 4 + 6 - 5 & 2 - 11 + 24 - 28 + 15 & 2 - 3 \\ & \underline{2 - 8 + 12 - 10} & \\ & -3 + 12 - 18 + 15 & \\ & \underline{-3 + 12 - 18 + 15} & \end{array}$$

$\therefore$ the H.C.F. $= x^3 - 4x^2 + 6x - 5$.

$$\begin{aligned} x^4 - 2x^3 - 2x^2 + 7x - 10 &= (x+2)(x^3 - 4x^2 + 6x - 5).\\ x^4 - 7x^3 + 18x^2 - 23x + 15 &= (x-3)(x^3 - 4x^2 + 6x - 5).\\ 2x^4 - 11x^3 + 24x^2 - 28x + 15 &= (2x-3)(x^3 - 4x^2 + 6x - 5). \end{aligned}$$

$\therefore$ the L.C.M. $= (x+2)(x-3)(2x-3)(x^3 - 4x^2 + 6x - 5)$.

10. Find the H.C.F. and the L.C.M. of

$$3x^4 + 2x^3y - 15x^2y^2 + 13xy^3 - 15y^4;$$
$$3x^4 - 19x^3y + 34x^2y^2 - 29xy^3 + 20y^4;$$
$$6x^4 + x^3y - 23x^2y^2 + 20xy^3 - 25y^4.$$

$$\begin{array}{r|r|l} 3+2-15+13-15. & 3-19+34-29+20 & 1\\ \underline{3-7+\ 6-\ 5} & \underline{3+\ 2-15+13-15} & \\ 9-21+18-15 & -7\overline{)-21+49-42+35} & \\ \underline{9-21+18-15} & 3-\ 7+\ 6-\ 5 & 1+3 \end{array}$$

$\therefore$ the H.C.F. of $3x^4+2x^3y-15x^2y^2+13xy^3-15y^4$ and $3x^4-19x^3y+34x^2y^2-29xy^3+20y^4 = 3x^3-7x^2y+6xy^2-5y^3$.

$$\begin{array}{r|r|l} 3-7+6-5 & 6+\ 1-23+20-25 & 2+5\\ & \underline{6-14+12-10} & \\ & 15-35+30-25 & \\ & \underline{15-35+30-25} & \end{array}$$

$\therefore$ the H.C.F. $= 3x^3 - 7x^2y + 6xy^2 - 5y^3$.

$$3x^4 + 2x^3y - 15x^2y^2 + 13xy^3 - 15y^4$$
$$= (x + 3y)(3x^3 - 7x^2y + 6xy^2 - 5y^3).$$
$$3x^4 - 19x^3y + 34x^2y^2 - 29xy^3 + 20y^4$$
$$= (x - 4y)(3x^3 - 7x^2y + 6xy^2 - 5y^3).$$
$$6x^4 + x^3y - 23x^2y^2 + 20xy^3 - 25y^4$$
$$= (2x + 5y)(3x^3 - 7x^2y + 6xy^2 - 5y^3).$$

$\therefore$ the L.C.M. $= (x+3y)(x-4y)(2x+5y)(3x^3-7x^2y+6xy^2-5y^3)$.

11. Find the H.C.F. and the L.C.M. of

$$20x^4 - 31x^3y + 2x^2y^2 + 13xy^3 - 4y^4;$$
$$4x^4 + x^3y - 5x^2y^2 - xy^3 + y^4;$$
$$4x^4 - 11x^3y + 4x^2y^2 + 5xy^3 - 2y^4.$$

$$\begin{array}{r|r|l} 4+1-5-1+1 & 20-31+\ 2+13-4 & 5\\ \underline{4-3-2+1} & \underline{20+\ 5-25-\ 5+5} & \\ 4-3-2+1 & -9\overline{)-36+27+18-9} & \\ \underline{4-3-2+1} & 4-\ 3-\ 2+1 & 1+1 \end{array}$$

$\therefore$ the H.C.F. of $20x^4 - 31x^3y + 2x^2y^2 + 13xy^3 - 4y^4$ and
$4x^4 + x^3y - 5x^2y^2 - xy^3 + y^4 = 4x^3 - 3x^2y - 2xy^2 + y^3$
$= (x - y)(4x^2 + xy - y^2)$.

$$20x^4-31x^3y+2x^2y^2+13xy^3-4y^4 = (x - y)(4x^2 + xy - y^2)(5x - 4y).$$
$$4x^4 + x^3y - 5x^2y^2 - xy^3 + y^4 = (x - y)(4x^2 + xy - y^2)(x + y).$$
$$4x^4 - 11x^3y + 4x^2y^2 + 5xy^3 - 2y^4 = (x - y)(4x^2 + xy - y^2)(x - 2y).$$

$\therefore$ the H.C.F. $= (x - y)(4x^2 + xy - y^2)$,
and the L.C.M. $= (x - y)(x + y)(x - 2y)(5x - 4y)(4x^2 + xy - y^2)$.

Exercise 54. Page 124

Reduce to lowest terms:

1. $\dfrac{10\,a^4b^3}{15\,a^3b^4}=\dfrac{2\times 5\,a^4b^3}{3\times 5\,a^3b^4}=\dfrac{2\,a}{3\,b}.$

2. $\dfrac{6\,x^4y^5z^6}{12\,x^6y^5z^4}=\dfrac{2\times 3\,x^4y^5z^6}{2^2\times 3\,x^6y^5z^4}=\dfrac{z^2}{2\,x^2}.$

3. $\dfrac{64\,a^6b^3c}{48\,a^4bc^5}=\dfrac{2^6a^6b^3c}{2^4\times 3\,a^4bc^5}=\dfrac{4\,a^2b^2}{3\,c^4}.$

4. $\dfrac{36\,x^2y^4z^6}{60\,x^4y^2z^5}=\dfrac{2^2\times 3^2x^2y^4z^6}{2^2\times 3\times 5\,x^4y^2z^5}=\dfrac{3\,y^2z}{5\,x^2}.$

5. $\dfrac{60\,a^7b^6c^5d^4}{80\,a^8b^4c^4d^6}=\dfrac{2^2\times 3\times 5\,a^7b^6c^5d^4}{2^4\times 5\,a^8b^4c^4d^6}=\dfrac{3\,b^2c}{4\,ad^2}.$

6. $\dfrac{75\,a^4x^5y^4z^5}{50\,a^2x^7y^5z^3}=\dfrac{3\times 5^2a^4x^5y^4z^5}{2\times 5^2a^2x^7y^5z^3}=\dfrac{3\,a^2z^2}{2\,x^2y}.$

7. $\dfrac{a^2-2\,ab}{ab-2\,b^2}=\dfrac{a\,(a-2\,b)}{b\,(a-2\,b)}=\dfrac{a}{b}.$

8. $\dfrac{10\,a^2-2\,ab}{15\,ab-3\,b^2}=\dfrac{2\,a\,(5\,a-b)}{3\,b\,(5\,a-b)}=\dfrac{2\,a}{3\,b}.$

9. $\dfrac{14\,a^5+7\,a^4b}{10\,a^4c+5\,a^3bc}=\dfrac{7\,a^4\,(2\,a+b)}{5\,a^3c\,(2\,a+b)}=\dfrac{7\,a}{5\,c}.$

10. $\dfrac{12\,a^4+27\,a^3b}{16\,a^3b+36\,a^2b^2}=\dfrac{3\,a^3\,(4\,a+9\,b)}{4\,a^2b\,(4\,a+9\,b)}=\dfrac{3\,a}{4\,b}.$

11. $\dfrac{45\,a^3x^3-105\,a^5x^2}{33\,ab^3x-77\,a^3b^3}=\dfrac{15\,a^3x^2\,(3\,x-7\,a^2)}{11\,ab^3\,(3\,x-7\,a^2)}=\dfrac{15\,a^2x^2}{11\,b^3}.$

12. $$\dfrac{49\,a^7b^2x^6-81\,ab^{12}x^2}{56\,a^6b^4x^5+72\,a^3b^9x^3}=\dfrac{ab^2x^2\,(49\,a^6x^4-81\,b^{10})}{8\,a^3b^4x^3\,(7\,a^3x^2+9\,b^5)}$$
$$=\dfrac{ab^2x^2\,(7\,a^3x^2+9\,b^5)\,(7\,a^3x^2-9\,b^5)}{8\,a^3b^4x^3\,(7\,a^3x^2+9\,b^5)}$$
$$=\dfrac{7\,a^3x^2-9\,b^5}{8\,a^2b^2x}.$$

13. $\dfrac{x^3+x^2y}{x^2+2\,xy+y^2}=\dfrac{x^2\,(x+y)}{(x+y)^2}=\dfrac{x^2}{x+y}.$

14. $\dfrac{x^3+2\,x^2}{x^2+4\,x+4}=\dfrac{x^2\,(x+2)}{(x+2)^2}=\dfrac{x^2}{x+2}.$

15.
$$\begin{aligned}\frac{x^4-y^4}{x^4+2x^2y^2+y^4}&=\frac{(x^2+y^2)(x^2-y^2)}{(x^2+y^2)^2}\\&=\frac{(x^2+y^2)(x+y)(x-y)}{(x^2+y^2)^2}\\&=\frac{(x+y)(x-y)}{x^2+y^2}.\end{aligned}$$

16.
$$\begin{aligned}\frac{7x^2+3}{245x^5+210x^3+45x}&=\frac{7x^2+3}{5x(49x^4+42x^2+9)}\\&=\frac{7x^2+3}{5x(7x^2+3)^2}\\&=\frac{1}{5x(7x^2+3)}.\end{aligned}$$

17.
$$\frac{15a^2-27ab}{25a^2-90ab+81b^2}=\frac{3a(5a-9b)}{(5a-9b)^2}=\frac{3a}{5a-9b}.$$

18.
$$\begin{aligned}\frac{x-xy+z-zy}{1-3y+3y^2-y^3}&=\frac{x(1-y)+z(1-y)}{(1-y)^3}\\&=\frac{(x+z)(1-y)}{(1-y)^3}\\&=\frac{x+z}{(1-y)^2}.\end{aligned}$$

19.
$$\frac{x^3+y^3}{2(x+y)^2}=\frac{(x+y)(x^2-xy+y^2)}{2(x+y)^2}=\frac{x^2-xy+y^2}{2(x+y)}.$$

20.
$$\begin{aligned}\frac{x^8y-x^2y^5z^2}{x^3y^4-y^6z}&=\frac{x^2y(x^6-y^4z^2)}{y^4(x^3-y^2z)}\\&=\frac{x^2y(x^3+y^2z)(x^3-y^2z)}{y^4(x^3-y^2z)}\\&=\frac{x^2(x^3+y^2z)}{y^3}.\end{aligned}$$

21.
$$\frac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}=\frac{(x+a)(x+b)}{(x+a)(x+c)}=\frac{x+b}{x+c}.$$

22.
$$\frac{4a^2-7ab+3b^2}{5a^2-3ab-2b^2}=\frac{(a-b)(4a-3b)}{(a-b)(5a+2b)}=\frac{4a-3b}{5a+2b}.$$

23.
$$\frac{3x^2-10xy+8y^2}{5x^2-13xy+6y^2}=\frac{(x-2y)(3x-4y)}{(x-2y)(5x-3y)}=\frac{3x-4y}{5x-3y}.$$

24. $\dfrac{2x^3+5x^2-12x}{7x^3+25x^2-12x}=\dfrac{x(2x^2+5x-12)}{x(7x^2+25x-12)}=\dfrac{x(x+4)(2x-3)}{x(x+4)(7x-3)}=\dfrac{2x-3}{7x-3}.$

25. $\dfrac{a^4-a^3-a+1}{a^4-2a^3-a^2-2a+1}.$

$$\begin{array}{r|r|l}
a^4-a^3 \qquad\qquad -a+1 & a^4-2a^3-a^2-2a+1 & 1 \\
\underline{a^4+a^3+a^2 \qquad\qquad\quad} & \underline{a^4-a^3 \qquad\quad -a+1} & \\
-2a^3-a^2-a+1 & -a\,\big)\,\underline{-a^3-a^2-a} & \\
\underline{-2a^3-2a^2-2a \quad\;} & a^2+a+1 & a^2-2a+1 \\
a^2+a+1 & & \\
\underline{a^2+a+1} & &
\end{array}$$

$\therefore$ the H.C.F. $= a^2+a+1.$

$$\therefore \frac{a^4-a^3-a+1}{a^4-2a^3-a^2-2a+1}=\frac{(a^2+a+1)(a^2-2a+1)}{(a^2+a+1)(a^2-3a+1)}$$

$$=\frac{(a^2+a+1)(a-1)^2}{(a^2+a+1)(a^2-3a+1)}$$

$$=\frac{(a-1)^2}{a^2-3a+1}.$$

26. $\dfrac{(x+a)^2-(b+c)^2}{(x+b)^2-(a+c)^2}=\dfrac{(x+a+b+c)(x+a-b-c)}{(x+b+a+c)(x+b-a-c)}=\dfrac{x+a-b-c}{x-a+b-c}.$

27. $\dfrac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}$

$$=\frac{7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6}{5x^4y+10x^3y^2+10x^2y^3+5xy^4}$$

$$=\frac{7xy(x^5+3x^4y+5x^3y^2+5x^2y^3+3xy^4+y^5)}{5xy(x^3+2x^2y+2xy^2+y^3)}$$

$$=\frac{7xy(x^3+2x^2y+2xy^2+y^3)(x^2+xy+y^2)}{5xy(x^3+2x^2y+2xy^2+y^3)}$$

$$=\frac{7(x^2+xy+y^2)}{5}.$$

28. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}=\dfrac{a^4(a-b)-b^4(a-b)}{a(a^3-a^2b-ab^2+b^3)}$

$$=\frac{(a^4-b^4)(a-b)}{a[a^2(a-b)-b^2(a-b)]}$$

$$=\frac{(a^2+b^2)(a^2-b^2)(a-b)}{a(a^2-b^2)(a-b)}$$

$$=\frac{a^2+b^2}{a}.$$

29.
$$\frac{x^4+4x+3}{x^5-5x-4}=\frac{(x+1)(x^3-x^2+x+3)}{(x+1)(x^4-x^3+x^2-x-4)}$$
$$=\frac{(x+1)^2(x^2-2x+3)}{(x+1)^2(x^3-2x^2+3x-4)}$$
$$=\frac{x^2-2x+3}{x^3-2x^2+3x-4}.$$

30.
$$\frac{(a^6-b^6)(a-b)}{(a^3-b^3)(a^4-b^4)}=\frac{(a^3+b^3)(a^3-b^3)(a-b)}{(a-b)(a^2+ab+b^2)(a^2+b^2)(a^2-b^2)}$$
$$=\frac{(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)(a-b)}{(a-b)(a^2+ab+b^2)(a^2+b^2)(a+b)(a-b)}$$
$$=\frac{a^2-ab+b^2}{a^2+b^2}.$$

31.
$$\frac{x^2(y-z)+y^2(z-x)+z^2(x-y)}{x^2z-xz^2+xy^2-x^2y+yz^2-y^2z}$$
$$=\frac{x^2(y-z)+y^2(z-x)+z^2(x-y)}{x^2(z-y)+y^2(x-z)+z^2(y-x)}=-1.$$

32.
$$\frac{x^3-5x^2y-2xy^2+24y^3}{x^3+3x^2y-10xy^2-24y^3}.$$

$$\begin{array}{r|r|l}
1-5-2+24 & 1+3-10-24 & 1 \\
1-1-6 & 1-5-2+24 & \\
\hline
-4+4+24 & 8)\,8-8-48 & \\
-4+4+24 & 1-1-6 & 1-4 \\
\hline
\end{array}$$

$\therefore$ the H.C.F. $=x^2-xy-6y^2$.

$$\therefore\ \frac{x^3-5x^2y-2xy^2+24y^3}{x^3+3x^2y-10xy^2-24y^3}=\frac{(x^2-xy-6y^2)(x-4y)}{(x^2-xy-6y^2)(x+4y)}=\frac{x-4y}{x+4y}.$$

33.
$$\frac{2x^4-3x^3-47x^2-45x+18}{3x^4-13x^3-45x^2+94x-24}.$$

$$\begin{array}{r|r|l}
2-3-47-45+18 & 3-13-45+94-24 & \\
2-6-38+12 & 2\phantom{-13-45+94-24} & \\
\hline
3-9-57+18 & 6-26-90+188-48 & 3 \\
3-9-57+18 & 6-9-141-135+54 & \\
\hline
 & -17)\,-17+51+323-102 & \\
 & 1-3-19+6 & 2+3
\end{array}$$

$\therefore$ the H.C.F. $=x^3-3x^2-19x+6$.

$$\therefore\ \frac{2x^4-3x^3-47x^2-45x+18}{3x^4-13x^3-45x^2+94x-24}$$
$$=\frac{(x^3-3x^2-19x+6)(2x+3)}{(x^3-3x^2-19x+6)(3x-4)}=\frac{2x+3}{3x-4}.$$

Exercise 55. Page 127

Reduce to lowest terms:

1. $$\frac{a^2-4ab+4b^2}{2b^2+ab-a^2}=\frac{(a-2b)(a-2b)}{(2b-a)(b+a)}=\frac{(2b-a)(2b-a)}{(2b-a)(b+a)}=\frac{2b-a}{b+a}.$$

2. $$\begin{aligned}\frac{(4y-12)^3}{27y^2-3y^4}&=\frac{4^3(y-3)^3}{3y^2(9-y^2)}\\&=\frac{4^3(y-3)^3}{3y^2(3+y)(3-y)}\\&=-\frac{64(y-3)^3}{3y^2(y+3)(y-3)}\\&=-\frac{64(y-3)^2}{3y^2(y+3)}.\end{aligned}$$

3. $$\begin{aligned}\frac{6y^2z-3x^2y^2}{x^4-4x^2z+4z^2}&=\frac{3y^2(2z-x^2)}{(x^2-2z)(x^2-2z)}\\&=\frac{3y^2(2z-x^2)}{(2z-x^2)(2z-x^2)}\\&=\frac{3y^2}{2z-x^2}.\end{aligned}$$

4. $$\begin{aligned}\frac{a^8-x^4y^4}{15x^2y^2-15a^2xy}&=\frac{(a^4+x^2y^2)(a^4-x^2y^2)}{15xy(xy-a^2)}\\&=\frac{(a^4+x^2y^2)(a^2+xy)(a^2-xy)}{15xy(xy-a^2)}\\&=-\frac{(a^4+x^2y^2)(a^2+xy)(a^2-xy)}{15xy(a^2-xy)}\\&=-\frac{(a^4+x^2y^2)(a^2+xy)}{15xy}.\end{aligned}$$

5. $$\begin{aligned}\frac{(9a^2x^2-15ax^3)^3}{675a^4x^8-243a^6x^6}&=\frac{3^3a^3x^6(3a-5x)^3}{27a^4x^6(25x^2-9a^2)}\\&=\frac{27a^3x^6(3a-5x)^3}{27a^4x^6(5x+3a)(5x-3a)}\\&=-\frac{27a^3x^6(5x-3a)^3}{27a^4x^6(5x+3a)(5x-3a)}\\&=-\frac{(5x-3a)^2}{a(5x+3a)}.\end{aligned}$$

6. $$\frac{6x^2-3y-2x^4+x^2y}{3y^2+3x^4-9x^2-x^2y^2}=\frac{3(2x^2-y)-x^2(2x^2-y)}{x^2(3x^2-y^2)-3(3x^2-y^2)}$$
$$=\frac{(3-x^2)(2x^2-y)}{(x^2-3)(3x^2-y^2)}$$
$$=-\frac{(x^2-3)(2x^2-y)}{(x^2-3)(3x^2-y^2)}$$
$$=-\frac{2x^2-y}{3x^2-y^2}.$$

7. $$\frac{100a^3b-64a^7b}{48a^4b^2-120a^2b^2+75b^2}=\frac{4a^3b(25-16a^4)}{3b^2(16a^4-40a^2+25)}$$
$$=\frac{4a^3b(5+4a^2)(5-4a^2)}{3b^2(4a^2-5)(4a^2-5)}$$
$$=\frac{4a^3b(5+4a^2)(5-4a^2)}{3b^2(5-4a^2)(5-4a^2)}$$
$$=\frac{4a^3(5+4a^2)}{3b(5-4a^2)}.$$

8. $$\frac{2a^2c^2-3a^2b^2-2b^2c^2+3b^4}{3b^2c^2-2a^2b^2-3a^2c^2+2b^4}=\frac{2c^2(a^2-b^2)-3b^2(a^2-b^2)}{3c^2(b^2-a^2)+2b^2(b^2-a^2)}$$
$$=\frac{(2c^2-3b^2)(a^2-b^2)}{(3c^2+2b^2)(b^2-a^2)}$$
$$=\frac{(3b^2-2c^2)(b^2-a^2)}{(3c^2+2b^2)(b^2-a^2)}$$
$$=\frac{3b^2-2c^2}{2b^2+3c^2}.$$

9. $$\frac{48a^4b^3x^4-144a^2c^2x^5}{720ab^2c^6x^4-80a^5b^8c^2x^2}=\frac{48a^2x^4(a^2b^3-3c^2x)}{80ab^2c^2x^2(9c^4x^2-a^4b^6)}$$
$$=\frac{48a^2x^4(a^2b^3-3c^2x)}{80ab^2c^2x^2(3c^2x+a^2b^3)(3c^2x-a^2b^3)}$$
$$=-\frac{48a^2x^4(3c^2x-a^2b^3)}{80ab^2c^2x^2(3c^2x+a^2b^3)(3c^2x-a^2b^3)}$$
$$=-\frac{3ax^2}{5b^2c^2(3c^2x+a^2b^3)}.$$

10. $$\frac{6a^2c^2+18bc^2-2a^4-6a^2b}{4a^4+2a^2b^2-12a^2c^2-6b^2c^2}=\frac{6c^2(a^2+3b)-2a^2(a^2+3b)}{2a^2(2a^2+b^2)-6c^2(2a^2+b^2)}$$
$$=\frac{(6c^2-2a^2)(a^2+3b)}{(2a^2-6c^2)(2a^2+b^2)}$$
$$=-\frac{(6c^2-2a^2)(a^2+3b)}{(6c^2-2a^2)(2a^2+b^2)}$$
$$=-\frac{a^2+3b}{2a^2+b^2}.$$

Exercise 56. Page 128

Reduce to an integral or a mixed expression :

1. $\dfrac{12x^2-5y}{6x}=2x-\dfrac{5y}{6x}.$

2. $\dfrac{25x^3-3a+5c}{5x}$

$$=5x^2-\frac{3a-5c}{5x}.$$

3. $\dfrac{x^2+3x+2}{x+3}=x+\dfrac{2}{x+3}.$

4. $\dfrac{x^2+3y^2}{x+y}.$

$$\begin{array}{rl}
x^2+3y^2 & \underline{)\,x+y} \\
\underline{x^2+xy} & x-y \\
-xy+3y^2 & \\
\underline{-xy-y^2} & \\
4y^2 &
\end{array}$$

$$\therefore\ \frac{x^2+3y^2}{x+y}=x-y+\frac{4y^2}{x+y}.$$

5. $\dfrac{a^4-b^4}{a^2+b^2}=\dfrac{(a^2+b^2)(a^2-b^2)}{a^2+b^2}$

$$=a^2-b^2.$$

6. $\dfrac{a^2+6a+12b-5b^2}{a-2b+6}.$

$$\begin{array}{rl}
a^2+6a\ +12b-5b^2 & \underline{)\,a-2b+6} \\
\underline{a^2+6a\ -\ 2ab} & a+2b \\
2ab+12b-5b^2 & \\
\underline{2ab+12b-4b^2} & \\
-\ b^2 &
\end{array}$$

$$\therefore\ \frac{a^2+6a+12b-5b^2}{a-2b+6}$$

$$=a+2b-\frac{b^2}{a-2b+6}.$$

7. $\dfrac{4ab-2b^2-a^2}{2a-b}=2b-\dfrac{a^2}{2a-b}.$

8. $\dfrac{axy+4x+5y}{x+y}.$

$$\begin{array}{rl}
axy+4x+5y & \underline{)\,x+y} \\
\underline{4x+4y} & 4 \\
axy\qquad+\ y &
\end{array}$$

$$\therefore\ \frac{axy+4x+5y}{x+y}=4+\frac{axy+y}{x+y}.$$

9. $\dfrac{a^2+2ab+b^2}{a-2b}.$

$$\begin{array}{rl}
a^2+2ab+b^2 & \underline{)\,a-2b} \\
\underline{a^2-2ab} & a+4b \\
4ab+\ b^2 & \\
\underline{4ab-8b^2} & \\
9b^2 &
\end{array}$$

$$\therefore\ \frac{a^2+2ab+b^2}{a-2b}$$

$$=a+4b+\frac{9b^2}{a-2b}.$$

10. $\dfrac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}.$

$$\begin{array}{rl}
x^3-3x^2y+3xy^2-\ y^3 & \underline{)\,x^2-y^2} \\
\underline{x^3\qquad\quad-\ xy^2} & x-3y \\
-3x^2y+4xy^2-\ y^3 & \\
\underline{-3x^2y\qquad\quad+3y^3} & \\
4xy^2-4y^3 &
\end{array}$$

$$\therefore\ \frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}$$

$$=x-3y+\frac{4xy^2-4y^3}{x^2-y^2}$$

$$=x-3y+\frac{4y^2(x-y)}{(x+y)(x-y)}$$

$$=x-3y+\frac{4y^2}{x+y}.$$

Exercise 57. Page 129

Reduce to a fraction:

1. $$a-b+\frac{a^2+b^2}{a+b}=\frac{(a-b)(a+b)+a^2+b^2}{a+b}=\frac{a^2-b^2+a^2+b^2}{a+b}=\frac{2a^2}{a+b}.$$

2. $$\begin{aligned}a-b-\frac{a-b-c}{2}&=\frac{2(a-b)-(a-b-c)}{2}\\&=\frac{2a-2b-a+b+c}{2}\\&=\frac{a-b+c}{2}.\end{aligned}$$

3. $$a^2-ax+x^2-\frac{a^3}{a+x}=\frac{(a^2-ax+x^2)(a+x)-a^3}{a+x}=\frac{a^3+x^3-a^3}{a+x}=\frac{x^3}{a+x}.$$

4. $$\begin{aligned}3a+4+\frac{28-5a}{2a-7}&=\frac{(3a+4)(2a-7)+28-5a}{2a-7}\\&=\frac{6a^2-13a-28+28-5a}{2a-7}\\&=\frac{6a^2-18a}{2a-7}.\end{aligned}$$

5. $$\frac{a-7b}{a+4b}-a+4b=\frac{a-7b-(a-4b)(a+4b)}{a+4b}=\frac{a-7b-a^2+16b^2}{a+4b}.$$

6. $$\frac{a+b+c}{a+c}-1=\frac{a+b+c-(a+c)}{a+c}=\frac{a+b+c-a-c}{a+c}=\frac{b}{a+c}.$$

7. $$\begin{aligned}a^2+ax+x^2+\frac{2x^3}{a-x}&=\frac{(a^2+ax+x^2)(a-x)+2x^3}{a-x}\\&=\frac{a^3-x^3+2x^3}{a-x}\\&=\frac{a^3+x^3}{a-x}.\end{aligned}$$

8. $$1-\frac{a^2-2ax+x^2}{a^2+x^2}=\frac{a^2+x^2-(a^2-2ax+x^2)}{a^2+x^2}=\frac{2ax}{a^2+x^2}.$$

Exercise 58. Page 131

Express with lowest common denominator:

1. $\frac{x}{a-b}$; $\frac{y}{a+b}$.

The L.C.D. is a^2-b^2.

The respective quotients are $a+b$; $a-b$.

The respective products are $x(a+b)$; $y(a-b)$.

The required fractions are $\frac{x(a+b)}{a^2-b^2}$; $\frac{y(a-b)}{a^2-b^2}$.

2. $\frac{4x-5}{3}$; $\frac{2x+7}{4}$.

The L.C.D is 12.

The respective quotients are 4; 3.

The respective products are $4(4x-5)$; $3(2x+7)$.

The required fractions are $\frac{4(4x-5)}{12}$; $\frac{3(2x+7)}{12}$.

3. $\frac{2a-3b}{2x}$; $\frac{3a-4b}{3x}$.

The L.C.D. is $6x$.

The respective quotients are 3; 2.

The respective products are $3(2a-3b)$; $2(3a-4b)$.

The required fractions are $\frac{3(2a-3b)}{6x}$; $\frac{2(3a-4b)}{6x}$.

4. $\frac{4a-5b}{7ac}$; $\frac{2a+3b}{5a^2c^2}$.

The L.C.D. is $35a^2c^2$.

The respective quotients are $5ac$; 7.

The respective products are $5ac(4a-5b)$; $7(2a+3b)$.

The required fractions are $\frac{5ac(4a-5b)}{35a^2c^2}$; $\frac{7(2a+3b)}{35a^2c^2}$.

5. $\frac{2}{x-2}$; $\frac{3}{x-3}$.

The L.C.D. is $(x-2)(x-3)=x^2-5x+6$.

The respective quotients are $x-3$; $x-2$.

The respective products are $2(x-3)$; $3(x-2)$.

The required fractions are $\frac{2(x-3)}{x^2-5x+6}$; $\frac{3(x-2)}{x^2-5x+6}$.

6. $\frac{5a}{x+4}$; $\frac{3a}{x-2}$.

The L.C.D. is $(x+4)(x-2) = x^2 + 2x - 8$.
The respective quotients are $x - 2$; $x + 4$.
The respective products are $5a(x-2)$; $3a(x+4)$.
The required fractions are $\frac{5a(x-2)}{x^2+2x-8}$; $\frac{3a(x+4)}{x^2+2x-8}$.

7. $\frac{1}{1+2c}$; $\frac{1}{1-4c^2}$.

The L.C.D. is $1 - 4c^2$.
The respective quotients are $1 - 2c$; 1.
The respective products are $1 - 2c$; 1.
The required fractions are $\frac{1-2c}{1-4c^2}$; $\frac{1}{1-4c^2}$.

8. $\frac{1}{x-2a}$; $\frac{1}{x-3a}$.

The L.C.D. is $(x-2a)(x-3a) = x^2 - 5ax + 6a^2$.
The respective quotients are $x - 3a$; $x - 2a$.
The respective products are $x - 3a$; $x - 2a$.
The required fractions are $\frac{x-3a}{x^2-5ax+6a^2}$; $\frac{x-2a}{x^2-5ax+6a^2}$.

9. $\frac{1}{x+y}$; $\frac{2y}{x^2-y^2}$.

The L.C.D. is $x^2 - y^2$.
The respective quotients are $x - y$; 1.
The respective products are $x - y$; $2y$.
The required fractions are $\frac{x-y}{x^2-y^2}$; $\frac{2y}{x^2-y^2}$.

10. $\frac{a}{a-b}$; $\frac{a^2}{a+b}$; $\frac{a^3}{a^2-b^2}$.

The L.C.D. is $a^2 - b^2$.
The respective quotients are $a + b$; $a - b$; 1.
The respective products are $a(a+b)$; $a^2(a-b)$; a^3.
The required fractions are $\frac{a(a+b)}{a^2-b^2}$; $\frac{a^2(a-b)}{a^2-b^2}$; $\frac{a^3}{a^2-b^2}$.

11. $\frac{2}{a-b}$; $\frac{2}{b-c}$; $\frac{2}{c-a}$.

The L.C.D. is $(a-b)(b-c)(c-a)$.

The respective quotients are

$$(b-c)(c-a);\ (a-b)(c-a);\ (a-b)(b-c).$$

The respective products are

$$2(b-c)(c-a);\ 2(a-b)(c-a);\ 2(a-b)(b-c).$$

The required fractions are

$$\frac{2(b-c)(c-a)}{(a-b)(b-c)(c-a)};\ \frac{2(a-b)(c-a)}{(a-b)(b-c)(c-a)};\ \frac{2(a-b)(b-c)}{(a-b)(b-c)(c-a)}.$$

12. $\frac{7a+4}{x-3};\ \frac{5a+2}{3x-9};\ \frac{4a-5}{4x-12}.$

The L.C.D. is $12(x-3)$.

The respective quotients are $12;\ 4;\ 3$.

The respective products are $12(7a+4);\ 4(5a+2);\ 3(4a-5)$.

The required fractions are $\frac{12(7a+4)}{12(x-3)};\ \frac{4(5a+2)}{12(x-3)};\ \frac{3(4a-5)}{12(x-3)}.$

13. $\frac{a}{a+1};\ \frac{a^2}{(a+1)^2};\ \frac{a^3}{(a+1)^3}.$

The L.C.D. is $(a+1)^3$.

The respective quotients are $(a+1)^2;\ a+1;\ 1$.

The respective products are $a(a+1)^2;\ a^2(a+1);\ a^3$.

The required fractions are $\frac{a(a+1)^2}{(a+1)^3};\ \frac{a^2(a+1)}{(a+1)^3};\ \frac{a^3}{(a+1)^3}.$

14. $\frac{4}{9-a^2};\ \frac{3-a}{3+a};\ \frac{3+a}{3-a}.$

The L.C.D. is $9-a^2$.

The respective quotients are $1;\ 3-a;\ 3+a$.

The respective products are $4;\ (3-a)^2;\ (3+a)^2$.

The required fractions are $\frac{4}{9-a^2};\ \frac{(3-a)^2}{9-a^2};\ \frac{(3+a)^2}{9-a^2}.$

15. $\frac{1}{x^2-5x+6};\ \frac{1}{x^2-7x+12};\ \frac{1}{x^2-6x+8}$

$$=\frac{1}{(x-2)(x-3)};\ \frac{1}{(x-3)(x-4)};\ \frac{1}{(x-2)(x-4)}.$$

The L.C.D. is $(x-2)(x-3)(x-4)$.

The respective quotients are $x-4;\ x-2;\ x-3$.

The respective products are $x-4;\ x-2;\ x-3$.

The required fractions are

$$\frac{x-4}{(x-2)(x-3)(x-4)};\ \frac{x-2}{(x-2)(x-3)(x-4)};\ \frac{x-3}{(x-2)(x-3)(x-4)}.$$

16. $$\frac{7x}{8x^2-10xy+3y^2};\ \frac{4x}{12x^2-xy-6y^2};\ \frac{3x}{6x^2+xy-2y^2}$$

$$=\frac{7x}{(2x-y)(4x-3y)};\ \frac{4x}{(4x-3y)(3x+2y)};\ \frac{3x}{(3x+2y)(2x-y)}.$$

The L.C.D. is $(2x-y)(3x+2y)(4x-3y)$.

The respective quotients are $3x+2y$; $2x-y$; $4x-3y$.

The respective products are

$$7x(3x+2y);\ 4x(2x-y);\ 3x(4x-3y).$$

The required fractions are $\dfrac{7x(3x+2y)}{(2x-y)(3x+2y)(4x-3y)}$;

$$\frac{4x(2x-y)}{(2x-y)(3x+2y)(4x-3y)};\ \frac{3x(4x-3y)}{(2x-y)(3x+2y)(4x-3y)}.$$

17. $$\frac{3a+2b}{6a^2+ab-35b^2};\ \frac{2a-3b}{12a^2-25ab-7b^2};\ \frac{4a+5b}{8a^2+22ab+5b^2}.$$

$$=\frac{3a+2b}{(2a+5b)(3a-7b)};\ \frac{2a-3b}{(3a-7b)(4a+b)};\ \frac{4a+5b}{(2a+5b)(4a+b)}.$$

The L.C.D. is $(2a+5b)(3a-7b)(4a+b)$.

The respective quotients are $4a+b$; $2a+5b$; $3a-7b$.

The respective products are

$$(3a+2b)(4a+b);\ (2a-3b)(2a+5b);\ (4a+5b)(3a-7b).$$

The required fractions are $\dfrac{(3a+2b)(4a+b)}{(2a+5b)(3a-7b)(4a+b)}$;

$$\frac{(2a-3b)(2a+5b)}{(2a+5b)(3a-7b)(4a+b)};\ \frac{(4a+5b)(3a-7b)}{(2a+5b)(3a-7b)(4a+b)}.$$

Exercise 59. Page 132

Find the algebraic sum of:

1. $$\frac{4a-23b}{4}-\frac{4a-25b}{6}+\frac{19b-3a}{12}.$$

The L.C.D. is 12.

The respective multipliers are 3; 2; 1.

$$\begin{aligned} 3(4a-23b) &= 12a-69b = \text{1st numerator.}\\ -2(4a-25b) &= -8a+50b = \text{2d numerator.}\\ 1(19b-3a) &= \underline{-3a+19b} = \text{3d numerator.}\\ & \quad\ a \qquad\quad = \text{the sum of the numerators.} \end{aligned}$$

Hence, the sum of the fractions is $\dfrac{a}{12}$.

2. $\dfrac{29a}{90}-\dfrac{3a-20b}{60}+\dfrac{3a-10b+5c}{20}-\dfrac{4a-2b+3c}{12}.$

The L.C.D. is 180.

The respective multipliers are 2; 3; 9; 15.

$$\begin{array}{rll}
2(29a) = & 58a & = \text{1st numerator.}\\
-3(3a-20b) = & -9a+60b & = \text{2d numerator.}\\
9(3a-10b+5c) = & 27a-90b+45c & = \text{3d numerator.}\\
-15(4a-2b+3c) = & \underline{-60a+30b-45c} & = \text{4th numerator.}\\
 & 16a & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{16a}{180}=\dfrac{4a}{45}.$

3. $\dfrac{2}{3a}-\dfrac{1}{2b}-\dfrac{2a+3}{6a^2}-\dfrac{1}{2x^2}+\dfrac{3a-2b}{6ab}.$

The L.C.D. is $6a^2bx^2$.

The respective multipliers are $2abx^2$; $3a^2x^2$; bx^2; $3a^2b$; ax^2.

$$\begin{array}{rlll}
2abx^2(2) = & 4abx^2 & & = \text{1st numerator.}\\
-3a^2x^2(1) = & -3a^2x^2 & & = \text{2d numerator.}\\
-bx^2(2a+3) = & -2abx^2 & -3bx^2 & = \text{3d numerator.}\\
-3a^2b(1) = & & -3a^2b & = \text{4th numerator.}\\
ax^2(3a-2b) = & \underline{-2abx^2+3a^2x^2} & & = \text{5th numerator.}\\
 & & -3bx^2-3a^2b & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{-3bx^2-3a^2b}{6a^2bx^2}=-\dfrac{x^2+a^2}{2a^2x^2}.$

4. $\dfrac{5}{2(x+1)}-\dfrac{1}{10(x-1)}-\dfrac{24}{5(2x+3)}.$

The L.C.D. is $10(x+1)(x-1)(2x+3)$.

The respective multipliers are

$5(x-1)(2x+3)$; $(x+1)(2x+3)$; $2(x+1)(x-1)$.

$$\begin{array}{rll}
5(x-1)(2x+3)(5) = & 50x^2+25x-75 & = \text{1st numerator.}\\
-(x+1)(2x+3) = & -2x^2-5x-3 & = \text{2d numerator.}\\
-2(x+1)(x-1)(24) = & \underline{-48x^2+48} & = \text{3d numerator.}\\
 & 20x-30 & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is

$$\frac{20x-30}{10(x+1)(x-1)(2x+3)}=\frac{2x-3}{(x+1)(x-1)(2x+3)}.$$

5. $$\frac{2+3x-5x^2}{x^3-3x^2+3x-1}-\frac{2x+5}{x^2-2x+1}+\frac{1}{x-1}$$
$$=\frac{2+3x-5x^2}{(x-1)^3}-\frac{2x+5}{(x-1)^2}+\frac{1}{x-1}.$$

The L.C.D. is $(x-1)^3$.

The respective multipliers are 1; $(x-1)$; $(x-1)^2$.

$$\begin{array}{rl} (1)(2+3x-5x^2) = & 2+3x-5x^2 = \text{1st numerator.} \\ -(x-1)(2x+5) = & 5-3x-2x^2 = \text{2d numerator.} \\ (x-1)^2(1) = & \underline{1-2x+\ \ x^2} = \text{3d numerator.} \\ & 8-2x-6x^2 = \text{the sum of the numerators.} \end{array}$$

Hence, the sum of the fractions is

$$\frac{8-2x-6x^2}{(x-1)^3}=\frac{(8+6x)(1-x)}{(x-1)^3}=-\frac{8+6x}{(x-1)^2}.$$

6. $$\frac{1}{a^2-7a+12}+\frac{2}{a^2-4a+3}-\frac{3}{a^2-5a+4}$$
$$=\frac{1}{(a-3)(a-4)}+\frac{2}{(a-1)(a-3)}-\frac{3}{(a-1)(a-4)}.$$

The L.C.D. is $(a-1)(a-3)(a-4)$.

The respective multipliers are $a-1$; $a-4$; $a-3$.

$$\begin{array}{rl} (a-1)(1) = & a-1 = \text{1st numerator.} \\ (a-4)(2) = & 2a-8 = \text{2d numerator.} \\ -(a-3)(3) = & \underline{-3a+9} = \text{3d numerator.} \\ & 0 = \text{the sum of the numerators.} \end{array}$$

Hence, the sum of the fractions is 0.

7. $$\frac{x+1}{x^2+x+1}+\frac{x-1}{x^2-x+1}+\frac{2}{x^4+x^2+1}.$$

The L.C.D. is x^4+x^2+1.

The respective multipliers are x^2-x+1; x^2+x+1; 1.

$$\begin{array}{rl} (x^2-x+1)(x+1) = & x^3+1 = \text{1st numerator.} \\ (x^2+x+1)(x-1) = & x^3-1 = \text{2d numerator.} \\ (1)(2) \quad = & \underline{\quad +2} = \text{3d numerator.} \\ & 2x^3+2 = \text{the sum of the numerators.} \end{array}$$

Hence, the sum of the fractions is

$$\frac{2x^3+2}{x^4+x^2+1}=\frac{2(x+1)(x^2-x+1)}{(x^2+x+1)(x^2-x+1)}=\frac{2(x+1)}{x^2+x+1}$$

8. $$\frac{x^2+ax+a^2}{x^3-a^3}-\frac{x^2-ax+a^2}{x^3+a^3}=\frac{1}{x-a}-\frac{1}{x+a}.$$

The L.C.D. is x^2-a^2.

The respective multipliers are $x+a$; $x-a$.

$$\begin{array}{rl}
(x+a)(1) = & x+a = \text{1st numerator.}\\
-(x-a)(1) = & \underline{-x+a} = \text{2d numerator.}\\
& 2a = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{2a}{x^2-a^2}$.

9. $$\frac{x^2+y^2}{xy}-\frac{x^2}{xy+y^2}-\frac{y^2}{x^2+xy}=\frac{x^2+y^2}{xy}-\frac{x^2}{y(x+y)}-\frac{y^2}{x(x+y)}.$$

The L.C.D. is $xy(x+y)$.

The respective multipliers are $x+y$; x; y.

$$\begin{array}{rl}
(x+y)(x^2+y^2) = & x^3+x^2y+xy^2+y^3 = \text{1st numerator.}\\
-x(x^2) = & -x^3 = \text{2d numerator.}\\
-y(y^2) = & \underline{\qquad\qquad\qquad -y^3} = \text{3d numerator.}\\
& x^2y+xy^2 = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{x^2y+xy^2}{xy(x+y)}=\dfrac{xy(x+y)}{xy(x+y)}=1$.

10. $$\frac{x^2-2x+3}{x^3+1}+\frac{x-2}{x^2-x+1}-\frac{1}{x+1}.$$

The L.C.D. is x^3+1.

The respective multipliers are 1; $x+1$; x^2-x+1.

$$\begin{array}{rl}
(1)(x^2-2x+3) = & x^2-2x+3 = \text{1st numerator.}\\
(x+1)(x-2) = & x^2-x-2 = \text{2d numerator.}\\
-(x^2-x+1)(1) = & \underline{-x^2+x-1} = \text{3d numerator.}\\
& x^2-2x = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{x^2-2x}{x^3+1}$.

11. $$\frac{x-y}{x^2-xy+y^2}+\frac{1}{x+y}+\frac{xy}{x^3+y^3}.$$

The L.C.D. is x^3+y^3.

The respective multipliers are $x + y$; $x^2 - xy + y^2$; 1.

$$\begin{array}{rrl}
(x+y)(x-y) = & x^2 \phantom{{}-xy} - y^2 & = \text{1st numerator.}\\
(x^2-xy+y^2)(1) = & x^2 - xy + y^2 & = \text{2d numerator.}\\
(1)(xy) = & + xy \phantom{{}+y^2} & = \text{3d numerator.}\\
\hline
 & 2x^2 \phantom{{}-xy+y^2} & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{2x^2}{x^3+y^3}$.

12. $\dfrac{1}{x-y} + \dfrac{x-y}{x^2+xy+y^2} + \dfrac{xy-2x^2}{x^3-y^3}$.

The L.C.D. is $x^3 - y^3$.

The respective multipliers are $x^2 + xy + y^2$; $x - y$; 1.

$$\begin{array}{rrl}
(x^2+xy+y^2)(1) = & x^2 + xy + y^2 & = \text{1st numerator.}\\
(x-y)(x-y) = & x^2 - 2xy + y^2 & = \text{2d numerator.}\\
(1)(xy-2x^2) = & -2x^2 + xy \phantom{{}+y^2} & = \text{3d numerator.}\\
\hline
 & + 2y^2 & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{2y^2}{x^3-y^3}$.

13. $\dfrac{1}{x-3} + \dfrac{x-1}{x^2+3x+9} + \dfrac{x^2+x-3}{x^3-27}$.

The L.C.D. is $x^3 - 27$.

The respective multipliers are $x^2 + 3x + 9$; $x - 3$; 1.

$$\begin{array}{rrl}
(x^2+3x+9)(1) = & x^2 + 3x + 9 & = \text{1st numerator.}\\
(x-3)(x-1) = & x^2 - 4x + 3 & = \text{2d numerator.}\\
(1)(x^2+x-3) = & x^2 + x - 3 & = \text{3d numerator.}\\
\hline
 & 3x^2 \phantom{{}+4x} + 9 & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{3x^2+9}{x^3-27}$.

14. $\dfrac{2a}{a^4-a^2+1} - \dfrac{1}{a^2-a+1} + \dfrac{1}{a^2+a+1}$.

The L.C.D. is $(a^4 - a^2 + 1)(a^4 + a^2 + 1) = a^8 + a^4 + 1$.

The respective multipliers are a^4+a^2+1; $(a^4-a^2+1)(a^2+a+1)$; $(a^4 - a^2 + 1)(a^2 - a + 1)$.

$$\begin{array}{rrl}
(a^4+a^2+1)(2a) = & +2a^5 + 2a^3 + 2a \phantom{{}-1} & = \text{1st numerator.}\\
-(a^4-a^2+1)(a^2+a+1)(1) = & -a^6 - a^5 + a^3 - a - 1 & = \text{2d numerator.}\\
(a^4-a^2+1)(a^2-a+1)(1) = & a^6 - a^5 + a^3 - a + 1 & = \text{3d numerator.}\\
\hline
 & 4a^3 \phantom{{}-a+1} & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{4a^3}{a^8+a^4+1}$.

15. $\dfrac{2}{(x-1)^3}+\dfrac{1}{(x-1)^2}+\dfrac{2}{x-1}-\dfrac{1}{x}.$

The L.C.D. is $x(x-1)^3$.

The respective multipliers are x; $x(x-1)$; $x(x-1)^2$; $(x-1)^3$.

$$\begin{array}{rl}
(x)(2) = & +2x = \text{1st numerator.}\\
x(x-1)(1) = & +x^2-x = \text{2d numerator.}\\
x(x-1)^2(2) = & 2x^3-4x^2+2x = \text{3d numerator.}\\
-(x-1)^3(1) = & \underline{-x^3+3x^2-3x+1} = \text{4th numerator.}\\
& x^3 \qquad\qquad +1 = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{x^3+1}{x(x-1)^3}$.

16. $\dfrac{x-1}{x^2-7x+10}-\dfrac{x+2}{x^2-9x+14}-\dfrac{x-3}{x^2-12x+35}$

$$=\frac{x-1}{(x-2)(x-5)}-\frac{x+2}{(x-2)(x-7)}-\frac{x-3}{(x-5)(x-7)}.$$

The L.C.D. is $(x-2)(x-5)(x-7)$.

The respective multipliers are $x-7$; $x-5$; $x-2$.

$$\begin{array}{rl}
(x-7)(x-1) = & x^2-8x+7 = \text{1st numerator.}\\
-(x-5)(x+2) = & -x^2+3x+10 = \text{2d numerator.}\\
-(x-2)(x-3) = & \underline{-x^2+5x-6} = \text{3d numerator.}\\
& -x^2 \qquad\quad +11 = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{11-x^2}{(x-2)(x-5)(x-7)}$.

17. $\dfrac{2+x}{3+x^2}-\dfrac{3-x}{9+6x+x^2}+\dfrac{1-2x^2}{(3+x)(3+x^2)}$

$$=\frac{2+x}{3+x^2}-\frac{3-x}{(3+x)^2}+\frac{1-2x^2}{(3+x)(3+x^2)}.$$

The L.C.D. is $(3+x)^2(3+x^2)$.

The respective multipliers are $(3+x)^2$; $3+x^2$; $3+x$.

$$\begin{array}{rl}
(3+x)^2(2+x) = & 18+21x+8x^2+x^3 = \text{1st numerator.}\\
-(3+x^2)(3-x) = & -9+3x-3x^2+x^3 = \text{2d numerator.}\\
(3+x)(1-2x^2) = & \underline{3+x-6x^2-2x^3} = \text{3d numerator.}\\
& 12+25x-x^2 \qquad = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{12+25x-x^2}{(3+x)^2(3+x^2)}$.

18. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ac}{(b+c)(a+b)}+\dfrac{c^2-ab}{(a+c)(b+c)}.$

The L.C.D. is $(a+b)(b+c)(a+c)$.

The respective multipliers are $b+c$; $a+c$; $a+b$.

$$\begin{array}{rllll}
(b+c)(a^2-bc) = & a^2b-b^2c+a^2c-bc^2 & & = \text{1st numerator.}\\
(a+c)(b^2-ac) = & \quad +b^2c-a^2c & +ab^2-ac^2 & = \text{2d numerator.}\\
(a+b)(c^2-ab) = & -a^2b & +bc^2-ab^2+ac^2 & = \text{3d numerator,}\\
\hline
 & \qquad 0 & & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is 0.

19. $\dfrac{5x-2}{x^2-3x-4}+\dfrac{2x+1}{x^2-x-12}-\dfrac{3x-1}{x^2+x-20}$

$$=\frac{5x-2}{(x+1)(x-4)}+\frac{2x+1}{(x+3)(x-4)}-\frac{3x-1}{(x-4)(x+5)}.$$

The L.C.D. is $(x+1)(x+3)(x-4)(x+5)$.

The respective multipliers are

$$(x+3)(x+5);\ (x+1)(x+5);\ (x+1)(x+3).$$

$$\begin{array}{rll}
(x+3)(x+5)(5x-2)= & 5x^3+38x^2+59x-30 & = \text{1st numerator.}\\
(x+1)(x+5)(2x+1)= & 2x^3+13x^2+16x+\ \ 5 & = \text{2d numerator.}\\
-(x+1)(x+3)(3x-1)= & -3x^3-11x^2-\ \ 5x+\ \ 3 & = \text{3d numerator.}\\
\hline
 & 4x^3+40x^2+70x-22 & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{4x^3+40x^2+70x-22}{(x+1)(x+3)(x-4)(x+5)}.$

20. $\dfrac{7+3x^2}{4-x^2}-\dfrac{5-2x^2}{4+4x+x^2}-\dfrac{3-2x+x^2}{4-4x+x^2}$

$$=\frac{7+3x^2}{(2+x)(2-x)}-\frac{5-2x^2}{(2+x)^2}-\frac{3-2x+x^2}{(2-x)^2}.$$

The L.C.D. is $(2+x)^2(2-x)^2$.

The respective multipliers are $(2+x)(2-x)$; $(2-x)^2$; $(2+x)^2$.

$$\begin{array}{rll}
(2+x)(2-x)(7+3x^2)= & 28\qquad\quad +5x^2\qquad\quad -3x^4 & = \text{1st numerator.}\\
-(2-x)^2(5-2x^2)= & -20+20x+3x^2-\ 8x^3+2x^4 & = \text{2d numerator.}\\
-(2+x)^2(3-2x+x^2)= & -12-\ 4x+\ x^2-\ 2x^3-\ x^4 & = \text{3d numerator.}\\
\hline
 & -\ 4+16x+9x^2-10x^3-2x^4 & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{-4+16x+9x^2-10x^3-2x^4}{(2+x)^2(2-x)^2}.$

21. $$\frac{a-5x}{4a^2-20ax+25x^2}-\frac{3a^2-4ax+x^2}{24a^3-180a^2x+450ax^2-375x^3}$$
$$=\frac{a-5x}{(2a-5x)^2}-\frac{(a-x)(3a-x)}{3(2a-5x)^3}.$$

The L.C.D. is $3(2a-5x)^3$.

The respective multipliers are $3(2a-5x)$; 1.

$$\begin{array}{rl}
3(2a-5x)(a-5x)= & 6a^2-45ax+75x^2=\text{1st numerator.}\\
-1(3a^2-4ax+x^2)= & \underline{-3a^2+4ax-x^2}=\text{2d numerator.}\\
 & 3a^2-41ax+74x^2=\text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{3a^2-41ax+74x^2}{3(2a-5x)^3}$.

22. $$\frac{x+a}{x^2-(b+c)x+bc}+\frac{x+b}{x^2-(a+c)x+ac}+\frac{x+c}{x^2-(a+b)x+ab}$$
$$=\frac{x+a}{(x-b)(x-c)}+\frac{x+b}{(x-a)(x-c)}+\frac{x+c}{(x-a)(x-b)}.$$

The L.C.D. is $(x-a)(x-b)(x-c)$.

The respective multipliers are $x-a$; $x-b$; $x-c$.

$$\begin{array}{rl}
(x-a)(x+a)= & x^2-a^2=\text{1st numerator.}\\
(x-b)(x+b)= & x^2-b^2=\text{2d numerator.}\\
(x-c)(x+c)= & \underline{x^2-c^2}=\text{3d numerator.}\\
 & 3x^2-a^2-b^2-c^2=\text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{3x^2-a^2-b^2-c^2}{(x-a)(x-b)(x-c)}$.

23. $$\frac{1}{a^2+5a+6}+\frac{2a}{a^2+4a+3}+\frac{1}{a^2+3a+2}+\frac{2a+1}{a+3}$$
$$=\frac{1}{(a+2)(a+3)}+\frac{2a}{(a+1)(a+3)}+\frac{1}{(a+1)(a+2)}+\frac{2a+1}{a+3}.$$

The L.C.D. is $(a+1)(a+2)(a+3)$.

The respective multipliers are $a+1$; $a+2$; $a+3$; $(a+1)(a+2)$.

$$\begin{array}{rl}
(a+1)(1)= & +a+1=\text{1st numerator.}\\
(a+2)(2a)= & +2a^2+4a=\text{2d numerator.}\\
(a+3)(1)= & +a+3=\text{3d numerator.}\\
(a+1)(a+2)(2a+1)= & \underline{2a^3+7a^2+7a+2}=\text{4th numerator.}\\
 & 2a^3+9a^2+13a+6=\text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is

$$\frac{2a^3+9a^2+13a+6}{(a+1)(a+2)(a+3)}=\frac{(a+1)(a+2)(2a+3)}{(a+1)(a+2)(a+3)}=\frac{2a+3}{a+3}.$$

24. $\dfrac{x^2-7xy+12y^2}{4x^2-11xy+3y^2}-\dfrac{2x^2+7xy-4y^2}{8x^2-6xy+y^2}=\dfrac{x^2-7xy+12y^2}{4x^2-11xy+3y^2}-\dfrac{(x+4y)(2x-y)}{(4x-y)(2x-y)}=\dfrac{x^2-7xy+12y^2}{4x^2-11xy+3y^2}-\dfrac{x+4y}{4x-y}.$

The L.C.D. is $(4x-y)(4x^2-11xy+3y^2)$.

The respective multipliers are $4x-y$; $4x^2-11xy+3y^2$.

$$\begin{array}{rrrrrl}
(4x-y)(x^2-7xy+12y^2)= & 4x^3 & -29x^2y & +55xy^2 & -12y^3 & = \text{1st numerator.}\\
-(4x^2-11xy+3y^2)(x+4y)= & -4x^3 & -5x^2y & +41xy^2 & -12y^3 & = \text{2d numerator.}\\
\hline
 & & -34x^2y & +96xy^2 & -24y^3 & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{96xy^2-34x^2y-24y^3}{(4x-y)(4x^2-11xy+3y^2)}$.

25. $\dfrac{a^2+ab}{a^2+ab+b^2}+\dfrac{a^2-ab}{a^2-ab+b^2}-\dfrac{a^4-a^2b^2-b^4}{a^4+a^2b^2+b^4}.$

The L.C.D is $a^4+a^2b^2+b^4$.

The respective multipliers are a^2-ab+b^2; a^2+ab+b^2; 1.

$$\begin{array}{rrrrrl}
(a^2-ab+b^2)(a^2+ab)= & a^4 & & +ab^3 & & = \text{1st numerator.}\\
(a^2+ab+b^2)(a^2-ab)= & a^4 & & -ab^3 & & = \text{2d numerator.}\\
-(1)(a^4-a^2b^2-b^4)= & -a^4 & +a^2b^2 & & +b^4 & = \text{3d numerator.}\\
\hline
 & a^4 & +a^2b^2 & & +b^4 & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{a^4+a^2b^2+b^4}{a^4+a^2b^2+b^4}=1.$

26. $$\frac{x^4-2x^2-3}{15x^6-17x^2-18+25x^4}-\frac{x^2-4x+1}{12x^4-x^2-6}=\frac{(x^2+1)(x^2-3)}{(3x^2+2)(5x^4+5x^2-9)}-\frac{x^2-4x+1}{(4x^2-3)(3x^2+2)}.$$

The L.C.D is $(3x^2+2)(4x^2-3)(5x^4+5x^2-9)$.
The respective multipliers are $4x^2-3$; $5x^4+5x^2-9$.

$$\begin{array}{rl}
(4x^2-3)(x^4-2x^2-3)= & 4x^6 \quad -11x^4 \quad -6x^2 \quad +9 = \text{1st numerator.}\\
-(5x^4+5x^2-9)(x^2-4x+1)= & \underline{-5x^6+20x^5-10x^4+20x^3+4x^2-36x+9} = \text{2d numerator.}\\
 & -x^6+20x^5-21x^4+20x^3-2x^2-36x+18 = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{-x^6+20x^5-21x^4+20x^3-2x^2-36x+18}{(3x^2+2)(4x^2-3)(5x^4+5x^2-9)}$.

27. $$\frac{2-3x}{6-4x}+\frac{3-4x}{6+4x}-\frac{1-2x+x^2}{9-12x+4x^2}-\frac{2-5x^2}{36-16x^2}=\frac{2-3x}{2(3-2x)}+\frac{3-4x}{2(3+2x)}-\frac{(1-x)^2}{(3-2x)^2}-\frac{2-5x^2}{4(3+2x)(3-2x)}.$$

The L.C.D. is $4(3+2x)(3-2x)^2$.
The respective multipliers are $2(3+2x)(3-2x)$; $2(3-2x)^2$; $4(3+2x)$; $3-2x$.

$$\begin{array}{rl}
2(3+2x)(3-2x)(2-3x)= & 36-54x-16x^2+24x^3 = \text{1st numerator.}\\
2(3-2x)^2(3-4x)= & 54-144x+120x^2-32x^3 = \text{2d numerator.}\\
-4(3+2x)(1-x)^2= & -12+16x+4x^2-8x^3 = \text{3d numerator.}\\
-(3-2x)(2-5x^2)= & \underline{-6+4x+15x^2-10x^3} = \text{4th numerator.}\\
 & 72-178x+123x^2-26x^3 = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{72-178x+123x^2-26x^3}{4(3+2x)(3-2x)^2}$.

28. $\dfrac{x-y}{x+y}+\dfrac{y-z}{y+z}+\dfrac{z-x}{z+x}+\dfrac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}.$

The L.C.D. is $(x+y)(y+z)(z+x)$.
The respective multipliers are $(y+z)(z+x)$; $(x+y)(z+x)$; $(x+y)(y+z)$; 1.

$$\begin{array}{rl}
(y+z)(z+x)(x-y) = & x^2y+x^2z-xy^2+xz^2-y^2z-yz^2 = \text{1st numerator.}\\
(x+y)(z+x)(y-z) = & x^2y-x^2z+xy^2-xz^2+y^2z-yz^2 = \text{2d numerator.}\\
(x+y)(y+z)(z-x) = & -x^2y-x^2z-xy^2+xz^2+y^2z+yz^2 = \text{3d numerator.}\\
(1)(x-y)(y-z)(z-x) = & \underline{-x^2y+x^2z+xy^2-xz^2-y^2z+yz^2} = \text{4th numerator.}\\
& \qquad\qquad\quad 0 \qquad\qquad\qquad = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is 0.

29. $\dfrac{y-z}{x^2-(y-z)^2}+\dfrac{z-x}{y^2-(z-x)^2}+\dfrac{x-y}{z^2-(x-y)^2}$

$$=\frac{y-z}{(x+y-z)(x-y+z)}+\frac{z-x}{(-x+y+z)(x+y-z)}+\frac{x-y}{(x-y+z)(-x+y+z)}.$$

The L.C.D. is $(x+y-z)(x-y+z)(-x+y+z)$.
The respective multipliers are $-x+y+z$; $x-y+z$; $x+y-z$.

$$\begin{array}{rl}
(-x+y+z)(y-z) = & \quad +y^2-z^2-xy+xz \qquad\;\; = \text{1st numerator.}\\
(x-y+z)(z-x) = & -x^2 \qquad +z^2+xy \qquad -yz = \text{2d numerator.}\\
(x+y-z)(x-y) = & \underline{\;x^2-y^2 \qquad\qquad\qquad -xz+yz} = \text{3d numerator.}\\
& \qquad\qquad 0 \qquad\qquad\qquad = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is 0.

30. $\dfrac{2}{x-y}+\dfrac{2}{y-z}+\dfrac{2}{z-x}+\dfrac{(x-y)^2+(y-z)^2+(z-x)^2}{(x-y)(y-z)(z-x)}.$

The L.C.D. is $(x-y)(y-z)(z-x)$.

The respective multipliers are $(y-z)(z-x)$; $(x-y)(z-x)$; $(x-y)(y-z)$; 1.

$$\begin{array}{rl}
(y-z)(z-x)(2) = & - 2z^2 - 2xy + 2yz + 2xz = \text{1st numerator.}\\
(x-y)(z-x)(2) = & -2x^2 + 2xy - 2yz + 2xz = \text{2d numerator.}\\
(x-y)(y-z)(2) = & - 2y^2 + 2xy + 2yz - 2xz = \text{3d numerator.}\\
(1)[(x-y)^2+(y-z)^2+(z-x)^2] = & \underline{2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz} = \text{4th numerator.}\\
 & \qquad\qquad 0 \qquad\qquad = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is 0.

31. $\dfrac{a^3}{(a-b)(a-c)}-\dfrac{b^3}{(a-b)(b-c)}+\dfrac{c^3}{(a-c)(b-c)}.$

The L.C.D. is $(a-b)(a-c)(b-c)$.

The respective multipliers are $b-c$; $a-c$; $a-b$.

$$\begin{aligned}
(b-c)(a^3) &= a^3b-a^3c &&= \text{1st numerator.}\\
-(a-c)(b^3) &= -ab^3+b^3c &&= \text{2d numerator.}\\
(a-b)(c^3) &= +ac^3-bc^3 &&= \text{3d numerator.}\\
\hline
& a^3b-a^3c-ab^3+b^3c+ac^3-bc^3 &&= \text{the sum of the numerators.}
\end{aligned}$$

Hence, the sum of the fractions is

$$\frac{a^3b-a^3c-ab^3+b^3c+ac^3-bc^3}{(a-b)(a-c)(b-c)}.$$

Now $a^3b-a^3c-ab^3+b^3c+ac^3-bc^3$

$$\begin{aligned}
&=(a^3b-ab^3)-(a^3c-b^3c)+(ac^3-bc^3)\\
&=ab(a^2-b^2)-c(a^3-b^3)+c^3(a-b)\\
&=ab(a+b)(a-b)-c(a^2+ab+b^2)(a-b)+c^3(a-b)\\
&=(a-b)[ab(a+b)-c(a^2+ab+b^2)+c^3]\\
&=(a-b)(a^2b+ab^2-a^2c-abc-b^2c+c^3)\\
&=(a-b)[(a^2b-abc)+(ab^2-b^2c)-(a^2c-c^3)]\\
&=(a-b)[ab(a-c)+b^2(a-c)-c(a^2-c^2)]\\
&=(a-b)[ab(a-c)+b^2(a-c)-c(a+c)(a-c)]\\
&=(a-b)(a-c)(ab+b^2-ac-c^2)\\
&=(a-b)(a-c)[(ab-ac)+(b^2-c^2)]\\
&=(a-b)(a-c)[a(b-c)+(b+c)(b-c)]\\
&=(a-b)(a-c)(b-c)(a+b+c).
\end{aligned}$$

Hence, the sum of the fractions is

$$\frac{(a-b)(a-c)(b-c)(a+b+c)}{(a-b)(a-c)(b-c)}=a+b+c$$

Exercise 60. Page 135

Find the algebraic sum of:

1. $\dfrac{1}{(x-2)(x-3)}+\dfrac{2}{(x-1)(3-x)}+\dfrac{1}{(x-1)(x-2)}$

$$=\frac{1}{(x-2)(x-3)}-\frac{2}{(x-1)(x-3)}+\frac{1}{(x-1)(x-2)}.$$

The L.C.D. is $(x-1)(x-2)(x-3)$.

The respective multipliers are $x-1$; $x-2$; $x-3$.

$$\begin{aligned}(x-1)(1) &= x-1 = \text{1st numerator.}\\ -(x-2)(2) &= -2x+4 = \text{2d numerator.}\\ (x-3)(1) &= \underline{x-3} = \text{3d numerator.}\\ &0 = \text{the sum of the numerators.}\end{aligned}$$

Hence, the sum of the fractions is 0.

2. $$\frac{1}{x^2(x^2+y^2)}-\frac{1}{2x^3(y-x)}+\frac{1}{2x^3(x+y)}$$
$$=\frac{1}{x^2(x^2+y^2)}+\frac{1}{2x^3(x-y)}+\frac{1}{2x^3(x+y)}.$$

The L.C.D. is $2x^3(x+y)(x-y)(x^2+y^2)=2x^3(x^4-y^4)$.
The respective multipliers are

$$2x(x^2-y^2);\ (x+y)(x^2+y^2);\ (x-y)(x^2+y^2).$$

$$\begin{aligned}2x(x^2-y^2) &= 2x^3 -2xy^2 = \text{1st numerator.}\\ (x+y)(x^2+y^2) &= x^3+x^2y+xy^2+y^3 = \text{2d numerator.}\\ (x-y)(x^2+y^2) &= \underline{x^3-x^2y+xy^2-y^3} = \text{3d numerator.}\\ &4x^3 = \text{the sum of the numerators.}\end{aligned}$$

Hence, the sum of the fractions is $\dfrac{4x^3}{2x^3(x^4-y^4)}=\dfrac{2}{x^4-y^4}$.

3. $$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}$$
$$=\frac{1}{a(a-b)(a-c)}-\frac{1}{b(b-c)(a-b)}+\frac{1}{c(a-c)(b-c)}.$$

The L.C.D. is $abc(a-b)(a-c)(b-c)$.
The respective multipliers are $bc(b-c)$; $ac(a-c)$; $ab(a-b)$.

$$\begin{aligned}bc(b-c) &= b^2c-bc^2 = \text{1st numerator.}\\ -ac(a-c) &= -a^2c+ac^2 = \text{2d numerator.}\\ ab(a-b) &= \underline{+a^2b-ab^2} = \text{3d numerator.}\\ &b^2c-bc^2-a^2c+ac^2+a^2b-ab^2 = \text{the sum of the numerators.}\end{aligned}$$

Hence, the sum of the fractions is

$$\frac{b^2c-bc^2-a^2c+ac^2+a^2b-ab^2}{abc(a-b)(a-c)(b-c)}=\frac{(a-b)(a-c)(b-c)}{abc(a-b)(a-c)(b-c)}=\frac{1}{abc}.$$

4. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}=\dfrac{a^2}{(a-b)(a-c)}-\dfrac{b^2}{(b-c)(a-b)}+\dfrac{c^2}{(a-c)(b-c)}.$

The L.C.D. is $(a-b)(a-c)(b-c)$.
The respective multipliers are $b-c$; $a-c$; $a-b$.

$$\begin{array}{rrl}
(b-c)(a^2) = & a^2b-a^2c \qquad\qquad\qquad\quad & = \text{1st numerator.}\\
-(a-c)(b^2) = & -ab^2+b^2c \qquad\quad\; & = \text{2d numerator.}\\
(a-b)(c^2) = & \underline{\qquad\qquad\qquad\qquad +ac^2-bc^2} & = \text{3d numerator.}\\
 & a^2b-a^2c-ab^2+b^2c+ac^2-bc^2 & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{a^2b-a^2c-ab^2+b^2c+ac^2-bc^2}{(a-b)(a-c)(b-c)}=\dfrac{(a-b)(a-c)(b-c)}{(a-b)(a-c)(b-c)}=1.$

5. $\dfrac{a+b}{b}-\dfrac{2a}{a+b}+\dfrac{a^2b-a^3}{a^2b-b^3}=\dfrac{a+b}{b}-\dfrac{2a}{a+b}+\dfrac{a^2(b-a)}{b(a^2-b^2)}=\dfrac{a+b}{b}-\dfrac{2a}{a+b}-\dfrac{a^2}{b(a+b)}.$

The L.C.D. is $b(a+b)$.
The respective multipliers are $a+b$; b; 1.

$$\begin{array}{rrl}
(a+b)(a+b) = & a^2+2ab+b^2 & = \text{1st numerator.}\\
-(b)(2a) = & -2ab \qquad & = \text{2d numerator.}\\
-(1)(a^2) = & \underline{-a^2 \qquad\qquad\quad} & = \text{3d numerator.}\\
 & b^2 & = \text{the sum of the numerators.}
\end{array}$$

Hence, the sum of the fractions is $\dfrac{b^2}{b(a+b)}=\dfrac{b}{a+b}.$

6. $\frac{x^4}{(x-y)(x-z)}+\frac{y^4}{(y-z)(y-x)}+\frac{z^4}{(z-x)(z-y)}=\frac{x^4}{(x-y)(x-z)}-\frac{y^4}{(y-z)(x-y)}+\frac{z^4}{(x-z)(y-z)}.$

The L.C.D. is $(x-y)(x-z)(y-z)$.

The respective multipliers are $y-z$; $x-z$; $x-y$.

$$\begin{array}{rl}(y-z)(x^4)= & x^4y-x^4z \qquad\qquad\qquad\qquad = \text{1st numerator.}\\ -(x-z)(y^4)= & \qquad\qquad -xy^4+y^4z \qquad\quad = \text{2d numerator.}\\ (x-y)(z^4)= & \underline{\qquad\qquad\qquad\qquad +xz^4-yz^4} = \text{3d numerator.}\\ & x^4y-x^4z-xy^4+y^4z+xz^4-yz^4 = \text{the sum of the numerators.}\end{array}$$

Hence, the sum of the fractions is

$$\frac{x^4y-x^4z-xy^4+y^4z+xz^4-yz^4}{(x-y)(x-z)(y-z)}=\frac{(x-y)(x-z)(y-z)(x^2+y^2+z^2+xy+xz+yz)}{(x-y)(x-z)(y-z)}$$
$$=x^2+y^2+z^2+xy+xz+yz.$$

7. $\frac{a(b+c-a)}{(a-b)(a-c)}+\frac{b(c+a-b)}{(b-c)(b-a)}+\frac{c(a+b-c)}{(c-a)(c-b)}=\frac{a(b+c-a)}{(a-b)(a-c)}-\frac{b(c+a-b)}{(b-c)(a-b)}+\frac{c(a+b-c)}{(a-c)(b-c)}.$

The L.C.D. is $(a-b)(a-c)(b-c)$.

The respective multipliers are $b-c$; $a-c$; $a-b$.

$$\begin{array}{rl}(b-c)(a)(b+c-a)= & ab^2-a^2b-ac^2+a^2c \qquad\qquad\qquad = \text{1st numerator.}\\ -(a-c)(b)(c+a-b)= & ab^2-a^2b \qquad\qquad\qquad +bc^2-b^2c = \text{2d numerator.}\\ (a-b)(c)(a+b-c)= & \underline{\qquad\qquad\quad -ac^2+a^2c+bc^2-b^2c} = \text{3d numerator.}\\ & 2ab^2-2a^2b-2ac^2+2a^2c+2bc^2-2b^2c = \text{the sum of the numerators.}\end{array}$$

Hence, the sum of the fractions is

$$\frac{2ab^2-2a^2b-2ac^2+2a^2c+2bc^2-2b^2c}{(a-b)(a-c)(b-c)}=\frac{-2(a-b)(a-c)(b-c)}{(a-b)(a-c)(b-c)}=-2.$$

8. $$\frac{a^2}{(a-b)(a-c)(1+ax)}+\frac{b^2}{(b-c)(b-a)(1+bx)}+\frac{c^2}{(c-a)(c-b)(1+cx)}$$
$$=\frac{a^2}{(a-b)(a-c)(1+ax)}-\frac{b^2}{(b-c)(a-b)(1+bx)}+\frac{c^2}{(a-c)(b-c)(1+cx)}.$$

The L.C.D. is $(a-b)(a-c)(b-c)(1+ax)(1+bx)(1+cx)$.

The respective multipliers are $(b-c)(1+bx)(1+cx)$; $(a-c)(1+ax)(1+cx)$; $(a-b)(1+ax)(1+bx)$.

$$\begin{array}{rllllll}
(b-c)(1+bx)(1+cx)(a^2) = & a^2b-a^2c & & +a^2b^2x-a^2c^2x & & +a^2b^2cx^2-a^2bc^2x^2 & \\
-(a-c)(1+ax)(1+cx)(b^2) = & & -ab^2+b^2c & -a^2b^2x & +b^2c^2x-a^2b^2cx^2 & & +ab^2c^2x^2 \\
(a-b)(1+ax)(1+bx)(c^2) = & & +ac^2-bc^2 & +a^2c^2x-b^2c^2x & & +a^2bc^2x^2-ab^2c^2x^2 & \\
\hline
& \multicolumn{6}{l}{a^2b-a^2c-ab^2+b^2c+ac^2-bc^2 = \text{the sum of the numerators.}}
\end{array}$$

Hence, the sum of the fractions is

$$\frac{a^2b-a^2c-ab^2+b^2c+ac^2-bc^2}{(a-b)(a-c)(b-c)(1+ax)(1+bx)(1+cx)}=\frac{(a-b)(a-c)(b-c)}{(a-b)(a-c)(b-c)(1+ax)(1+bx)(1+cx)}$$
$$=\frac{1}{(1+ax)(1+bx)(1+cx)}.$$

9. $$\frac{3}{(x-y)(y-z)}-\frac{4}{(y-x)(z-x)}-\frac{6}{(x-z)(z-y)}$$
$$=\frac{3}{(x-y)(y-z)}+\frac{4}{(x-y)(z-x)}-\frac{6}{(z-x)(y-z)}.$$

The L.C.D. is $(x-y)(y-z)(z-x)$.
The respective multipliers are $z-x$; $y-z$; $x-y$.

$$\begin{array}{rrrrl} (z-x)(3)= & -3x & & +3z & = \text{1st numerator.}\\ (y-z)(4)= & & +4y & -4z & = \text{2d numerator.}\\ -(x-y)(6)= & -6x & +6y & & = \text{3d numerator.}\\ \hline & -9x & +10y & -z & = \text{the sum of the numerators.} \end{array}$$

Hence, the sum of the fractions is $\dfrac{10y-9x-z}{(x-y)(y-z)(z-x)}$.

10. $$\frac{7x-13}{2x-1}+\frac{28-13x}{3-2x}+\frac{28x+43}{4x^2-8x+3}$$
$$=\frac{7x-13}{2x-1}+\frac{13x-28}{2x-3}+\frac{28x+43}{(2x-1)(2x-3)}.$$

The L.C.D. is $(2x-1)(2x-3)$.
The respective multipliers are $2x-3$; $2x-1$; 1.

$$\begin{array}{rrrrl} (2x-3)(7x-13)= & 14x^2 & -47x & +39 & = \text{1st numerator.}\\ (2x-1)(13x-28)= & 26x^2 & -69x & +28 & = \text{2d numerator.}\\ (1)(28x+43)= & & +28x & +43 & = \text{3d numerator.}\\ \hline & 40x^2 & -88x & +110 & = \text{the sum of the numerators.} \end{array}$$

Hence, the sum of the fractions is $\dfrac{40x^2-88x+110}{(2x-1)(2x-3)}$.

11. $$\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(a-c)^2}{(a+b)^2-c^2}-\frac{c^2-(a-b)^2}{a^2-(b+c)^2}$$
$$=\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(a-c)^2}{(a+b)^2-c^2}+\frac{c^2-(a-b)^2}{(b+c)^2-a^2}$$
$$=\frac{(a+b-c)(a-b+c)}{(a+b+c)(a-b+c)}+\frac{(b+a-c)(b-a+c)}{(a+b+c)(a+b-c)}$$
$$+\frac{(c+a-b)(c-a+b)}{(b+c+a)(b+c-a)}$$
$$=\frac{a+b-c}{a+b+c}+\frac{b-a+c}{a+b+c}+\frac{c+a-b}{a+b+c}=\frac{a+b+c}{a+b+c}=1.$$

Exercise 61. Page 137

Find the product of:

1. $$\frac{12x^2}{5y^2} \times \frac{10xy}{9z^2} = \frac{12 \times 10x^3y}{5 \times 9y^2z^2} = \frac{8x^3}{3yz^2}.$$

2. $$\frac{a^2}{bc} \times \frac{b^2}{ac} \times \frac{c^2}{ab} = \frac{a^2b^2c^2}{a^2b^2c^2} = 1.$$

3. $$\frac{a^2b}{x^2y} \times \frac{b^2c}{y^3z} \times \frac{x^2y^2z}{abc} = \frac{a^2b^3cx^2y^2z}{abcx^2y^4z} = \frac{ab^2}{y^2}.$$

4. $$\frac{5a^2b}{3cd} \times \frac{4b^2c}{10a^2} \times \frac{9c^2d}{16b^3} = \frac{5 \times 4 \times 9a^2b^3c^3d}{3 \times 10 \times 16a^2b^3cd} = \frac{3c^2}{8}.$$

5. $$\frac{x^2-y^2}{x^2+y^2} \times \frac{4x}{x+y} = \frac{(x+y)(x-y)}{x^2+y^2} \times \frac{4x}{x+y} = \frac{4x(x-y)}{x^2+y^2}.$$

6. $$\frac{x^2-y^2}{x^2+y^2} \times \frac{3x^2}{4x+4y} = \frac{(x+y)(x-y)}{x^2+y^2} \times \frac{3x^2}{4(x+y)} = \frac{3x^2(x-y)}{4(x^2+y^2)}.$$

7. $$\frac{ab+ac}{bd-cd} \times \frac{ab-ac}{bd+cd} = \frac{a(b+c)}{d(b-c)} \times \frac{a(b-c)}{d(b+c)} = \frac{a^2}{d^2}.$$

8. $$\frac{4x+4y}{3x^2-3y^2} \times \frac{5x^2-5y^2}{7x+7y} = \frac{4(x+y)}{3(x^2-y^2)} \times \frac{5(x^2-y^2)}{7(x+y)} = \frac{20}{21}.$$

9. $$\frac{3x^2-x}{a} \times \frac{2a}{2x^2-4x} = \frac{x(3x-1)}{a} \times \frac{2a}{2x(x-2)} = \frac{3x-1}{x-2}.$$

10. $$\frac{3x^2}{5x-10} \times \frac{3x-6}{4x^3} = \frac{3x^2}{5(x-2)} \times \frac{3(x-2)}{4x^3} = \frac{9}{20x}.$$

11. $$\frac{(x-y)^2}{x+y} \times \frac{y}{x(x-y)} = \frac{y(x-y)}{x(x+y)}.$$

12. $$\frac{x^4-y^4}{x^2-2xy+y^2} \times \frac{x-y}{x^2+xy} = \frac{(x^2+y^2)(x+y)(x-y)}{(x-y)(x-y)} \times \frac{x-y}{x(x+y)}$$
$$= \frac{x^2+y^2}{x}.$$

13. $$\frac{x(a-x)}{a^2+2ax+x^2} \times \frac{a(a+x)}{a^2-2ax+x^2} = \frac{x(a-x)}{(a+x)^2} \times \frac{a(a+x)}{(a-x)^2}$$
$$= \frac{ax}{(a+x)(a-x)}$$
$$= \frac{ax}{a^2-x^2}.$$

14. $$\frac{a^6-b^6}{a^4+2a^2b^2+b^4}\times\frac{a^2+b^2}{a^2-ab+b^2}\times\frac{a+b}{a^3-b^3}$$
$$=\frac{(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2)}{(a^2+b^2)^2}\times\frac{a^2+b^2}{a^2-ab+b^2}$$
$$\times\frac{a+b}{(a-b)(a^2+ab+b^2)}$$
$$=\frac{(a+b)^2}{a^2+b^2}$$
$$=\frac{a^2+2ab+b^2}{a^2+b^2}.$$

15. $$\frac{a^3-b^3}{a^3+b^3}\times\frac{a+b}{a-b}\times\left(\frac{a^2-ab+b^2}{a^2+ab+b^2}\right)^2$$
$$=\frac{(a-b)(a^2+ab+b^2)}{(a+b)(a^2-ab+b^2)}\times\frac{a+b}{a-b}\times\frac{(a^2-ab+b^2)^2}{(a^2+ab+b^2)^2}$$
$$=\frac{a^2-ab+b^2}{a^2+ab+b^2}.$$

16. $$\frac{a^3+3a^2b+3ab^2+b^3}{a^2-b^2}\times\frac{a+b}{a^2+2ab+b^2}$$
$$=\frac{(a+b)^3}{(a+b)(a-b)}\times\frac{a+b}{(a+b)^2}$$
$$=\frac{a+b}{a-b}.$$

17. $$\frac{2a^3+13a^2x-15ax^2-126x^3}{a^2x+6ax^2-7x^3}\times\frac{2a^3+19a^2x+35ax^2}{a^3-a^2x-4ax^2-6x^3}$$
$$=\frac{(a-3x)(a+6x)(2a+7x)}{x(a-x)(a+7x)}\times\frac{a(a+7x)(2a+5x)}{(a-3x)(a^2+2ax+2x^2)}$$
$$=\frac{a(a+6x)(2a+7x)(2a+5x)}{x(a-x)(a^2+2ax+2x^2)}.$$

18. $$\frac{a^2-x^2}{a+b}\times\frac{a^2-b^2}{ax+x^2}\times\left(a+\frac{ax}{a-x}\right)$$
$$=\frac{(a+x)(a-x)}{a+b}+\frac{(a+b)(a-b)}{x(a+x)}\times\frac{a^2}{a-x}$$
$$=\frac{a^2(a-b)}{x}.$$

19. $$\left(\frac{x}{a}-\frac{a}{x}+\frac{y}{b}-\frac{b}{y}\right)\left(\frac{x}{a}-\frac{a}{x}-\frac{y}{b}+\frac{b}{y}\right)$$
$$=\left(\frac{x^2-a^2}{ax}+\frac{y^2-b^2}{by}\right)\left(\frac{x^2-a^2}{ax}-\frac{y^2-b^2}{by}\right)$$
$$=\frac{(x^2-a^2)^2}{a^2x^2}-\frac{(y^2-b^2)^2}{b^2y^2}$$
$$=\frac{x^4}{a^2x^2}-\frac{2a^2x^2}{a^2x^2}+\frac{a^4}{a^2x^2}-\frac{y^4}{b^2y^2}+\frac{2b^2y^2}{b^2y^2}-\frac{b^4}{b^2y^2}$$
$$=\frac{x^2}{a^2}-2+\frac{a^2}{x^2}-\frac{y^2}{b^2}+2-\frac{b^2}{y^2}$$
$$=\frac{x^2}{a^2}+\frac{a^2}{x^2}-\frac{y^2}{b^2}-\frac{b^2}{y^2}.$$

20. $$\left(\frac{a}{bc}+\frac{b}{ac}-\frac{c}{ab}-\frac{2}{c}\right)\left(1-\frac{2c}{a-b+c}\right)$$
$$=\frac{a^2+b^2-c^2-2ab}{abc}\times\frac{a-b-c}{a-b+c}$$
$$=\frac{(a-b+c)(a-b-c)}{abc}\times\frac{a-b-c}{a-b+c}$$
$$=\frac{(a-b-c)^2}{abc}.$$

21. $$\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}-\frac{x}{a}-\frac{a}{x}+1\right)\left(\frac{x}{a}-\frac{a}{x}\right)$$
$$=\frac{x^4+a^4-ax^3-a^3x+a^2x^2}{a^2x^2}\times\frac{x^2-a^2}{ax}$$
$$=\frac{x^6-ax^5+a^5x-a^6}{a^3x^3}.$$

22. $$\frac{x^2+2xy+y^2-4z^2}{4x^2-(y+2z)^2}\times\frac{(2x-y)^2-4z^2}{x^2-(y-2z)^2}\times\frac{(x-y)^2-4z^2}{(x+y)^2-4z^2}$$
$$=\frac{(x+y+2z)(x+y-2z)}{(2x+y+2z)(2x-y-2z)}\times\frac{(2x-y+2z)(2x-y-2z)}{(x+y-2z)(x-y+2z)}$$
$$\times\frac{(x-y+2z)(x-y-2z)}{(x+y+2z)(x+y-2z)}$$
$$=\frac{(2x-y+2z)(x-y-2z)}{(2x+y+2z)(x+y-2z)}.$$

23. $$\frac{x^2-3ax+2a^2}{x^2-2ax-3a^2}\times\frac{x^2-7ax+12a^2}{x^2-8ax+12a^2}\times\frac{x^2-11ax+30a^2}{x^2-5ax+4a^2}$$
$$=\frac{(x-a)(x-2a)}{(x+a)(x-3a)}\times\frac{(x-3a)(x-4a)}{(x-2a)(x-6a)}\times\frac{(x-5a)(x-6a)}{(x-a)(x-4a)}$$
$$=\frac{x-5a}{x+a}.$$

24. $$\frac{(x+3y)^2-z^2}{x^2-(3y+z)^2}\times\left(1-\frac{6y}{x+3y-z}\right)\times\frac{x^2-4y^2}{x^3-8y^3}$$
$$=\frac{(x+3y+z)(x+3y-z)}{(x+3y+z)(x-3y-z)}\times\frac{x-3y-z}{x+3y-z}$$
$$\times\frac{(x+2y)(x-2y)}{(x-2y)(x^2+2xy+4y^2)}$$
$$=\frac{x+2y}{x^2+2xy+4y^2}.$$

25. $$\frac{2a^2b^3c}{3x^2y^3}\times\frac{a^mb^nc^r}{x^my^n}\times\frac{6x^{m-1}y^{n-2}}{a^{m+1}b^{n+2}c^{r+3}}$$
$$=\frac{2\times 6a^{m+2}b^{n+3}c^{r+1}x^{m-1}y^{n-2}}{3a^{m+1}b^{n+2}c^{r+3}x^{m+2}y^{n+3}}$$
$$=\frac{4ab}{c^2x^3y^5}.$$

26. $$\frac{a^{x-1}b^{x-2}c^{x-3}}{x^{n-1}y^{n+2}z^{n-3}}\times\frac{x^{n+1}y^{n-2}z^{n+4}}{a^{x-2}b^{x+1}c^{x-1}}\times\frac{a^2c^2}{b^2x^2}$$
$$=\frac{a^{x+1}b^{x-2}c^{x-1}x^{n+1}y^{n-2}z^{n+4}}{a^{x-2}b^{x+3}c^{x-1}x^{n+1}y^{n+2}z^{n-3}}$$
$$=\frac{a^3z^7}{b^5y^4}.$$

27. $$\frac{a^{m-n}b^{n-p}c^{p-m}}{x^{n-p}y^{p-m}z^{m-n}}\times\frac{a^{n-p}b^{p-m}c^{m-n}}{x^{p-1}y^{m-2}z^{n-3}}\times\frac{x^ny^pz^m}{a^mb^nc^p}$$
$$=\frac{a^{m-p}b^{n-m}c^{p-n}x^ny^pz^m}{a^mb^nc^px^{n-1}y^{p-2}z^{m-3}}$$
$$=\frac{xy^2z^3}{a^pb^mc^n}.$$

Exercise 62. Page 140

Find the quotient of:

1. $$\frac{18\,xy}{25\,uz} \div \frac{6\,mn}{35\,x^2y^2} = \frac{18\,xy}{25\,uz} \times \frac{35\,x^2y^2}{6\,mn} = \frac{21\,x^3y^3}{5\,mnuz}.$$

2. $$\frac{14\,a^2b^3c}{39\,x^2y^5z^6} \div \frac{35\,x^7y^4z^8}{9\,a^4b^5z^2} = \frac{14\,a^2b^3c}{39\,x^2y^5z^6} \times \frac{9\,a^4b^5z^2}{35\,x^7y^4z^8} = \frac{6\,a^6b^8c}{65\,x^9y^9z^{12}}.$$

3. $$\frac{25\,a^4b^5c^6}{49\,x^4y^5z^6} \div \frac{30\,a^4bc^8}{77\,xy^7z^2} = \frac{25\,a^4b^5c^6}{49\,x^4y^5z^6} \times \frac{77\,xy^7z^2}{30\,a^4bc^8} = \frac{55\,b^4y^2}{42\,c^2x^3z^4}.$$

4. $$\begin{aligned}\frac{3p-3q}{5p+5q} \div \frac{9q-9p}{10q+10p} &= \frac{3p-3q}{5p+5q} \times \frac{10q+10p}{9q-9p}\\ &= \frac{3(p-q)}{5(p+q)} \times \frac{10(p+q)}{-9(p-q)}\\ &= -\frac{2}{3}.\end{aligned}$$

5. $$\begin{aligned}\frac{x^2+y^2}{x^2-y^2} \div \frac{3x^2+3y^2}{x+y} &= \frac{x^2+y^2}{x^2-y^2} \times \frac{x+y}{3x^2+3y^2}\\ &= \frac{x^2+y^2}{(x+y)(x-y)} \times \frac{x+y}{3(x^2+y^2)}\\ &= \frac{1}{3(x-y)}.\end{aligned}$$

6. $$\begin{aligned}\frac{6ab-6b^2}{a(a+b)} \div \frac{2b^2}{a(a^2-b^2)} &= \frac{6ab-6b^2}{a(a+b)} \times \frac{a(a^2-b^2)}{2b^2}\\ &= \frac{6b(a-b)}{a(a+b)} \times \frac{a(a+b)(a-b)}{2b^2}\\ &= \frac{3(a-b)^2}{b}.\end{aligned}$$

7. $$\begin{aligned}\frac{a^2-4x^2}{a^2+4ax} \div \frac{a^2-2ax}{ax+4x^2} &= \frac{a^2-4x^2}{a^2+4ax} \times \frac{ax+4x^2}{a^2-2ax}\\ &= \frac{(a+2x)(a-2x)}{a(a+4x)} \times \frac{x(a+4x)}{a(a-2x)}\\ &= \frac{x(a+2x)}{a^2}.\end{aligned}$$

8. $$\frac{8a^3}{a^3-b^3}\div\frac{4a^3}{a^2+ab+b^2}=\frac{8a^3}{a^3-b^3}\times\frac{a^2+ab+b^2}{4a^3}$$
$$=\frac{8a^3}{(a-b)(a^2+ab+b^2)}\times\frac{a^2+ab+b^2}{4a^3}$$
$$=\frac{2}{a-b}.$$

9. $$\frac{x^2+y^2+2xy-z^2}{z^2-x^2-y^2+2xy}\div\frac{x+y+z}{y+z-x}$$
$$=\frac{x^2+y^2+2xy-z^2}{z^2-x^2-y^2+2xy}\times\frac{y+z-x}{x+y+z}$$
$$=\frac{(x+y+z)(x+y-z)}{(z+x-y)(z-x+y)}\times\frac{y+z-x}{x+y+z}$$
$$=\frac{x+y-z}{x-y+z}.$$

10. $$\frac{a^2-3ab+2b^2}{a^2-6ab+9b^2}\div\frac{a^2-2ab+b^2}{a^2-5ab+6b^2}$$
$$=\frac{a^2-3ab+2b^2}{a^2-6ab+9b^2}\times\frac{a^2-5ab+6b^2}{a^2-2ab+b^2}$$
$$=\frac{(a-b)(a-2b)}{(a-3b)^2}\times\frac{(a-2b)(a-3b)}{(a-b)^2}$$
$$=\frac{(a-2b)^2}{(a-b)(a-3b)}$$
$$=\frac{a^2-4ab+4b^2}{a^2-4ab+3b^2}.$$

11. $$\frac{a^4-3a^3x+3a^2x^2-ax^3}{a^3b-b^4}\div\frac{a^4-2a^3x+a^2x^2}{a^2b^2+ab^3+b^4}$$
$$=\frac{a^4-3a^3x+3a^2x^2-ax^3}{a^3b-b^4}\times\frac{a^2b^2+ab^3+b^4}{a^4-2a^3x+a^2x^2}$$
$$=\frac{a(a-x)^3}{b(a-b)(a^2+ab+b^2)}\times\frac{b^2(a^2+ab+b^2)}{a^2(a-x)^2}$$
$$=\frac{b(a-x)}{a(a-b)}.$$

12. $$\frac{64\,a^3-96\,a^2b+36\,ab^2}{36\,a^2b-729\,b^3}\div\frac{48\,a^2b-27\,b^3}{8\,a^3-72\,a^2b+162\,ab^2}$$
$$=\frac{64\,a^3-96\,a^2b+36\,ab^2}{36\,a^2b-729\,b^3}\times\frac{8\,a^3-72\,a^2b+162\,ab^2}{48\,a^2b-27\,b^3}$$
$$=\frac{4\,a(4\,a-3\,b)^2}{9\,b(2\,a+9\,b)(2\,a-9\,b)}\times\frac{2\,a(2\,a-9\,b)^2}{3\,b(4\,a+3\,b)(4\,a-3\,b)}$$
$$=\frac{8\,a^2(4\,a-3\,b)(2\,a-9\,b)}{27\,b^2(4\,a+3\,b)(2\,a+9\,b)}.$$

13. $$\left(\frac{1+x}{1-x}-\frac{1-y}{1+y}\right)\div\left[1+\frac{(1+x)(1-y)}{(1-x)(1+y)}\right]$$
$$=\frac{(1+x)(1+y)-(1-x)(1-y)}{(1-x)(1+y)}\div\frac{(1-x)(1+y)+(1+x)(1-y)}{(1-x)(1+y)}$$
$$=\frac{2(x+y)}{(1-x)(1+y)}\div\frac{2(1-xy)}{(1-x)(1+y)}$$
$$=\frac{2(x+y)}{(1-x)(1+y)}\times\frac{(1-x)(1+y)}{2(1-xy)}$$
$$=\frac{x+y}{1-xy}.$$

14. $$\left(\frac{a+b}{1-ab}-a\right)\frac{a}{b}\div\left[1+\frac{a(a+b)}{1-ab}\right]=\frac{a(1+a^2)}{1-ab}\div\frac{1+a^2}{1-ab}$$
$$=\frac{a(1+a^2)}{1-ab}\times\frac{1-ab}{1+a^2}$$
$$=a.$$

15. $$\left(2\,x-\frac{x^2-y^2}{x}\right)\left(3\,y+\frac{x^2+y^2}{y}\right)\div\left(\frac{x^2}{y^2}+5+\frac{4\,y^2}{x^2}\right)$$
$$=\frac{x^2+y^2}{x}\times\frac{x^2+4\,y^2}{y}\div\frac{x^4+5\,x^2y^2+4\,y^4}{x^2y^2}$$
$$=\frac{x^2+y^2}{x}\times\frac{x^2+4\,y^2}{y}\times\frac{x^2y^2}{x^4+5\,x^2y^2+4\,y^4}$$
$$=\frac{x^2+y^2}{x}\times\frac{x^2+4\,y^2}{y}\times\frac{x^2y^2}{(x^2+y^2)(x^2+4\,y^2)}$$
$$=xy.$$

16. $$\frac{x^2-3ax+2a^2}{x^2-2ax-3a^2}\times\frac{x^2-8ax+15a^2}{x^2-8ax+12a^2}\div\frac{x^2-6ax+5a^2}{x^2-11ax+30a^2}$$
$$=\frac{x^2-3ax+2a^2}{x^2-2ax-3a^2}\times\frac{x^2-8ax+15a^2}{x^2-8ax+12a^2}\times\frac{x^2-11ax+30a^2}{x^2-6ax+5a^2}$$
$$=\frac{(x-a)(x-2a)}{(x+a)(x-3a)}\times\frac{(x-3a)(x-5a)}{(x-2a)(x-6a)}\times\frac{(x-5a)(x-6a)}{(x-a)(x-5a)}$$
$$=\frac{x-5a}{x+a}.$$

17. $$\left(\frac{4}{x+1}-\frac{2}{x+2}\right)\div\left[\frac{(x+1)^2-4}{x^2+x-2}\right]$$
$$=\frac{2(x+3)}{(x+1)(x+2)}\div\frac{(x-1)(x+3)}{(x-1)(x+2)}$$
$$=\frac{2(x+3)}{(x+1)(x+2)}\times\frac{(x-1)(x+2)}{(x-1)(x+3)}$$
$$=\frac{2}{x+1}.$$

18. $$\frac{y^2+(a+c)y+ac}{y^2+(b+c)y+bc}\div\frac{y^2-a^2}{y^2-b^2}=\frac{y^2+(a+c)y+ac}{y^2+(b+c)y+bc}\times\frac{y^2-b^2}{y^2-a^2}$$
$$=\frac{(y+a)(y+c)}{(y+b)(y+c)}\times\frac{(y+b)(y-b)}{(y+a)(y-a)}$$
$$=\frac{y-b}{y-a}.$$

19. $$\frac{(x+y)^2-(a+b)^2}{(x+a)^2-(y+b)^2}\div\frac{(x-a)^2-(b-y)^2}{(x-y)^2-(b-a)^2}$$
$$=\frac{(x+y)^2-(a+b)^2}{(x+a)^2-(y+b)^2}\times\frac{(x-y)^2-(b-a)^2}{(x-a)^2-(b-y)^2}$$
$$=\frac{(x+y+a+b)(x+y-a-b)}{(x+a+y+b)(x+a-y-b)}\times\frac{(x-y+b-a)(x-y-b+a)}{(x-a+b-y)(x-a-b+y)}$$
$$=1.$$

Exercise 63. Page 142

Simplify:

1. $$\frac{1+\dfrac{1}{a-1}}{1-\dfrac{1}{a+1}}=\frac{\dfrac{a-1+1}{a-1}}{\dfrac{a+1-1}{a+1}}=\frac{\dfrac{a}{a-1}}{\dfrac{a}{a+1}}=\frac{a}{a-1}\times\frac{a+1}{a}=\frac{a+1}{a-1}.$$

2. $$\frac{1+\dfrac{y}{x-y}}{1-\dfrac{y}{x+y}}=\frac{\dfrac{x-y+y}{x-y}}{\dfrac{x+y-y}{x+y}}=\frac{\dfrac{x}{x-y}}{\dfrac{x}{x+y}}=\frac{x}{x-y}\times\frac{x+y}{x}=\frac{x+y}{x-y}.$$

3. $$\frac{m+\dfrac{mn}{m-n}}{m-\dfrac{mn}{m+n}}=\frac{\dfrac{m^2-mn+mn}{m-n}}{\dfrac{m^2+mn-mn}{m+n}}=\frac{\dfrac{m^2}{m-n}}{\dfrac{m^2}{m+n}}=\frac{m^2}{m-n}\times\frac{m+n}{m^2}=\frac{m+n}{m-n}.$$

4. $$\frac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}}.$$

Multiply both terms by $(x+y)(x-y)$.

$$\frac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}}=\frac{x^2+2xy+y^2-x^2+2xy-y^2}{x^2-2xy+y^2+x^2+2xy+y^2}=\frac{4xy}{2x^2+2y^2}=\frac{2xy}{x^2+y^2}.$$

5. $$\frac{x+\dfrac{y-x}{1+xy}}{1-\dfrac{(y-x)x}{1+xy}}.$$

Multiply both terms by $1+xy$.

$$\frac{x+\dfrac{y-x}{1+xy}}{1-\dfrac{(y-x)x}{1+xy}}=\frac{x+x^2y+y-x}{1+xy-xy+x^2}=\frac{x^2y+y}{1+x^2}=\frac{y(x^2+1)}{1+x^2}=y.$$

6. $$\frac{\dfrac{2m+n}{m+n}-1}{1-\dfrac{n}{m+n}}.$$

Multiply both terms by $m+n$.

$$\frac{\dfrac{2m+n}{m+n}-1}{1-\dfrac{n}{m+n}}=\frac{2m+n-m-n}{m+n-n}=\frac{m}{m}=1.$$

7. $$\frac{a-1+\dfrac{6}{a-6}}{a-2+\dfrac{3}{a-6}}.$$

Multiply both terms by $a-6$.

$$\frac{a-1+\dfrac{6}{a-6}}{a-2+\dfrac{3}{a-6}}=\frac{a^2-7a+6+6}{a^2-8a+12+3}$$

$$=\frac{a^2-7a+12}{a^2-8a+15}$$

$$=\frac{(a-3)(a-4)}{(a-3)(a-5)}$$

$$=\frac{a-4}{a-5}.$$

8. $$\frac{1}{1+\dfrac{c}{1+c+\dfrac{2c}{1-c}}}=\frac{1}{1+\dfrac{c-c^2}{1-c^2+2c}}$$

$$=\frac{1-c^2+2c}{1-c^2+2c+c-c^2}$$

$$=\frac{1-c^2+2c}{1+3c-2c^2}$$

$$=\frac{1+2c-c^2}{1+3c-2c^2}.$$

9. $$\frac{1}{a+\dfrac{1}{1+\dfrac{a+1}{3-a}}}=\frac{1}{a+\dfrac{3-a}{3-a+a+1}}=\frac{1}{a+\dfrac{3-a}{4}}=\frac{4}{4a+3-a}=\frac{4}{3a+3}.$$

10. $$\frac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}=\frac{a}{b+\dfrac{cf}{df+e}}=\frac{adf+ae}{bdf+be+cf}.$$

11. $$\frac{\dfrac{a^2+b^2}{b}-a}{\dfrac{1}{b}-\dfrac{1}{a}}\times\frac{a^2-b^2}{a^3+b^3}$$

$$=\frac{a^3+ab^2-a^2b}{a-b}\times\frac{a^2-b^2}{a^3+b^3}$$

$$=\frac{a(a^2-ab+b^2)}{a-b}\times\frac{(a+b)(a-b)}{(a+b)(a^2-ab+b^2)}$$

$$=a.$$

12. $$\frac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\times\left(1+\frac{b^2+c^2-a^2}{2\,bc}\right)$$

$$=\frac{b+c+a}{b+c-a}\times\frac{2\,bc+b^2+c^2-a^2}{2\,bc}$$

$$=\frac{b+c+a}{b+c-a}\times\frac{(b+c+a)(b+c-a)}{2\,bc}$$

$$=\frac{(b+c+a)^2}{2\,bc}.$$

13. $$\frac{\dfrac{x^3-y^3}{x^2+y^2}\times\dfrac{x^2-y^2}{x^3+y^3}\times\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right)\div\left(\dfrac{1}{y}-\dfrac{1}{x}\right)}{\dfrac{(x+y)^2-xy}{(x-y)^2+xy}}$$

$$=\frac{\dfrac{x^3-y^3}{x^2+y^2}\times\dfrac{x^2-y^2}{x^3+y^3}\times\dfrac{x^2+y^2}{x^2y^2}\div\dfrac{x-y}{xy}}{\dfrac{x^2+xy+y^2}{x^2-xy+y^2}}$$

$$=\frac{x^3-y^3}{x^2+y^2}\times\frac{x^2-y^2}{x^3+y^3}\times\frac{x^2+y^2}{x^2y^2}\times\frac{xy}{x-y}\times\frac{x^2-xy+y^2}{x^2+xy+y^2}$$

$$=\frac{(x-y)(x^2+xy+y^2)}{x^2+y^2}\times\frac{(x+y)(x-y)}{(x+y)(x^2-xy+y^2)}$$

$$\times\frac{x^2+y^2}{x^2y^2}\times\frac{xy}{x-y}\times\frac{x^2-xy+y^2}{x^2+xy+y^2}$$

$$=\frac{x-y}{xy}.$$

14. $$\frac{\dfrac{1+x}{1-x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1-x^4}-\dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}}$$

$$=\frac{\left(\dfrac{1+x}{1-x}-\dfrac{1-x}{1+x}\right)+\left(\dfrac{4x}{1+x^2}+\dfrac{8x}{1-x^4}\right)}{\left(\dfrac{1+x^2}{1-x^2}-\dfrac{1-x^2}{1+x^2}\right)+\dfrac{4x^2}{1+x^4}}$$

$$=\frac{\dfrac{1+2x+x^2-1+2x-x^2}{1-x^2}+\dfrac{4x-4x^3+8x}{1-x^4}}{\dfrac{1+2x^2+x^4-1+2x^2-x^4}{1-x^4}+\dfrac{4x^2}{1+x^4}}$$

$$=\frac{\dfrac{4x}{1-x^2}+\dfrac{12x-4x^3}{1-x^4}}{\dfrac{4x^2}{1-x^4}+\dfrac{4x^2}{1+x^4}}$$

$$=\frac{\dfrac{4x+4x^3+12x-4x^3}{1-x^4}}{\dfrac{4x^2+4x^6+4x^2-4x^6}{1-x^8}}$$

$$=\frac{\dfrac{16x}{1-x^4}}{\dfrac{8x^2}{1-x^8}}$$

$$=\frac{16x}{1-x^4}\times\frac{1-x^8}{8x^2}$$

$$=\frac{2(1+x^4)}{x}.$$

15. $$\frac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}+\frac{\dfrac{1}{1+x}}{\dfrac{x}{1-x}}+\frac{\dfrac{1}{1-x}}{\dfrac{x}{1+x}}=\frac{1}{1+x-1}+\frac{1-x}{x(1+x)}+\frac{1+x}{x(1-x)}$$

$$=\frac{1}{x}+\frac{1-x}{x(1+x)}+\frac{1+x}{x(1-x)}$$

$$=\frac{1-x^2}{x(1-x^2)}+\frac{(1-x)^2}{x(1-x^2)}+\frac{(1+x)^2}{x(1-x^2)}$$

$$=\frac{1-x^2}{x(1-x^2)}+\frac{1-2x+x^2}{x(1-x^2)}+\frac{1+2x+x^2}{x(1-x^2)}$$

$$=\frac{3+x^2}{x(1-x^2)}.$$

16. $$\frac{\left[\frac{(a+b)^2}{4\,ab}-1\right]\left[\frac{(a-b)^2}{4\,ab}+1\right]}{(a+b)^3-3\,a^2b-3\,ab^2}\times\frac{[(a+b)^2-ab]\,[(a-b)^2+ab]}{(a-b)^3+3\,ab\,(a-b)}\times 4\left[\left(\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}\right)\div\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right)\right]$$

$$=\frac{\frac{a^2+2\,ab+b^2-4\,ab}{4\,ab}\times\frac{a^2-2\,ab+b^2+4\,ab}{4\,ab}}{a^3+3\,a^2b+3\,ab^2+b^3-3\,a^2b-3\,ab^2}\times\frac{(a^2+2\,ab+b^2-ab)\,(a^2-2\,ab+b^2+ab)}{a^3-3\,a^2b+3\,ab^2-b^3+3\,a^2b-3\,ab^2}$$

$$\times\, 4\left[\frac{a^4+2\,a^2b^2+b^4-a^4+2\,a^2b^2-b^4}{(a^2+b^2)\,(a^2-b^2)}\div\frac{a^2+2\,ab+b^2-a^2+2\,ab-b^2}{(a+b)\,(a-b)}\right]$$

$$=\frac{\frac{a^2-2\,ab+b^2}{4\,ab}\times\frac{a^2+2\,ab+b^2}{4\,ab}}{a^3+b^3}\times\frac{(a^2+ab+b^2)(a^2-ab+b^2)}{a^3-b^3}\times 4\left[\frac{4\,a^2b^2}{(a^2+b^2)\,(a^2-b^2)}\div\frac{4\,ab}{(a+b)\,(a-b)}\right]$$

$$=\frac{a^2-2\,ab+b^2}{4\,ab}\times\frac{a^2+2\,ab+b^2}{4\,ab}\times\frac{1}{a^3+b^3}\times\frac{(a^2+ab+b^2)(a^2-ab+b^2)}{a^3-b^3}\times\frac{16\,a^2b^2}{(a^2+b^2)(a^2-b^2)}\div\frac{4\,ab}{(a+b)(a-b)}$$

$$=\frac{(a-b)^2}{4\,ab}\times\frac{(a+b)^2}{4\,ab}\times\frac{1}{(a+b)\,(a^2-ab+b^2)}\times\frac{(a^2+ab+b^2)\,(a^2-ab+b^2)}{(a-b)\,(a^2+ab+b^2)}$$

$$\times\frac{16\,a^2b^2}{(a^2+b^2)\,(a+b)\,(a-b)}\times\frac{(a+b)\,(a-b)}{4\,ab}$$

$$=\frac{a^2-b^2}{4\,ab\,(a^2+b^2)}.$$

Exercise 64. Page 145

1. Solve $\frac{4x-1}{3}-\frac{3}{4}=\frac{x-4}{6}+\frac{3x+5}{4}.$

Multiply by 12, $16x-4-9=2x-8+9x+15.$

Transpose, $16x-2x-9x=15-8+4+9.$

Combine, $5x=20.$

$\therefore x=4.$

2. Solve $\frac{4x+9}{10}-\frac{x+5}{5}=\frac{7x-1}{25}-\frac{x+3}{20}.$

Multiply by 100,

$$40x+90-20x-100=28x-4-5x-15.$$

Transpose, $40x-20x-28x+5x=100-90-4-15.$

Combine, $-3x=-9.$

$\therefore x=3.$

3. Solve $\frac{7}{x}+\frac{1}{3}=\frac{23-x}{3x}+\frac{7}{12}-\frac{1}{4x}.$

Multiply by $12x$, $84+4x=92-4x+7x-3.$

Transpose, $4x+4x-7x=92-3-84.$

Combine, $x=5.$

4. Solve $\frac{5}{x+3}+\frac{3}{2(x+3)}=\frac{1}{2}-\frac{7}{2(x+3)}.$

Multiply by $2(x+3)$, $10+3=x+3-7.$

Transpose, $-x=3-7-10-3.$

Combine, $-x=-17.$

$\therefore x=17.$

5. Solve $\frac{3x-5}{5x-5}+\frac{5x-1}{7x-7}+\frac{x-4}{x-1}=2.$

Factor denominators,

$$\frac{3x-5}{5(x-1)}+\frac{5x-1}{7(x-1)}+\frac{x-4}{x-1}=2.$$

Multiply by $35(x-1)$,

$$21x-35+25x-5+35x-140=70x-70.$$

Transpose, $21x+25x+35x-70x=35+5+140-70.$

Combine, $11x=110.$

$\therefore x=10.$

6. Solve $\frac{8x+2}{x-2}-\frac{2x-1}{3x-6}+\frac{3x+2}{5x-10}=10.$

Factor denominators,

$$\frac{8x+2}{x-2}-\frac{2x-1}{3(x-2)}+\frac{3x+2}{5(x-2)}=10.$$

Multiply by $15(x-2)$,

$$120x+30-10x+5+9x+6=150x-300.$$

Transpose, $120x-10x+9x-150x=-300-30-5-6.$

Combine, $-31x=-341.$

$\therefore x=11.$

7. Solve $\frac{3x-1}{2x-6}+\frac{5x-7}{3x-9}+\frac{7x+1}{4x-12}=11.$

Factor denominators,

$$\frac{3x-1}{2(x-3)}+\frac{5x-7}{3(x-3)}+\frac{7x+1}{4(x-3)}=11.$$

Multiply by $12(x-3)$,

$$18x-6+20x-28+21x+3=132x-396.$$

Transpose, $18x+20x+21x-132x=6+28-3-396.$

Combine, $-73x=-365.$

$\therefore x=5.$

8. Solve $\frac{2x+1}{3}+\frac{3x+1}{4}=28-\frac{5x-2}{7}.$

Multiply by 84, $56x+28+63x+21=2352-60x+24.$

Transpose, $56x+63x+60x=2352+24-28-21.$

Combine, $179x=2327.$

$\therefore x=13.$

9. Solve $\frac{5(x+3)}{12}+\frac{5}{9}-\frac{11-3x}{36}=5x-20-\frac{13-x}{12}-\frac{21-3x}{18}.$

Multiply by 36,

$$15x+45+20-11+3x=180x-720-39+3x-42+6x.$$

Transpose,

$$15x+3x-180x-3x-6x=11-45-20-720-39-42.$$

Combine, $-171x=-855.$

$\therefore x=5.$

10. Solve $\frac{x-3}{4}-\frac{2x-5}{6}-\frac{41}{60}-\frac{3x-8}{5}+\frac{5x+6}{15}=0.$

Multiply by 60,

$$15x-45-20x+50-41-36x+96+20x+24=0.$$

Transpose, $15x-20x-36x+20x=45-50+41-96-24.$

Combine, $-21x=-84.$

$\therefore x=4.$

11. Solve $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}.$

Multiply by 12, $6x+18-4x+8=3x-5+3.$

Transpose, $6x-4x-3x=3-5-18-8.$

Combine, $-x=-28.$

$\therefore x=28.$

12. Solve $\frac{4x}{15}-\frac{9}{10}-\frac{16x-81}{24}=\frac{2x-3}{15}-\frac{4x-9}{20}+\frac{9}{40}.$

Multiply by 120,

$$32x-108-80x+405=16x-24-24x+54+27.$$

Transpose, $32x-80x-16x+24x=108-405-24+54+27.$

Combine, $-40x=-240.$

$\therefore x=6.$

13. Solve $\frac{5x-16}{x-5}=\frac{5x-3}{x+5}.$

Multiply by $(x-5)(x+5)$, $5x^2+9x-80=5x^2-28x+15.$

Transpose, $5x^2-5x^2+9x+28x=15+80.$

Combine, $37x=95.$

$\therefore x=\frac{95}{37}.$

14. Solve $\frac{3x+7}{4x+5}=\frac{3x+5}{4x+3}.$

Multiply by $(4x+5)(4x+3)$,

$$12x^2+37x+21=12x^2+35x+25.$$

Transpose, $12x^2-12x^2+37x-35x=25-21.$

Combine, $2x=4.$

$\therefore x=2.$

15. Solve $\dfrac{29-10x}{9-5x}=\dfrac{5+36x}{18x}$.

Multiply by $18x(9-5x)$, $522x-180x^2=45+299x-180x^2$.

Transpose, $180x^2-180x^2+522x-299x=45$.

Combine, $223x=45$.

$\therefore x=\frac{45}{223}$.

16. Solve $\dfrac{47+6x}{11+2x}=\dfrac{2(8+3x)}{3+2x}$.

Multiply by $(11+2x)(3+2x)$,

$141+112x+12x^2=176+98x+12x^2$.

Transpose, $12x^2-12x^2+112x-98x=176-141$.

Combine, $14x=35$.

$\therefore x=\frac{5}{2}$.

17. Solve $\dfrac{5-2x}{x-1}-\dfrac{2-7x}{x+1}=\dfrac{5x^2+4}{x^2-1}$.

Multiply by x^2-1, $5+3x-2x^2+7x^2-9x+2=5x^2+4$.

Transpose, $7x^2-2x^2-5x^2+3x-9x=4-5-2$.

Combine, $-6x=-3$.

$\therefore x=\frac{1}{2}$.

18. Solve $\dfrac{2x+1}{2x-1}-\dfrac{8}{4x^2-1}=\dfrac{2x-1}{2x+1}$.

Multiply by $4x^2-1$, $4x^2+4x+1-8=4x^2-4x+1$.

Transpose, $4x^2-4x^2+4x+4x=1-1+8$.

Combine, $8x=8$.

$\therefore x=1$.

19. Solve $\dfrac{1+3x}{5+7x}-\dfrac{9-11x}{5-7x}=\dfrac{14(2x-3)^2}{25-49x^2}$.

Multiply by $25-49x^2$,

$5+8x-21x^2-45-8x+77x^2=56x^2-168x+126$.

Transpose,

$77x^2-21x^2-56x^2+8x-8x+168x=126-5+45$.

Combine, $168x=166$.

$\therefore x=\frac{83}{84}$.

20. Solve $\frac{11x-13}{14}-\frac{22x-75}{28}=\frac{13x+7}{2(3x+7)}$.

Multiply by $28(3x+7)$,

$$66x^2+76x-182-66x^2+71x+525=182x+98.$$

Transpose, $66x^2-66x^2+76x+71x-182x=98+182-525.$

Combine, $-35x=-245.$

$\therefore x=7.$

21. Solve $\frac{4x+3}{15x-35}-\frac{11x-5}{9x+21}=\frac{375x-86x^2-35}{10(9x^2-49)}$.

Factor denominators,

$$\frac{4x+3}{5(3x-7)}-\frac{11x-5}{3(3x+7)}=\frac{375x-86x^2-35}{10(3x+7)(3x-7)}.$$

Multiply by $30(3x+7)(3x-7)$,

$$72x^2+222x+126-330x^2+920x-350=1125x-258x^2-105.$$

Transpose,

$$72x^2-330x^2+258x^2+222x+920x-1125x=350-126-105.$$

Combine, $17x=119.$

$\therefore x=7.$

22. Solve $\frac{2}{x-14}-\frac{5}{x-13}=\frac{2}{x-9}-\frac{5}{x-11}$.

Simplify each member, $\frac{-3x+44}{x^2-27x+182}=\frac{-3x+23}{x^2-20x+99}.$

Clear of fractions,

$$-3x^3+104x^2-1177x+4356=-3x^3+104x^2-1167x+4186.$$

Transpose,

$$-3x^3+3x^3+104x^2-104x^2-1177x+1167x=4186-4356.$$

Combine, $-10x=-170.$

$\therefore x=17.$

23. Solve $\frac{9}{x-51}-\frac{9}{x-15}=\frac{2}{x-81}-\frac{2}{x+81}$.

Simplify each member, $\frac{324}{x^2-66x+765}=\frac{324}{x^2-6561}.$

Since the numerators are equal, the denominators are equal.

$$\therefore\ x^2 - 66x + 765 = x^2 - 6561.$$

Transpose, $x^2 - x^2 - 66x = -765 - 6561.$

Combine, $-66x = -7326.$

$$\therefore\ x = 111.$$

24. Solve $\dfrac{x-8}{x-3} + \dfrac{x-3}{x-5} + \dfrac{x-9}{x-7} = \dfrac{x-1}{x-3} + \dfrac{x-13}{x-5} + \dfrac{x-6}{x-7}.$

Transpose, $\dfrac{x-3}{x-5} - \dfrac{x-13}{x-5} = \dfrac{x-1}{x-3} - \dfrac{x-8}{x-3} + \dfrac{x-6}{x-7} - \dfrac{x-9}{x-7}.$

Combine, $\dfrac{10}{x-5} = \dfrac{7}{x-3} + \dfrac{3}{x-7}.$

Multiply by $(x-5)(x-3)(x-7)$,

$$10x^2 - 100x + 210 = 7x^2 - 84x + 245 + 3x^2 - 24x + 45.$$

Transpose,

$$10x^2 - 7x^2 - 3x^2 + 84x + 24x - 100x = 245 + 45 - 210.$$

Combine, $8x = 80.$

$$\therefore\ x = 10.$$

25. Solve $\dfrac{x+2}{x+7} + \dfrac{x+7}{x+5} + \dfrac{x+1}{x+3} = \dfrac{x+9}{x+7} + \dfrac{x-3}{x+5} + \dfrac{x+4}{x+3}.$

Transpose, $\dfrac{x+7}{x+5} - \dfrac{x-3}{x+5} = \dfrac{x+9}{x+7} - \dfrac{x+2}{x+7} + \dfrac{x+4}{x+3} - \dfrac{x+1}{x+3}.$

Combine, $\dfrac{10}{x+5} = \dfrac{7}{x+7} + \dfrac{3}{x+3}.$

Simplify the right member,

$$\frac{10}{x+5} = \frac{10x+42}{x^2+10x+21}.$$

Multiply by $(x+5)(x^2+10x+21)$,

$$10x^2 + 100x + 210 = 10x^2 + 92x + 210.$$

Transpose,

$$10x^2 - 10x^2 + 100x - 92x = 210 - 210.$$

Combine, $8x = 0.$

$$\therefore\ x = 0.$$

Exercise 65. Page 147

1. Solve $\dfrac{4x+3}{9} - \dfrac{8x+19}{18} + \dfrac{7x-29}{5x-12} = 0.$

Multiply by 18,
$$8x + 6 - 8x - 19 + \frac{126x - 522}{5x - 12} = 0.$$

Transpose, $\dfrac{126x - 522}{5x - 12} = 8x - 8x + 19 - 6.$

Combine, $\dfrac{126x - 522}{5x - 12} = 13.$

Multiply by $5x - 12$, $126x - 522 = 65x - 156.$

Transpose, $126x - 65x = 522 - 156.$

Combine, $61x = 366.$

$\therefore x = 6.$

2. Solve $\dfrac{16x+7}{24} - \dfrac{2x+1}{3} = \dfrac{x-16}{3(3x-59)}.$

Multiply by 24, $16x + 7 - 16x - 8 = \dfrac{8x - 128}{3x - 59}.$

Combine, $-1 = \dfrac{8x - 128}{3x - 59}.$

Multiply by $3x - 59$, $-3x + 59 = 8x - 128.$

Transpose, $-3x - 8x = -128 - 59.$

Combine, $-11x = -187.$

$\therefore x = 17.$

3. Solve $\dfrac{2x-3}{2x-4} - 6 = \dfrac{x+5}{3x-6} - \dfrac{11}{2}.$

Multiply by 6, $\dfrac{6x-9}{x-2} - 36 = \dfrac{2x+10}{x-2} - 33.$

Transpose, $\dfrac{6x-9}{x-2} - \dfrac{2x+10}{x-2} = 36 - 33.$

Combine, $\dfrac{4x-19}{x-2} = 3.$

Multiply by $x - 2$, $4x - 19 = 3x - 6.$

Transpose, $4x - 3x = 19 - 6.$

Combine, $x = 13.$

4. Solve $\frac{2x-13}{2x-16}+\frac{2(x-6)}{x-8}=\frac{7}{8}+\frac{2(5x-39)}{3x-24}.$

Multiply by 24, $\frac{24x-156}{x-8}+\frac{48x-288}{x-8}=21+\frac{80x-624}{x-8}.$

Multiply by $x-8$, $24x-156+48x-288=21x-168+80x-624.$

Transpose, $24x+48x-21x-80x=156+288-168-624.$

Combine, $-29x=-348.$

$\therefore x=12.$

5. Solve $\frac{x+1}{4(x+2)}+\frac{x+4}{5x+13}=\frac{9}{20}.$

Multiply by 20, $\frac{5x+5}{x+2}+\frac{20x+80}{5x+13}=9.$

Multiply by $(x+2)(5x+13)$,

$25x^2+90x+65+20x^2+120x+160=45x^2+207x+234.$

Transpose,

$25x^2+20x^2-45x^2+90x+120x-207x=234-65-160.$

Combine, $3x=9.$

$\therefore x=3.$

6. Solve $\frac{x-2}{2x+1}+\frac{x-1}{3(x-3)}=\frac{5}{6}.$

Multiply by 6, $\frac{6x-12}{2x+1}+\frac{2x-2}{x-3}=5.$

Multiply by $(2x+1)(x-3)$,

$6x^2-30x+36+4x^2-2x-2=10x^2-25x-15.$

Transpose,

$6x^2+4x^2-10x^2-30x-2x+25x=2-36-15.$

Combine, $-7x=-49.$

$\therefore x=7.$

7. Solve $\frac{10-x}{3}+\frac{13+x}{7}=\frac{7x+26}{x+21}-\frac{17+4x}{21}.$

Multiply by 21, $70-7x+39+3x=\frac{21(7x+26)}{x+21}-17-4x.$

Transpose, $70+39+17-7x+3x+4x=\frac{21(7x+26)}{x+21}.$

Combine, $126=\frac{21(7x+26)}{x+21}.$

Divide by 21, $6 = \dfrac{7x + 26}{x + 21}.$

Multiply by $x + 21$, $6x + 126 = 7x + 26.$

Transpose, $6x - 7x = 26 - 126.$

Combine, $-x = -100.$

$\therefore\ x = 100.$

8. Solve $\dfrac{7}{3} + \dfrac{13}{5x} = \dfrac{13x - 24}{3x} - \dfrac{37}{20} + \dfrac{10}{x}.$

Multiply by $60x$, $140x + 156 = 260x - 480 - 111x + 600.$

Transpose, $140x - 260x + 111x = 600 - 480 - 156.$

Combine, $-9x = -36.$

$\therefore\ x = 4.$

9. Solve $\dfrac{7x - 6}{35} - \dfrac{x - 5}{6x - 101} = \dfrac{x}{5}.$

Multiply by 35, $7x - 6 - \dfrac{35x - 175}{6x - 101} = 7x.$

Transpose, $7x - 7x - \dfrac{35x - 175}{6x - 101} = 6.$

Combine, $-\dfrac{35x - 175}{6x - 101} = 6.$

Multiply by $6x - 101$, $-35x + 175 = 36x - 606.$

Transpose, $-35x - 36x = -606 - 175.$

Combine, $-71x = -781.$

$\therefore\ x = 11.$

10. Solve $\dfrac{9x + 5}{14} + \dfrac{8x - 7}{6x + 2} = \dfrac{36x + 15}{56} + \dfrac{41}{56}.$

Multiply by 56,

$$36x + 20 + \frac{224x - 196}{3x + 1} = 36x + 15 + 41.$$

Transpose, $\dfrac{224x - 196}{3x + 1} = 36x - 36x + 15 + 41 - 20.$

Combine, $\dfrac{224x - 196}{3x + 1} = 36.$

Multiply by $3x + 1$, $224x - 196 = 108x + 36.$

Transpose, $224x - 108x = 196 + 36.$

Combine, $116x = 232.$

$\therefore\ x = 2.$

11. Solve $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{10x-11}{30}+\frac{1}{105}$.

Multiply by 210,

$$84-70x-\frac{105-30x^2}{x-1}=10+30x-70x+77+2.$$

Transpose,

$$84-10-77-2-70x-30x+70x=\frac{105-30x^2}{x-1}.$$

Combine, $-5-30x=\frac{105-30x^2}{x-1}$.

Divide by 5, $-1-6x=\frac{21-6x^2}{x-1}$.

Multiply by $x-1$, $1+5x-6x^2=21-6x^2$.

Transpose, $6x^2-6x^2+5x=21-1$.

Combine, $5x=20$.

$\therefore x=4$.

12. Solve $\frac{18x-22}{39-6x}+2x+\frac{1+16x}{24}=\frac{53}{12}-\frac{101-64x}{24}$.

Multiply by 24, $\frac{144x-176}{13-2x}+48x+1+16x=106-101+64x$.

Transpose,

$$\frac{144x-176}{13-2x}=106-101-1+64x-48x-16x.$$

Combine, $\frac{144x-176}{13-2x}=4$.

Multiply by $13-2x$, $144x-176=52-8x$.

Transpose, $144x+8x=52+176$.

Combine, $152x=228$.

$\therefore x=1\frac{1}{2}$.

13. Solve $\frac{x}{x-1}-\frac{5}{2x-2}=\frac{8}{3x-3}-\frac{x}{x-1}+\frac{5}{18}$.

Multiply by 18, $\frac{18x}{x-1}-\frac{45}{x-1}=\frac{48}{x-1}-\frac{18x}{x-1}+5$.

Multiply by $x-1$, $18x-45=48-18x+5x-5$.

Transpose, $18x+18x-5x=48+45-5$.

Combine, $31x=88$.

$\therefore x=\frac{88}{31}$.

14. Solve $\frac{13}{8} - \frac{5x}{4x+8} = \frac{2}{x+2} + \frac{1}{2}.$

Multiply by 8, $13 - \frac{10x}{x+2} = \frac{16}{x+2} + 4.$

Transpose, $13 - 4 = \frac{16}{x+2} + \frac{10x}{x+2}.$

Combine, $9 = \frac{16 + 10x}{x+2}.$

Multiply by $x + 2$, $9x + 18 = 16 + 10x.$

Transpose, $9x - 10x = 16 - 18.$

Combine, $-x = -2.$

$\therefore x = 2.$

Exercise 66. Page 148

1. Solve $\frac{c+d}{x} - a = \frac{c-d}{x} + b.$

Multiply by x, $c + d - ax = c - d + bx.$

Transpose, $-ax - bx = c - d - c - d.$

Combine, $-ax - bx = -2d.$

$(a+b)x = 2d.$

$\therefore x = \frac{2d}{a+b}.$

2. Solve $\frac{c+d}{ab+bx} = \frac{m-x}{an+nx}.$

Factor denominators, $\frac{c+d}{b(a+x)} = \frac{m-x}{n(a+x)}.$

Multiply by $bn(a+x)$, $cn + dn = bm - bx.$

Transpose, $bx = bm - cn - dn.$

$\therefore x = \frac{bm - cn - dn}{b}.$

3. Solve $\frac{a+bx}{a+b} = \frac{c+dx}{c+d}.$

Multiply by $(a+b)(c+d)$,

$ac + ad + bcx + bdx = ac + bc + adx + bdx.$

Transpose, $bcx + bdx - adx - bdx = ac + bc - ac - ad.$

Combine, $bcx - adx = bc - ad.$

$\therefore x = 1.$

4. Solve $\dfrac{6x+a}{4x+b}=\dfrac{3x-b}{2x-a}$.

Multiply by $(4x+b)(2x-a)$,

$$12x^2-4ax-a^2=12x^2-bx-b^2.$$

Transpose, $12x^2-12x^2-4ax+bx=a^2-b^2$.

Combine, $bx-4ax=a^2-b^2$.

$$\therefore\ x=\frac{a^2-b^2}{b-4a}.$$

5. Solve $\dfrac{3(x-a)}{b}-\dfrac{2(x-b)}{a}=1$.

Multiply by ab, $3ax-3a^2-2bx+2b^2=ab$.

Transpose, $3ax-2bx=3a^2+ab-2b^2$.

$$\therefore\ x=\frac{3a^2+ab-2b^2}{3a-2b}=a+b.$$

6. Solve $\dfrac{x-a}{x-m}+\dfrac{x-b}{x-n}=2$.

Multiply by $(x-m)(x-n)$,

$$x^2-ax-nx+an+x^2-bx-mx+bm=2x^2-2mx-2nx+2mn.$$

Transpose,

$$x^2+x^2-2x^2-ax-nx-bx-mx+2mx+2nx=2mn-an-bm.$$

Combine, $mx+nx-ax-bx=2mn-an-bm$.

$$\therefore\ x=\frac{2mn-an-bm}{m+n-a-b}.$$

7. Solve $\dfrac{ax}{b}-\dfrac{cx}{d}+\dfrac{fx}{g}=h$.

Multiply by bdg, $adgx-bcgx+bdfx=bdgh$.

$$\therefore\ x=\frac{bdgh}{adg-bcg+bdf}.$$

8. Solve $\dfrac{mx+n}{mx-n}=\dfrac{a}{b}$.

Multiply by $b(mx-n)$, $bmx+bn=amx-an$.

Transpose, $-amx+bmx=-an-bn$.

Change signs, $amx-bmx=an+bn$.

Factor, $m(a-b)x=n(a+b)$.

$$\therefore\ x=\frac{n(a+b)}{m(a-b)}.$$

9. Solve $\frac{5x}{3a+b} - 2 = \frac{8b}{5a}$.

Multiply by $5a(3a+b)$,

$$25ax - 30a^2 - 10ab = 24ab + 8b^2.$$

Transpose, $$25ax = 30a^2 + 10ab + 24ab + 8b^2.$$

Combine, $$25ax = 30a^2 + 34ab + 8b^2.$$

$$\therefore\ x = \frac{30a^2 + 34ab + 8b^2}{25a}.$$

10. Solve $\frac{x-2a}{x+3a} = 3 - \frac{2x^2 - 13a^2}{x^2 - 9a^2}$.

Multiply by $x^2 - 9a^2$, $x^2 - 5ax + 6a^2 = 3x^2 - 27a^2 - 2x^2 + 13a^2.$

Transpose, $x^2 - 3x^2 + 2x^2 - 5ax = 13a^2 - 6a^2 - 27a^2.$

Combine, $-5ax = -20a^2.$

$$\therefore\ x = 4a.$$

11. Solve $\frac{m}{x+m} + \frac{n}{x+n} = \frac{m+n}{x+p}$.

Multiply by $(x+m)(x+n)(x+p)$,

$$mx^2 + mnx + mpx + mnp + nx^2 + mnx + npx + mnp$$
$$= mx^2 + nx^2 + m^2x + n^2x + 2mnx + m^2n + mn^2.$$

Transpose,

$$mx^2 + nx^2 - mx^2 + nx^2 + mnx + mpx + mnx + npx - m^2x - n^2x - 2mnx$$
$$= m^2n + mn^2 - mnp - mnp.$$

Combine, $$mpx + npx - m^2x - n^2x = m^2n + mn^2 - 2mnp.$$

$$\therefore\ x = \frac{m^2n + mn^2 - 2mnp}{mp + np - m^2 - n^2}$$
$$= \frac{mn(m+n-2p)}{(m+n)p - m^2 - n^2}.$$

12. Solve $\frac{3x}{a+3b} + \frac{18bx}{a^2 - 9b^2} = 1$.

Multiply by $a^2 - 9b^2$, $3ax - 9bx + 18bx = a^2 - 9b^2.$

Combine, $$3ax + 9bx = a^2 - 9b^2.$$

$$\therefore\ x = \frac{a^2 - 9b^2}{3a + 9b} = \frac{a - 3b}{3}.$$

13. Solve $\frac{1}{a}+\frac{a}{x+a}=\frac{x+a}{ax}$.

Multiply by $ax(x+a)$, $x^2+ax+a^2x=x^2+2ax+a^2$.

Transpose, $x^2-x^2+ax+a^2x-2ax=a^2$.

Combine, $a^2x-ax=a^2$.

Divide by a, $ax-x=a$.

$$\therefore x=\frac{a}{a-1}.$$

14. Solve $\frac{mx-a-b}{nx-c-d}=\frac{mx-a-c}{nx-b-d}$.

Multiply by $(nx-c-d)(nx-b-d)$,

$$mnx^2-anx-bnx-bmx+ab+b^2-dmx+ad+bd$$
$$=mnx^2-anx-cnx-cmx+ac+c^2-dmx+ad+cd.$$

Transpose,

$$mnx^2-mnx^2-anx+anx-bnx+cnx-bmx+cmx-dmx+dmx$$
$$=ac-ab+c^2-b^2+ad-ad+cd-bd.$$

Combine,

$$-bnx+cnx-bmx+cmx=ac-ab+c^2-b^2+dc-bd.$$

Factor,

$$n(c-b)x+m(c-b)x=a(c-b)+(c+b)(c-b)+d(c-b).$$

Divide by $c-b$, $nx+mx=a+c+b+d$.

$$\therefore x=\frac{a+b+c+d}{m+n}.$$

15. Solve $\frac{a(x-a)}{a+2b}+\frac{b(x-b)}{2a+b}=\frac{x}{2}$.

Multiply by $2(a+2b)(2a+b)$,

$$4a^2x-4a^3+2abx-2a^2b+2abx-2ab^2+4b^2x-4b^3$$
$$=2a^2x+5abx+2b^2x.$$

Transpose,

$$4a^2x-2a^2x+2abx+2abx-5abx+4b^2x-2b^2x$$
$$=4a^3+2a^2b+2ab^2+4b^3.$$

Combine, $2a^2x-abx+2b^2x=4a^3+2a^2b+2ab^2+4b^3$.

$$\therefore x=\frac{4a^3+2a^2b+2ab^2+4b^3}{2a^2-ab+2b^2}$$
$$=2(a+b).$$

16. Solve $\dfrac{c-d}{ab-by}=\dfrac{m+y}{an-ny}$.

Multiply by $bn(a-y)$, $\qquad cn-dn=bm+by.$

Transpose, $\qquad -by=-cn+dn+bm.$

$$\therefore\ y=\frac{cn-dn-bm}{b}.$$

17. Solve $\dfrac{ax-b}{c}-\dfrac{bx+c}{a}=abc.$

Multiply by ac, $\qquad a^2x-ab-bcx-c^2=a^2bc^2.$

Transpose, $\qquad a^2x-bcx=ab+c^2+a^2bc^2.$

$$\therefore\ x=\frac{ab+c^2+a^2bc^2}{a^2-bc}.$$

18. Solve $\dfrac{5\,ax}{a-b}-3\,a=8\,x.$

Multiply by $a-b$, $\qquad 5\,ax-3\,a^2+3\,ab=8\,ax-8\,bx.$

Transpose, $\qquad 5\,ax-8\,ax+8\,bx=3\,a^2-3\,ab.$

Combine, $\qquad 8\,bx-3\,ax=3\,a^2-3\,ab.$

$$\therefore\ x=\frac{3\,a^2-3\,ab}{8\,b-3\,a}.$$

19. Solve $\dfrac{3\,a+x}{5\,a-b}=\dfrac{4\,x-b}{3\,a+2\,b}-1.$

Simplify the right member, $\qquad \dfrac{3\,a+x}{5\,a-b}=\dfrac{4\,x-3\,a-3\,b}{3\,a+2\,b}.$

Multiply by $(5\,a-b)(3\,a+2\,b)$,

$$9\,a^2+3\,ax+6\,ab+2\,bx=20\,ax-15\,a^2-4\,bx-12\,ab+3\,b^2.$$

Transpose,

$$3\,ax-20\,ax+2\,bx+4\,bx=-9\,a^2-15\,a^2-6\,ab-12\,ab+3\,b^2.$$

Combine, $\qquad 6\,bx-17\,ax=3\,b^2-18\,ab-24\,a^2.$

$$\therefore\ x=\frac{3\,b^2-18\,ab-24\,a^2}{6\,b-17\,a}.$$

20. Solve $\dfrac{m}{x-m}-\dfrac{n}{x-n}=\dfrac{m-n}{x-p}.$

Multiply by $(x-m)(x-n)(x-p)$,

$$\begin{aligned}&mx^2-mnx-mpx+mnp-nx^2+mnx+npx-mnp\\&\quad=mx^2-m^2x-nx^2+m^2n+n^2x-mn^2.\end{aligned}$$

Transpose,

$$\begin{aligned}&mx^2-nx^2-mx^2+nx^2-mnx+mnx-mpx+npx+m^2x-n^2x\\&\quad=-mnp+mnp+m^2n-mn^2.\end{aligned}$$

Combine, $-mpx + npx + m^2x - n^2x = m^2n - mn^2.$

Factor, $(n - m)px + (m^2 - n^2)x = mn(m - n),$

or $(n - m)px + (m + n)(m - n)x = mn(m - n).$

Divide by $m - n$, $-px + mx + nx = mn.$

$$\therefore x = \frac{mn}{m + n - p}.$$

21. Solve $\frac{x - c}{c - d} - \frac{x + c}{c + d} = \frac{2cx}{c^2 - d^2}.$

Multiply by $c^2 - d^2$,

$$cx - c^2 + dx - cd - cx - c^2 + dx + cd = 2cx.$$

Transpose, $cx - cx - 2cx + dx + dx = c^2 + c^2 + cd - cd.$

Combine, $2dx - 2cx = 2c^2.$

Divide by 2, $dx - cx = c^2.$

$$\therefore x = \frac{c^2}{d - c}.$$

22. Solve $\frac{x}{a + b} + abx = a + b + \frac{1}{ab}.$

Multiply by $ab(a + b)$,

$$abx + a^3b^2x + a^2b^3x = a^3b + 2a^2b^2 + ab^3 + a + b.$$

$$\begin{aligned}\therefore x &= \frac{a^3b + 2a^2b^2 + ab^3 + a + b}{ab + a^3b^2 + a^2b^3}\\ &= \frac{ab(a^2 + 2ab + b^2) + a + b}{ab(1 + a^2b + ab^2)}\\ &= \frac{(a^2b + ab^2 + 1)(a + b)}{ab(1 + a^2b + ab^2)}\\ &= \frac{a + b}{ab}.\end{aligned}$$

Second Method

Solve $\frac{x}{a + b} + abx = a + b + \frac{1}{ab}.$

Factor, $x\left(\frac{1}{a + b} + ab\right) = a + b + \frac{1}{ab}.$

$$x\left(\frac{1 + a^2b + ab^2}{a + b}\right) = \frac{a^2b + ab^2 + 1}{ab}.$$

$$\therefore x = \frac{a + b}{ab}.$$

23. Solve $\dfrac{1}{x-a} - \dfrac{1}{x-b} = \dfrac{a-b}{x^2-ab}$.

Simplify the left member,

$$\frac{a-b}{x^2-ax-bx+ab} = \frac{a-b}{x^2-ab}.$$

Since the numerators are equal, the denominators are equal.

$$\therefore\ x^2-ab = x^2-ax-bx+ab.$$

Transpose, $\quad x^2-x^2+ax+bx = ab+ab.$

Combine, $\quad ax+bx = 2\,ab.$

$$\therefore\ x = \frac{2\,ab}{a+b}.$$

24. Solve $\dfrac{7\,x}{9\,m^2} - \dfrac{3}{2\,m} = \dfrac{9\,n+14\,x}{18\,m^2-24\,mn}$.

Factor denominators, $\quad \dfrac{7\,x}{9\,m^2} - \dfrac{3}{2\,m} = \dfrac{9\,n+14\,x}{6\,m\,(3\,m-4\,n)}$.

Multiply by $18\,m^2(3\,m-4\,n)$,

$$42\,mx-56\,nx-81\,m^2+108\,mn = 27\,mn+42\,mx.$$

Transpose, $\quad 42\,mx-42\,mx-56\,nx = 81\,m^2-108\,mn+27\,mn.$

Combine, $\quad -56\,nx = 81\,m^2-81\,mn.$

$$\therefore\ x = \frac{81\,mn-81\,m^2}{56\,n}$$

$$= \frac{81\,m\,(n-m)}{56\,n}.$$

25. Solve $\dfrac{a\,(3-2\,x)}{b} + \dfrac{b\,(3\,x-2)}{a} - \dfrac{a-bx}{2\,(a+b)} = 2.$

Multiply by $2\,ab\,(a+b)$,

$$6\,a^3+6\,a^2b-4\,a^3x-4\,a^2bx+6\,ab^2x+6\,b^3x-4\,ab^2-4\,b^3-a^2b+ab^2x$$
$$= 4\,a^2b+4\,ab^2.$$

Transpose,

$$-4\,a^3x-4\,a^2bx+6\,ab^2x+ab^2x+6\,b^3x$$
$$= -6\,a^3+4\,a^2b-6\,a^2b+a^2b+4\,ab^2+4\,ab^2+4\,b^3.$$

Combine,

$$-4\,a^3x-4\,a^2bx+7\,ab^2x+6\,b^3x = -6\,a^3-a^2b+8\,ab^2+4\,b^3.$$

$$\therefore\ x = \frac{-6\,a^3-a^2b+8\,ab^2+4\,b^3}{-4\,a^3-4\,a^2b+7\,ab^2+6\,b^3}$$

$$= \frac{6\,a^3+a^2b-8\,ab^2-4\,b^3}{4\,a^3+4\,a^2b-7\,ab^2-6\,b^3}$$

$$=\frac{(3a+2b)(2a^2-ab-2b^2)}{(2a+3b)(2a^2-ab-2b^2)}$$

$$=\frac{3a+2b}{2a+3b}.$$

26. Solve $\dfrac{(a+1)x}{b}+\dfrac{(b+1)x}{a}+\dfrac{2ab}{a+b}=a+b+1.$

Multiply by $a+b$,

$$\frac{(a^2+a+b+ab)x}{b}+\frac{(ab+b^2+a+b)x}{a}+2ab=a^2+2ab+b^2+a+b.$$

Transpose and combine,

$$\frac{(a^2+a+b+ab)x}{b}+\frac{(ab+b^2+a+b)x}{a}=a^2+b^2+a+b.$$

Multiply by ab,

$$(a^3+a^2+ab+a^2b+ab^2+b^3+ab+b^2)x=ab(a^2+b^2+a+b).$$

Combine, $(a^3+a^2b+ab^2+b^3+a^2+2ab+b^2)x=ab(a^2+b^2+a+b).$

Factor the coefficient of x,

$$[a^2(a+b)+b^2(a+b)+(a+b)(a+b)]x=ab(a^2+b^2+a+b),$$

$$(a+b)(a^2+b^2+a+b)x=ab(a^2+b^2+a+b).$$

$$\therefore\ x=\frac{ab}{a+b}.$$

27. Solve $\dfrac{a+c}{a-b}-\dfrac{(3a-5c)x}{2a-3b}+\dfrac{(3a-2b)(x-1)}{a-b}=\dfrac{(5c-2b)x}{2a-3b}-\dfrac{a-c}{a-b}.$

Separate the third term into two parts,

$$\frac{a+c}{a-b}-\frac{(3a-5c)x}{2a-3b}+\frac{(3a-2b)x}{a-b}-\frac{3a-2b}{a-b}=\frac{(5c-2b)x}{2a-3b}-\frac{a-c}{a-b}.$$

Transpose,

$$\frac{(3a-2b)x}{a-b}-\frac{3a-2b}{a-b}+\frac{a+c}{a-b}+\frac{a-c}{a-b}=\frac{(5c-2b)x}{2a-3b}+\frac{(3a-5c)x}{2a-3b}.$$

Combine, $\dfrac{(3a-2b)x+2b-a}{a-b}=\dfrac{(3a-2b)x}{2a-3b}.$

Multiply by $(a-b)(2a-3b)$,

$$(6a^2-13ab+6b^2)x+(2b-a)(2a-3b)=(3a^2-5ab+2b^2)x.$$

Transpose,

$$(6a^2-13ab+6b^2-3a^2+5ab-2b^2)x=(a-2b)(2a-3b).$$

Combine, $(3a^2-8ab+4b^2)x=(a-2b)(2a-3b).$

Factor, $(a-2b)(3a-2b)x=(a-2b)(2a-3b).$

Divide by $a-2b$, $(3a-2b)x=2a-3b.$

$$\therefore\ x=\frac{2a-3b}{3a-2b}.$$

Exercise 67. Page 150

1. The sum of the fourth and fifth parts of a certain number is one less than twice the difference between the third and tenth parts. Find the number.

Let $x =$ the number.

Then $\frac{x}{4} + \frac{x}{5} =$ the sum of the fourth and fifth parts,

$\frac{x}{3} - \frac{x}{10} =$ the difference of the third and tenth parts,

$2\left(\frac{x}{3} - \frac{x}{10}\right) =$ twice the difference of the third and tenth parts,

and $2\left(\frac{x}{3} - \frac{x}{10}\right) - \left(\frac{x}{4} + \frac{x}{5}\right) =$ the given excess.

But $1 =$ the given excess.

$$\therefore\ 2\left(\frac{x}{3} - \frac{x}{10}\right) - \left(\frac{x}{4} + \frac{x}{5}\right) = 1.$$

Multiply by 60, $40x - 12x - 15x - 12x = 60.$

Combine, $x = 60.$

Therefore, the required number is 60.

2. The difference between the seventh and ninth parts of a certain number is 4. Find the number.

Let $x =$ the number.

Then $\frac{x}{7} - \frac{x}{9} =$ the difference between the seventh and ninth parts.

But $4 =$ the difference between the seventh and ninth parts.

$$\therefore\ \frac{x}{7} - \frac{x}{9} = 4.$$

Multiply by 63, $9x - 7x = 252.$

Combine, $2x = 252.$

$$\therefore\ x = 126.$$

Therefore, the required number is 126.

3. One third of a certain number is 4 less than the sum of its fourth and sixth parts. Find the number.

Let $x =$ the number.

Then $\frac{x}{4} + \frac{x}{6} =$ the sum of the fourth and sixth parts,

and $\frac{x}{4} + \frac{x}{6} - \frac{x}{3} =$ the given excess.

But $4 =$ the given excess.

$$\therefore \frac{x}{4} + \frac{x}{6} - \frac{x}{3} = 4.$$

Multiply by 12, $3x + 2x - 4x = 48.$

Combine, $x = 48.$

Therefore, the required number is 48.

4. The sum of the third and fourth parts of a certain number exceeds the difference between the fifth and sixth parts by 33. Find the number.

Let $x =$ the number.

Then $\frac{x}{3} + \frac{x}{4} =$ the sum of the third and fourth parts,

$\frac{x}{5} - \frac{x}{6} =$ the difference between the fifth and sixth parts,

and $\frac{x}{3} + \frac{x}{4} - \left(\frac{x}{5} - \frac{x}{6}\right) =$ the given excess.

But $33 =$ the given excess.

$$\therefore \frac{x}{3} + \frac{x}{4} - \left(\frac{x}{5} - \frac{x}{6}\right) = 33.$$

Multiply by 60, $20x + 15x - 12x + 10x = 1980.$

Combine, $33x = 1980.$

$$\therefore x = 60.$$

Therefore, the required number is 60.

5. There are two consecutive numbers, x and $x + 1$, such that half of the larger number exceeds one fifth of the smaller by 8. Find the numbers.

Let $x =$ the smaller number.

Then $x + 1 =$ the larger number,

$\frac{x+1}{2} =$ half of the larger number,

$\frac{x}{5} =$ one fifth of the smaller number,

and $\frac{x+1}{2} - \frac{x}{5} =$ the given excess.

But $8 =$ the given excess.

$$\therefore \frac{x+1}{2} - \frac{x}{5} = 8.$$

Multiply by 10, $5x + 5 - 2x = 80.$

Transpose, $5x - 2x = 80 - 5.$

Combine, $3x = 75.$

$\therefore x = 25.$

$x + 1 = 26.$

Therefore, the two numbers are 25 and 26.

6. The sum of two numbers is 63, and if the greater number is divided by the smaller the quotient is 2 and the remainder 9. Find the numbers.

Let $x =$ the greater number.

Then $63 - x =$ the smaller number.

Now the quotient $= \dfrac{\text{dividend} - \text{remainder}}{\text{divisor}}$,

and in this problem the dividend is x, the remainder 9, the divisor $63 - x$, and the quotient 2.

$$\therefore \frac{x-9}{63-x} = 2.$$

Multiply by $63 - x$, $x - 9 = 126 - 2x.$

Transpose, $x + 2x = 126 + 9.$

Combine, $3x = 135.$

$\therefore x = 45.$

$63 - x = 18.$

Therefore, the two numbers are 45 and 18.

7. The sum of two numbers is 75, and if the greater number is divided by the smaller the quotient is 3 and the remainder 7. Find the numbers.

Let $x =$ the greater number.

Then $75 - x =$ the smaller number.

$$\therefore \frac{x-7}{75-x} = 3.$$

Multiply by $75 - x$, $x - 7 = 225 - 3x.$

Transpose, $x + 3x = 225 + 7.$

Combine, $4x = 232.$

$\therefore x = 58.$

$75 - x = 17.$

Therefore, the two numbers are 58 and 17.

8. The sum of two numbers is 137, and if the greater number is divided by the smaller the quotient is 4 and the remainder 2. Find the numbers.

Let $x =$ the greater number.

Then $137 - x =$ the smaller number.

$$\therefore \frac{x-2}{137-x} = 4.$$

Multiply by $137 - x$, $x - 2 = 548 - 4x.$

Transpose, $x + 4x = 548 + 2.$

Combine, $5x = 550.$

$$\therefore x = 110.$$

$$137 - x = 27.$$

Therefore, the two numbers are 110 and 27.

9. The difference between two numbers is 19, and if the greater number is divided by the smaller the quotient is 2 and the remainder 8. Find the numbers.

Let $x =$ the greater number.

Then $x - 19 =$ the smaller number.

$$\therefore \frac{x-8}{x-19} = 2.$$

Multiply by $x - 19$, $x - 8 = 2x - 38.$

Transpose, $x - 2x = 8 - 38.$

Combine $-x = -30.$

$$\therefore x = 30.$$

$$x - 19 = 11.$$

Therefore, the two numbers are 30 and 11.

10. The difference between two numbers is 63, and if the greater number is divided by the smaller the quotient is 2 and the remainder 19. Find the numbers.

Let $x =$ the greater number.

Then $x - 63 =$ the smaller number.

$$\therefore \frac{x-19}{x-63} = 2.$$

Multiply by $x - 63$, $x - 19 = 2x - 126.$

Transpose, $x - 2x = 19 - 126.$

Combine, $-x = -107.$

$$\therefore x = 107.$$

$$x - 63 = 44.$$

Therefore, the two numbers are 107 and 44.

11. Divide 227 into two parts such that the smaller part is contained in the larger 5 times with a remainder of 5.

Let $x =$ the larger part.
Then $227 - x =$ the smaller part.

$$\therefore \frac{x-5}{227-x} = 5.$$

Multiply by $227 - x$, $x - 5 = 1135 - 5x.$
Transpose, $x + 5x = 1135 + 5.$
Combine, $6x = 1140.$

$$\therefore x = 190.$$
$$227 - x = 37.$$

Therefore, the two parts are 190 and 37.

12. Divide 367 into two parts such that the smaller part is contained in the larger 4 times with a remainder of 52.

Let $x =$ the larger part.
Then $367 - x =$ the smaller part.

$$\therefore \frac{x-52}{367-x} = 4.$$

Multiply by $367 - x$, $x - 52 = 1468 - 4x.$
Transpose, $x + 4x = 1468 + 52.$
Combine, $5x = 1520.$

$$\therefore x = 304.$$
$$367 - x = 63.$$

Therefore, the two parts are 304 and 63.

13. Divide 421 into two parts such that the smaller part is contained in the larger 7 times with a remainder of 5.

Let $x =$ the larger part.
Then $421 - x =$ the smaller part.

$$\therefore \frac{x-5}{421-x} = 7.$$

Multiply by $421 - x$, $x - 5 = 2947 - 7x.$
Transpose, $x + 7x = 2947 + 5.$
Combine, $8x = 2952.$

$$\therefore x = 369.$$
$$421 - x = 52.$$

Therefore, the two parts are 369 and 52.

14. Seven years ago a boy was half as old as he will be 1 year hence. How old is he now?

Let $x =$ the number of years old he is now.

Then $x - 7 =$ the number of years old he was 7 years ago,

and $x + 1 =$ the number of years old he will be 1 year hence.

$$\therefore x - 7 = \frac{x+1}{2}.$$

Multiply by 2, $2x - 14 = x + 1.$

Transpose, $2x - x = 1 + 14.$

Combine, $x = 15.$

Therefore, the boy is 15 years old.

15. A son is two fifths as old as his father. In 10 years he will be half as old as his father. Find the age of each.

Let $x =$ the number of years in the age of the father.

Then $\frac{2x}{5} =$ the number of years in the age of the son,

$x + 10 =$ the number of years in the age of the father in 10 years,

and $\frac{2x}{5} + 10 =$ the number of years in the age of the son in 10 years.

$$\therefore \frac{2x}{5} + 10 = \frac{x+10}{2}.$$

Multiply by 10, $4x + 100 = 5x + 50.$

Transpose, $4x - 5x = 50 - 100.$

Combine, $-x = -50.$

$$\therefore x = 50.$$

$$\frac{2x}{5} = 20.$$

Therefore, the father is 50 years old and the son is 20 years old.

16. A is one third as old as B. In 4 years A will be half as old as B. Find the age of each.

Let $x =$ the number of years in B's age.

Then $\frac{x}{3} =$ the number of years in A's age,

$\frac{x}{3} + 4 =$ the number of years in A's age in 4 years,

and $x + 4 =$ the number of years in B's age in 4 years.

$$\therefore \frac{x}{3} + 4 = \frac{x+4}{2}.$$

Multiply by 6, $2x + 24 = 3x + 12.$

Transpose, $2x - 3x = 12 - 24.$

Combine, $-x = -12.$

$$\therefore x = 12.$$

$$\frac{x}{3} = 4.$$

Therefore, A is 4 years old and B is 12 years old.

17. The sum of the ages of A and B is 70 years, and 15 years hence A's age will be one third of B's. Find their ages.

Let $x =$ the number of years in A's age.

Then $70 - x =$ the number of years in B's age,

$x + 15 =$ the number of years in A's age 15 years hence,

and $70 - x + 15 =$ the number of years in B's age 15 years hence.

$$\therefore x + 15 = \frac{70 - x + 15}{3}.$$

Multiply by 3, $3x + 45 = 70 - x + 15.$

Transpose, $3x + x = 70 + 15 - 45.$

Combine, $4x = 40.$

$$\therefore x = 10.$$

$$70 - x = 60.$$

Therefore, A is 10 years old and B is 60 years old.

18. A is 50 years old and B is half as old as A. In how many years will B be two thirds as old as A?

Let $x =$ the required number of years.

Then $50 + x =$ the number of years in A's age x years hence,

and $25 + x =$ the number of years in B's age x years hence.

$$\therefore 25 + x = \frac{2(50 + x)}{3}.$$

Multiply by 3, $75 + 3x = 100 + 2x.$

Transpose, $3x - 2x = 100 - 75.$

Combine, $x = 25.$

Therefore, the required number of years is 25.

19. A is 72 years old and B is two thirds as old as A. How many years ago was A five times as old as B?

Let x = the required number of years.

Then $72 - x$ = the number of years in A's age x years ago,

and $48 - x$ = the number of years in B's age x years ago.

$$\therefore\ 72 - x = 5(48 - x).$$

Expand, $72 - x = 240 - 5x.$

Transpose, $5x - x = 240 - 72.$

Combine, $4x = 168.$

$$\therefore\ x = 42.$$

Therefore, the required number of years is 42.

20. The sum of the ages of a father and son is 80 years. The son's age increased by 10 years is one half of the father's age. Find their ages.

Let x = the number of years in the age of the son.

Then $80 - x$ = the number of years in the age of the father.

$$\therefore\ x + 10 = \frac{80 - x}{2}.$$

Multiply by 2, $2x + 20 = 80 - x.$

Transpose, $2x + x = 80 - 20.$

Combine, $3x = 60.$

$$\therefore\ x = 20.$$

$$80 - x = 60.$$

Therefore, the age of the son is 20 years and the age of the father is 60 years.

21. B is one half as old as A. Seven years ago B was two fifths as old as A. Find their ages.

Let x = the number of years in A's age.

Then $\frac{x}{2}$ = the number of years in B's age,

$\frac{x}{2} - 7$ = the number of years in B's age 7 years ago,

and $x - 7$ = the number of years in A's age 7 years ago.

$$\therefore\ \frac{x}{2} - 7 = \frac{2(x - 7)}{5}.$$

Multiply by 10, $5x - 70 = 4x - 28.$

Transpose, $5x - 4x = 70 - 28.$

Combine, $x = 42.$

$$\frac{x}{2} = 21.$$

Therefore, A is 42 years old and B is 21 years old.

[illegible]. B's age is one sixth of A's age. In six years B's age will be one fourth of A's age. Find their ages.

Let $x =$ the number of years in A's age.

Then $\frac{x}{6} =$ the number of years in B's age,

$\frac{x}{6} + 6 =$ the number of years in B's age 6 years hence,

and $x + 6 =$ the number of years in A's age 6 years hence.

$$\therefore \frac{x}{6} + 6 = \frac{x+6}{4}.$$

Multiply by 12, $2x + 72 = 3x + 18.$

Transpose, $2x - 3x = 18 - 72.$

Combine, $-x = -54.$

$$\therefore x = 54.$$

$$\frac{x}{6} = 9.$$

Therefore, A is 54 years old and B is 9 years old.

23. A's age is two thirds of B's age. Eight years ago A's age was three fifths of B's age. Find their ages.

Let $x =$ the number of years in B's age.

Then $\frac{2x}{3} =$ the number of years in A's age,

$\frac{2x}{3} - 8 =$ the number of years in A's age 8 years ago,

and $x - 8 =$ the number of years in B's age 8 years ago.

$$\therefore \frac{2x}{3} - 8 = \frac{3(x-8)}{5}.$$

Multiply by 15, $10x - 120 = 9x - 72.$

Transpose, $10x - 9x = 120 - 72.$

Combine, $x = 48.$

$$\frac{2x}{3} = 32.$$

Therefore, A is 32 years old and B is 48 years old.

24. A is sixteen years older than B. Sixteen years ago A was three times as old as B. Find their ages.

Let $x =$ the number of years in A's age.

Then $x - 16 =$ the number of years in B's age,

$x - 16 - 16 =$ the number of years in B's age 16 years ago,

and $x - 16 =$ the number of years in A's age 16 years ago.

$$\therefore x - 16 = 3(x - 16 - 16).$$

Expand, $x - 16 = 3x - 48 - 48.$

Transpose, $x - 3x = 16 - 48 - 48.$

Combine, $-2x = -80.$

$$\therefore x = 40.$$

$$x - 16 = 24.$$

Therefore, A is 40 years old and B is 24 years old.

25. A rectangle has its length 14 feet more and its width 10 feet less than the side of the equivalent square. Find the dimensions of the rectangle.

Let x = the number of feet in a side of the square.

Then $x + 14$ = the number of feet in the length of the rectangle,

and $x - 10$ = the number of feet in the width of the rectangle.

Since the number of square units in the area of a rectangle is equal to the product of the number of linear units in the length multiplied by the number of linear units in the width of the rectangle,

$(x + 14)(x - 10)$ = the number of square feet in the area of the rectangle,

and $x \times x$ = the number of square feet in the area of the square.

But these areas are equal.

$$\therefore (x + 14)(x - 10) = x^2.$$

Expand, $x^2 + 4x - 140 = x^2.$

Transpose, $x^2 - x^2 + 4x = 140.$

Combine, $4x = 140.$

$$\therefore x = 35.$$

Then $x + 14 = 49,$

and $x - 10 = 25.$

Therefore, the dimensions of the rectangle are 49 feet and 25 feet.

26. A rectangle has its length 21 feet more and its width 12 feet less than the side of the equivalent square. Find the dimensions of the rectangle.

Let x = the number of feet in a side of the square.

Then $x + 21$ = the number of feet in the length of the rectangle,

$x - 12$ = the number of feet in the width of the rectangle,

$(x + 21)(x - 12)$ = the number of square feet in the area of the rectangle,

and x^2 = the number of square feet in the area of the square.

But these areas are equal.

$$\therefore\ (x+21)(x-12)=x^2.$$

Expand, $x^2+9x-252=x^2.$

Transpose, $x^2-x^2+9x=252.$

Combine, $9x=252.$

$$\therefore\ x=28.$$

Then $x+21=49,$

and $x-12=16.$

Therefore, the dimensions of the rectangle are 49 feet and 16 feet.

27. A rectangle has its length 9 feet more and its width 8 feet less than the side of the equivalent square. Find the dimensions of the rectangle.

Let $x=$ the number of feet in a side of the square.

Then $x+9=$ the number of feet in the length of the rectangle,

$x-8=$ the number of feet in the width of the rectangle,

$(x+9)(x-8)=$ the number of square feet in the area of the rectangle,

and $x^2=$ the number of square feet in the area of the square.

$$\therefore\ (x+9)(x-8)=x^2.$$

Expand, $x^2+x-72=x^2.$

Transpose, $x^2-x^2+x=72.$

Combine, $x=72.$

Then $x+9=81,$

and $x-8=64.$

Therefore, the dimensions of the rectangle are 81 feet and 64 feet.

28. The length of a floor exceeds the width by 7 feet. If each dimension were 3 feet more, the area would be 102 square feet more. Find the dimensions.

Let $x=$ the number of feet in the width of the floor.

Then $x+7=$ the number of feet in the length of the floor,

$x+3=$ the number of feet in the width increased by 3 feet,

$x+10=$ the number of feet in the length increased by 3 feet,

$(x+3)(x+10)=$ the number of square feet in the area of the increased floor,

and $x(x+7)=$ the number of square feet in the area of the floor.

$$\therefore\ (x+3)(x+10)-x(x+7)=102.$$

Expand, $x^2+13x+30-x^2-7x=102.$

Transpose, $x^2-x^2+13x-7x=102-30.$

Combine, $6x=72.$

$$\therefore\ x=12.$$

$$x+7=19.$$

Therefore, the dimensions of the floor are 19 feet and 12 feet.

29. The length of a rectangle exceeds the width by 9 inches. If the length were diminished by 5 inches and the width by 4 inches, the area would be diminished by 205 square inches. Find the dimensions.

Let $x =$ the number of inches in the width of the rectangle.

Then $x + 9 =$ the number of inches in the length of the rectangle,

$x + 4 =$ the number of inches in the length diminished by 5 inches,

$x - 4 =$ the number of inches in the width diminished by 4 inches,

$(x + 4)(x - 4) =$ the number of square inches in the area of the diminished rectangle,

and $x(x + 9) =$ the number of square inches in the area of the rectangle.

$$\therefore x(x + 9) - (x + 4)(x - 4) = 205.$$

Expand, $$x^2 + 9x - x^2 + 16 = 205.$$

Transpose, $$x^2 - x^2 + 9x = 205 - 16.$$

Combine, $$9x = 189.$$

$$\therefore x = 21.$$

$$x + 9 = 30.$$

Therefore, the dimensions of the rectangle are 30 inches and 21 inches.

30. The length of a rectangular field exceeds the width by 82 yards. If the length were diminished by 60 yards and the width increased by 40 yards, the area would be diminished by 1480 square yards. Find the dimensions.

Let $x =$ the number of yards in the width of the field.

Then $x + 82 =$ the number of yards in the length of the field,

$x + 22 =$ the number of yards in the length diminished by 60 yards,

$x + 40 =$ the number of yards in the width increased by 40 yards,

$(x + 22)(x + 40) =$ the number of square yards in the area of the diminished field,

and $x(x + 82) =$ the number of square yards in the area of the field.

$$\therefore x(x + 82) - (x + 22)(x + 40) = 1480.$$

Expand, $$x^2 + 82x - x^2 - 62x - 880 = 1480.$$

Transpose, $$x^2 - x^2 + 82x - 62x = 1480 + 880.$$

Combine, $$20x = 2360.$$

$$\therefore x = 118.$$

$$x + 82 = 200.$$

Therefore, the dimensions of the field are 200 yards and 118 yards.

31. A can do a piece of work in 3 days, and B can do the work in 4 days. How many days will it take them to do it working together?

Let x = the number of days it will take them working together.

Then $\frac{1}{x}$ = the part both together can do in one day,

$\frac{1}{3}$ = the part A can do in one day,

$\frac{1}{4}$ = the part B can do in one day,

and $\frac{1}{3}+\frac{1}{4}$ = the part both together can do in one day.

$$\therefore \frac{1}{3}+\frac{1}{4}=\frac{1}{x}.$$

Multiply by $12x$, $4x+3x=12.$

Combine, $7x=12.$

$$\therefore x=1\tfrac{5}{7}.$$

Therefore, A and B together can do the work in $1\frac{5}{7}$ days.

32. A can do a piece of work in 5 days, B in 6 days, and C in 7 days. How many days will it take them to do it working together?

Let x = the number of days it will take them working together.

Then $\frac{1}{x}$ = the part the three together can do in one day,

$\frac{1}{5}$ = the part A can do in one day,

$\frac{1}{6}$ = the part B can do in one day,

$\frac{1}{7}$ = the part C can do in one day,

and $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$ = the part the three together can do in one day.

$$\therefore \frac{1}{5}+\frac{1}{6}+\frac{1}{7}=\frac{1}{x}.$$

Multiply by $210x$, $42x+35x+30x=210.$

Combine, $107x=210.$

$$\therefore x=1\tfrac{103}{107}.$$

Therefore, A, B, and C together can do the work in $1\frac{103}{107}$ days.

33. A can do a piece of work in 9 days, B in 8 days, and C in $7\frac{1}{2}$ days. How many days will it take them to do it working together?

Let x = the number of days it will take them working together.

Then $\frac{1}{x}$ = the part the three together can do in one day,

$\frac{1}{9}$ = the part A can do in one day,

$\frac{1}{8}$ = the part B can do in one day,

$\frac{2}{15}$ = the part C can do in one day,

and $\frac{1}{9}+\frac{1}{8}+\frac{2}{15}$ = the part the three together can do in one day.

$$\therefore \frac{1}{9}+\frac{1}{8}+\frac{2}{15}=\frac{1}{x}.$$

Multiply by $360x$, $40x+45x+48x=360.$

Combine, $133x=360.$

$$\therefore x=2\tfrac{94}{133}.$$

Therefore, A, B, and C together can do the work in $2\frac{94}{133}$ days.

34. A can do a piece of work in $3\frac{1}{2}$ days, B in $4\frac{2}{3}$ days, and C in $5\frac{1}{4}$ days. How many days will it take them to do it working together?

Let x = the number of days it will take them working together.

Then $\frac{1}{x}$ = the part the three together can do in one day,

$\frac{2}{7}$ = the part A can do in one day,

$\frac{3}{14}$ = the part B can do in one day,

$\frac{4}{21}$ = the part C can do in one day,

and $\frac{2}{7}+\frac{3}{14}+\frac{4}{21}$ = the part the three together can do in one day.

$$\therefore \frac{2}{7}+\frac{3}{14}+\frac{4}{21}=\frac{1}{x}.$$

Multiply by $42x$, $12x+9x+8x=42.$

Combine, $29x=42.$

$$\therefore x=1\tfrac{13}{29}.$$

Therefore, A, B, and C together can do the work in $1\frac{13}{29}$ days.

35. A and B together can mow a field in 8 hours, A and C in 10 hours, and A alone in 15 hours. In how many hours can B and C together mow the field?

Let x = the number of hours it will take B and C together.

Then $\frac{1}{x}$ = the part B and C together can mow in one hour,

$\frac{1}{15}$ = the part A can mow in one hour,

$\frac{1}{8}$ = the part A and B together can mow in one hour,

$\frac{1}{10}$ = the part A and C together can mow in one hour,

$\frac{1}{8}-\frac{1}{15}$ = the part B can mow in one hour,

$\frac{1}{10}-\frac{1}{15}$ = the part C can mow in one hour,

and $\frac{1}{8}-\frac{1}{15}+\frac{1}{10}-\frac{1}{15}$ = the part B and C together can mow in one hour.

$$\therefore \frac{1}{8}-\frac{1}{15}+\frac{1}{10}-\frac{1}{15}=\frac{1}{x}.$$

Multiply by $120x$, $15x-8x+12x-8x=120.$

Combine, $11x=120.$

$$\therefore x=10\tfrac{10}{11}.$$

Therefore, B and C together can mow the field in $10\frac{10}{11}$ hours.

36. A can do a piece of work in $2a$ days, A and B together in b days, and A and C together in $a + \frac{b}{2}$ days. In how many days can the three do the work together?

Let $x =$ the number of days it will take them working together.

Then $\frac{1}{x} =$ the part the three together can do in one day,

$\frac{1}{2a} =$ the part A can do in one day,

$\frac{1}{b} =$ the part A and B together can do in one day,

$\frac{1}{a + \frac{b}{2}} = \frac{2}{2a + b} =$ the part A and C together can do in one day,

$\frac{2}{2a + b} - \frac{1}{2a} =$ the part C can do in one day,

and $\frac{1}{b} + \frac{2}{2a + b} - \frac{1}{2a} =$ the part the three together can do in one day.

$$\therefore \frac{1}{b} + \frac{2}{2a + b} - \frac{1}{2a} = \frac{1}{x}.$$

Multiply by $2abx(2a + b)$,

$$4a^2x + 2abx + 4abx - 2abx - b^2x = 2ab(2a + b).$$

Combine, $4a^2x + 4abx - b^2x = 2ab(2a + b).$

$$\therefore x = \frac{2ab(2a + b)}{4a^2 + 4ab - b^2}.$$

Therefore, A, B, and C working together can do the work in $\frac{2ab(2a + b)}{4a^2 + 4ab - b^2}$ days.

37. A and B together can dig a ditch in 40 days, B and C together in 48 days, and A and C together in 36 days. How many days will it take each alone to dig the ditch, and how many days if they all work together?

(i) Let $x =$ the number of days it will take the three together.

Then $\frac{1}{x} =$ the part the three together can dig in one day,

$\frac{2}{x} =$ the part the three together can dig in two days,

$\frac{1}{40} =$ the part A and B together can dig in one day,

$\frac{1}{48} =$ the part B and C together can dig in one day,

$\frac{1}{36} =$ the part A and C together can dig in one day,

and $\frac{1}{40} + \frac{1}{48} + \frac{1}{36} =$ the part the three together can dig in two days.

$$\therefore \frac{1}{40}+\frac{1}{48}+\frac{1}{36}=\frac{2}{x}.$$

Multiply by $720x$, $18x+15x+20x=1440.$

Combine, $53x=1440.$

$\therefore x=27\tfrac{9}{53}.$

Therefore, the three together can dig the ditch in $27\tfrac{9}{53}$ days.

(ii) Let $x =$ the number of days it will take A to dig the ditch.

Then $\frac{1}{x} =$ the part A can dig in one day,

$\tfrac{1}{48} =$ the part B and C together can dig in one day,

$\tfrac{53}{1440} =$ the part the three together can dig in one day,

and $\tfrac{53}{1440}-\tfrac{1}{48} =$ the part A can dig in one day.

$$\therefore \frac{53}{1440}-\frac{1}{48}=\frac{1}{x}.$$

Multiply by $1440x$, $53x-30x=1440.$

Combine, $23x=1440.$

$\therefore x=62\tfrac{14}{23}.$

Therefore, A can dig the ditch in $62\tfrac{14}{23}$ days.

(iii) Let $x =$ the number of days it will take B to dig the ditch.

Then $\frac{1}{x} =$ the part B can do in one day,

$\tfrac{1}{36} =$ the part A and C together can dig in one day,

$\tfrac{53}{1440} =$ the part the three together can dig in one day,

and $\tfrac{53}{1440}-\tfrac{1}{36} =$ the part B can dig in one day.

$$\therefore \frac{53}{1440}-\frac{1}{36}=\frac{1}{x}.$$

Multiply by $1440x$, $53x-40x=1440.$

Combine, $13x=1440.$

$\therefore x=110\tfrac{10}{13}.$

Therefore, B can dig the ditch in $110\tfrac{10}{13}$ days.

(iv) Let $x =$ the number of days it will take C to dig the ditch

Then $\frac{1}{x} =$ the part C can dig in one day,

$\tfrac{1}{40} =$ the part A and B together can dig in one day,

$\tfrac{53}{1440} =$ the part the three together can dig in one day,

and $\tfrac{53}{1440}-\tfrac{1}{40} =$ the part C can dig in one day.

$$\therefore \frac{53}{1440} - \frac{1}{40} = \frac{1}{x}.$$

Multiply by $1440x$, $53x - 36x = 1440.$

Combine, $17x = 1440.$

$$\therefore x = 84\tfrac{12}{17}.$$

Therefore, C can dig the ditch in $84\frac{12}{17}$ days.

38. A cistern can be filled by two pipes in 3 and 4 hours, respectively, and can be emptied by a waste pipe in 6 hours. In how many hours will it be filled if all the pipes together are open?

Let x = the number of hours if all the pipes are open.

Then $\frac{1}{x}$ = the part filled in one hour if all the pipes are open,

$\frac{1}{3}$ = the part filled in one hour by the first pipe,

$\frac{1}{4}$ = the part filled in one hour by the second pipe,

$\frac{1}{6}$ = the part emptied in one hour by the waste pipe,

and $\frac{1}{3} + \frac{1}{4} - \frac{1}{6}$ = the part filled in one hour if all the pipes are open.

$$\therefore \frac{1}{3} + \frac{1}{4} - \frac{1}{6} = \frac{1}{x}.$$

Multiply by $12x$, $4x + 3x - 2x = 12.$

Combine, $5x = 12.$

$$\therefore x = 2\tfrac{2}{5}.$$

Therefore, if all the pipes are open, the cistern will be filled in $2\frac{2}{5}$ hours.

39. A cistern can be filled by three pipes in 6, 10, and 15 hours, respectively. In how many hours will it be filled by all the pipes together?

Let x = the number of hours required to fill the cistern.

Then $\frac{1}{x}$ = the part filled in one hour by the three pipes together,

$\frac{1}{6}$ = the part filled in one hour by the first pipe,

$\frac{1}{10}$ = the part filled in one hour by the second pipe,

$\frac{1}{15}$ = the part filled in one hour by the third pipe,

and $\frac{1}{6} + \frac{1}{10} + \frac{1}{15}$ = the part filled in one hour by the three pipes together.

$$\therefore \frac{1}{6} + \frac{1}{10} + \frac{1}{15} = \frac{1}{x}.$$

Multiply by $30x$, $5x + 3x + 2x = 30.$

Combine, $10x = 30.$

$$\therefore x = 3.$$

Therefore, the three pipes together will fill the cistern in 3 hours.

40. A cistern can be filled by three pipes in 3 hours 30 minutes, 2 hours 48 minutes, and 2 hours 20 minutes, respectively. In what time will the cistern be filled if all the pipes are running together?

3 hours 30 minutes = $3\frac{1}{2}$ hours; 2 hours 48 minutes = $2\frac{4}{5}$ hours; 2 hours 20 minutes = $2\frac{1}{3}$ hours.

Let x = the number of hours required to fill the cistern.

Then $\frac{1}{x}$ = the part filled in one hour by the three pipes together,

$\frac{2}{7}$ = the part filled in one hour by the first pipe,

$\frac{5}{14}$ = the part filled in one hour by the second pipe,

$\frac{3}{7}$ = the part filled in one hour by the third pipe,

and $\frac{2}{7}+\frac{5}{14}+\frac{3}{7}$ = the part filled in one hour by the three pipes together.

$$\therefore \frac{2}{7}+\frac{5}{14}+\frac{3}{7}=\frac{1}{x}.$$

Multiply by $14x$, $4x+5x+6x=14.$

Combine, $15x=14.$

$$\therefore x=\tfrac{14}{15}.$$

Therefore, the three pipes together will fill the cistern in $\frac{14}{15}$ of an hour, or 56 minutes.

41. A cistern can be filled by two pipes in $3\frac{1}{3}$ and $3\frac{3}{4}$ hours, respectively, and can be emptied by a waste pipe in $2\frac{1}{2}$ hours. In how many hours will it be filled if all the pipes together are open?

Let x = the number of hours if all the pipes are open.

Then $\frac{1}{x}$ = the part filled in one hour if all the pipes are open,

$\frac{3}{10}$ = the part filled in one hour by the first pipe,

$\frac{4}{15}$ = the part filled in one hour by the second pipe,

$\frac{2}{5}$ = the part emptied in one hour by the waste pipe,

and $\frac{3}{10}+\frac{4}{15}-\frac{2}{5}$ = the part filled in one hour if all the pipes are open.

$$\therefore \frac{3}{10}+\frac{4}{15}-\frac{2}{5}=\frac{1}{x}.$$

Multiply by $30x$, $9x+8x-12x=30.$

Combine, $5x=30.$

$$\therefore x=6.$$

Therefore, if all the pipes are open, the cistern will be filled in 6 hours.

42. A cistern has three pipes. The first pipe will fill it in $5\frac{1}{2}$ hours, the second in $7\frac{1}{3}$ hours, and the three pipes together will fill it in $1\frac{5}{6}$ hours. In how many hours will the third pipe alone fill the cistern?

Let $x =$ the number of hours required for the third pipe.

Then $\frac{1}{x} =$ the part filled in one hour by the third pipe,

$\frac{2}{11} =$ the part filled in one hour by the first pipe,

$\frac{3}{22} =$ the part filled in one hour by the second pipe,

$\frac{6}{11} =$ the part filled in one hour by the three pipes together,

and $\frac{6}{11} - (\frac{2}{11} + \frac{3}{22}) =$ the part filled in one hour by the third pipe.

$$\therefore \frac{6}{11} - \left(\frac{2}{11} + \frac{3}{22}\right) = \frac{1}{x}.$$

Multiply by $22x$, $12x - 4x - 3x = 22.$

Combine, $5x = 22.$

$$\therefore x = 4\tfrac{2}{5}.$$

Therefore, the third pipe will fill the cistern in $4\frac{2}{5}$ hours.

43. A cistern can be filled by three pipes in $4\frac{1}{4}$, $5\frac{2}{3}$, and $3\frac{2}{5}$ hours, respectively. In how many hours will it be filled by the three pipes together?

Let $x =$ the number of hours required by the three pipes together.

Then $\frac{1}{x} =$ the part filled in one hour by the three pipes together,

$\frac{4}{17} =$ the part filled in one hour by the first pipe,

$\frac{3}{17} =$ the part filled in one hour by the second pipe,

$\frac{5}{17} =$ the part filled in one hour by the third pipe,

and $\frac{4}{17} + \frac{3}{17} + \frac{5}{17} =$ the part filled in one hour by the three pipes together.

$$\therefore \frac{4}{17} + \frac{3}{17} + \frac{5}{17} = \frac{1}{x}.$$

Multiply by $17x$, $4x + 3x + 5x = 17.$

Combine, $12x = 17.$

$$\therefore x = 1\tfrac{5}{12}.$$

Therefore, if all the pipes are open, the cistern will be filled in $1\frac{5}{12}$ hours.

44. An accommodation train which goes 32 miles an hour is followed, after 2 hours, by an express train which goes 48 miles an hour. In how many hours will the express train overtake the accommodation train?

Let $x =$ the number of hours the express train goes.
Then $x + 2 =$ the number of hours the accommodation train goes,
$48x =$ the number of miles the express train goes,
and $32(x + 2) =$ the number of miles the accommodation train goes.

But these distances are equal.

$$\therefore 48x = 32(x + 2).$$

Expand, $48x = 32x + 64.$

Transpose, $48x - 32x = 64.$

Combine, $16x = 64.$

$$\therefore x = 4.$$

Therefore, the express train will overtake the accommodation train in 4 hours.

45. A starts from Boston and travels at the rate of 8 miles an hour. Four hours afterwards B starts from the same place and follows A at the rate of 10 miles an hour. How many miles from Boston will B overtake A?

Let $x =$ the number of hours B travels.
Then $x + 4 =$ the number of hours A travels,
$8(x + 4) =$ the number of miles A travels,
and $10x =$ the number of miles B travels.

But these distances are equal.

$$\therefore 10x = 8(x + 4).$$

Expand, $10x = 8x + 32.$

Transpose, $10x - 8x = 32.$

Combine, $2x = 32.$

$$\therefore x = 16.$$

$$10x = 160.$$

Therefore, B will overtake A 160 miles from Boston.

46. A can run 10 yards a second and B can run 9 yards a second. If A gives B a start of 2 seconds, in how many seconds will A overtake B?

Let $x =$ the number of seconds A runs.
Then $x + 2 =$ the number of seconds B runs,
$9(x + 2) =$ the number of yards B runs,
and $10x =$ the number of yards A runs.

But these distances are equal.

$$\therefore\ 10x = 9(x + 2).$$

Expand, $10x = 9x + 18.$

Transpose, $10x - 9x = 18.$

Combine, $x = 18.$

Therefore, A will overtake B in 18 seconds.

47. A freight train which goes 18 miles an hour is followed after 4 hours by a passenger train which goes 24 miles an hour. In how many hours will the passenger train overtake the freight train?

Let $x =$ the number of hours the passenger train goes.

Then $x + 4 =$ the number of hours the freight train goes,

$18(x + 4) =$ the number of miles the freight train goes,

and $24x =$ the number of miles the passenger train goes.

But these distances are equal.

$$\therefore\ 24x = 18(x + 4).$$

Expand, $24x = 18x + 72.$

Transpose, $24x - 18x = 72.$

Combine, $6x = 72.$

$$\therefore\ x = 12.$$

Therefore, the passenger train will overtake the freight train in 12 hours.

48. In going a certain distance a train that makes 42 miles an hour takes 2 hours less than a train that makes 35 miles an hour. Find the distance.

Let $x =$ the number of hours the first train goes.

Then $x + 2 =$ the number of hours the second train goes,

$35(x + 2) =$ the number of miles the second train goes,

and $42x =$ the number of miles the first train goes.

But these distances are equal.

$$\therefore\ 42x = 35(x + 2).$$

Expand, $42x = 35x + 70.$

Transpose, $42x - 35x = 70.$

Combine, $7x = 70.$

$$\therefore\ x = 10.$$

$$42x = 420.$$

Therefore, the required distance is 420 miles.

49. In traveling a certain distance a horse that goes 12 miles an hour takes 1 hour less than a horse that goes 10 miles an hour. Find the distance.

Let $x =$ the number of hours the faster horse travels.
Then $x + 1 =$ the number of hours the slower horse travels,
$10(x + 1) =$ the number of miles the slower horse travels,
and $12x =$ the number of miles the faster horse travels.

But these distances are equal.

$$\therefore\ 12x = 10(x + 1).$$

Expand, $12x = 10x + 10.$
Transpose, $12x - 10x = 10.$
Combine, $2x = 10,$

$$\therefore\ x = 5.$$
$$12x = 60.$$

Therefore, the required distance is 60 miles.

50. In traveling a certain distance an automobilist who travels 28 miles an hour takes $5\frac{1}{4}$ hours less than a bicyclist who travels 16 miles an hour. Find the distance.

Let $x =$ the number of hours the automobilist travels.
Then $x + 5\frac{1}{4} =$ the number of hours the bicyclist travels,
$16(x + 5\frac{1}{4}) =$ the number of miles the bicyclist travels,
and $28x =$ the number of miles the automobilist travels.

But these distances are equal.

$$\therefore\ 28x = 16(x + 5\tfrac{1}{4}).$$

Expand, $28x = 16x + 84.$
Transpose, $28x - 16x = 84.$
Combine, $12x = 84.$

$$\therefore\ x = 7.$$
$$28x = 196.$$

Therefore, the required distance is 196 miles.

51. Find the time between 3 and 4 o'clock when the hands of a clock are together.

At 3 o'clock the hour-hand is 15 minute-spaces ahead of the minute-hand.

Let $x =$ the number of spaces the minute-hand moves over.
Then $x - 15 =$ the number of spaces the hour-hand moves over.
Now, as the minute-hand moves twelve times as fast as the hour-hand,
$12(x - 15) =$ the number of spaces the minute-hand moves over.

$$\therefore\ 12(x - 15) = x.$$

Expand, $12x - 180 = x.$

Transpose, $12x - x = 180.$

Combine, $11x = 180.$

$$\therefore\ x = 16\tfrac{4}{11}.$$

Therefore, the required time is $16\frac{4}{11}$ minutes past 3 o'clock.

52. Find the time between 6 and 7 o'clock when the hands of a clock are together.

At 6 o'clock the hour-hand is 30 minute-spaces ahead of the minute-hand.

Let $x =$ the number of spaces the minute-hand moves over.

Then $x - 30 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves twelve times as fast as the hour-hand,

$12(x - 30) =$ the number of spaces the minute-hand moves over.

$$\therefore\ 12(x - 30) = x.$$

Expand, $12x - 360 = x.$

Transpose, $12x - x = 360.$

Combine, $11x = 360.$

$$\therefore\ x = 32\tfrac{8}{11}.$$

Therefore, the required time is $32\frac{8}{11}$ minutes past 6 o'clock.

53. Find the time between 9 and 10 o'clock when the hands of a clock are at right angles to each other.

At 9 o'clock the minute-hand is 45 minute-spaces behind the hour-hand, and must be 15 minute-spaces behind to be at right angles.

Let $x =$ the number of spaces the minute-hand moves over.

Then $x - 45 + 15 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves twelve times as fast as the hour-hand,

$12(x - 45 + 15) =$ the number of spaces the minute-hand moves over.

$$\therefore\ 12(x - 45 + 15) = x.$$

Expand, $12x - 360 = x.$

Transpose, $12x - x = 360.$

Combine, $11x = 360.$

$$\therefore\ x = 32\tfrac{8}{11}.$$

Therefore, the required time is $32\frac{8}{11}$ minutes past 9 o'clock.

54. Find the time between 4 and 5 o'clock when the hands of a clock point in opposite directions.

At 4 o'clock the minute-hand is 20 minute-spaces behind the hour-hand, and must be 30 minute-spaces ahead that the hands may point in opposite directions.

Let $x =$ the number of spaces the minute-hand moves over.

Then $(x - 20 - 30) =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves twelve times as fast as the hour-hand,

$12(x - 20 - 30) =$ the number of spaces the minute-hand moves over.

$$\therefore\ 12(x - 20 - 30) = x.$$

Expand, $12x - 600 = x.$

Transpose, $12x - x = 600.$

Combine, $11x = 600.$

$$\therefore\ x = 54\tfrac{6}{11}.$$

Therefore, the required time is $54\frac{6}{11}$ minutes past 4 o'clock.

55. Find the times between 5 and 6 o'clock when the hands of a clock make right angles with each other.

At 5 o'clock the minute-hand is 25 minute-spaces behind the hour-hand, and must be 15 minute-spaces behind or 15 minute-spaces ahead to be at right angles.

Let $x =$ the number of spaces the minute-hand moves over.

Then $x - 25 + 15$ or $x - 25 - 15 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves twelve times as fast as the hour-hand,

$12(x - 25 + 15)$ or $12(x - 25 - 15) =$ the number of spaces the minute-hand moves over.

	$\therefore\ 12(x - 25 + 15) = x.$	$\therefore\ 12(x - 25 - 15) = x.$
Expand,	$12x - 120 = x.$	$12x - 480 = x.$
Transpose,	$12x - x = 120.$	$12x - x = 480.$
Combine,	$11x = 120.$	$11x = 480.$
	$\therefore\ x = 10\tfrac{10}{11}.$	$\therefore\ x = 43\tfrac{7}{11}.$

Therefore, the required times are $10\frac{10}{11}$ minutes past 5 o'clock and $43\frac{7}{11}$ minutes past 5 o'clock.

56. Find the time between 4 and 5 o'clock when the hands of a clock are together.

At 4 o'clock the minute-hand is 20 minute-spaces behind the hour-hand.

Let $x =$ the number of spaces the minute-hand moves over.

Then $x - 20 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves twelve times as fast as the hour-hand,

$12(x - 20) =$ the number of spaces the minute-hand moves over.

$$\therefore 12(x - 20) = x.$$

Expand, $12x - 240 = x.$

Transpose, $12x - x = 240.$

Combine, $11x = 240.$

$$\therefore x = 21\tfrac{9}{11}.$$

Therefore, the required time is $21\frac{9}{11}$ minutes past 4 o'clock.

57. Find the time between 11 and 12 o'clock when the hands of a clock point in opposite directions.

At 11 o'clock the minute-hand is 55 minute-spaces behind the hour-hand, and must be 30 minute-spaces behind that the hands may point in opposite directions.

Let $x =$ the number of spaces the minute-hand moves over.

Then $x - 55 + 30 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves twelve times as fast as the hour-hand,

$12(x - 55 + 30) =$ the number of spaces the minute-hand moves over.

$$\therefore 12(x - 55 + 30) = x.$$

Expand, $12x - 300 = x.$

Transpose, $12x - x = 300.$

Combine, $11x = 300.$

$$\therefore x = 27\tfrac{3}{11}.$$

Therefore, the required time is $27\frac{3}{11}$ minutes past 11 o'clock.

58. Find the times between 7 and 8 o'clock when the hands of a clock make right angles with each other.

At 7 o'clock the minute-hand is 35 minute-spaces behind the hour-hand, and must be 15 minute-spaces behind or 15 minute-spaces ahead that the hands may make right angles with each other.

Let $x =$ the number of spaces the minute-hand moves over.

Then $x-35+15$ or $x-35-15=$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves twelve times as fast as the hour-hand,

$12(x-35+15)$ or $12(x-35-15)=$ the number of spaces the minute-hand moves over.

	$\therefore\ 12(x-35+15)=x.$	$\therefore\ 12(x-35-15)=x.$
Expand,	$12x-240=x.$	$12x-600=x.$
Transpose,	$12x-x=240.$	$12x-x=600.$
Combine,	$11x=240.$	$11x=600.$
	$\therefore\ x=21\frac{9}{11}.$	$\therefore\ x=54\frac{6}{11}.$

Therefore, the required times are $21\frac{9}{11}$ minutes past 7 o'clock and $54\frac{6}{11}$ minutes past 7 o'clock.

59. Find the time between 1 and 2 o'clock when the hands of a watch point in opposite directions.

At 1 o'clock the minute-hand is 5 minute-spaces behind the hour-hand, and must be 30 minute-spaces ahead that the hands may point in opposite directions.

Let $x=$ the number of spaces the minute-hand moves over.

Then $x-5-30=$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves twelve times as fast as the hour-hand,

$12(x-5-30)=$ the number of spaces the minute-hand moves over.

$$\therefore\ 12(x-5-30)=x.$$

Expand, $12x-420=x.$

Transpose, $12x-x=420.$

Combine, $11x=420.$

$$\therefore\ x=38\tfrac{2}{11}.$$

Therefore, the required time is $38\frac{2}{11}$ minutes past 1 o'clock.

60. A hound takes 2 leaps while a rabbit takes 3; but 3 of the hound's leaps are equivalent to 5 of the rabbit's. The rabbit has a start of 70 of her own leaps. How many leaps must the hound take to catch the rabbit?

Let $2x=$ the number of leaps taken by the hound.

Then $3x=$ the number of leaps of the rabbit in the same time.

Also let $a=$ the number of feet in one leap of the rabbit.

Then $\frac{5a}{3}$ = the number of feet in one leap of the hound,

and $2x \times \frac{5a}{3} = \frac{10ax}{3}$ = the number of feet the hound runs.

As the rabbit has a start of 70 leaps, and takes $3x$ leaps more before being caught, and as each leap is a feet,

$(70 + 3x)a$ = the number of feet the hound runs.

$$\therefore \frac{10ax}{3} = (70 + 3x)a.$$

Multiply by 3, $10ax = (210 + 9x)a.$

Divide by a, $10x = 210 + 9x.$

Transpose, $10x - 9x = 210.$

Combine, $x = 210.$

$$2x = 420.$$

Therefore, the hound must take 420 leaps.

61. A hare takes 5 leaps to 3 of a hound; and 3 of the hound's leaps are equivalent to 7 of the hare's. The hare has a start of 60 of her own leaps. How many leaps must the hound take to catch the hare?

Let $3x$ = the number of leaps taken by the hound.

Then $5x$ = the number of leaps of the hare in the same time.

Also let a = the number of feet in one leap of the hare.

Then $\frac{7a}{3}$ = the number of feet in one leap of the hound,

and $3x \times \frac{7a}{3} = 7ax$ = the number of feet the hound runs.

As the hare has a start of 60 leaps, and takes $5x$ leaps more before being caught, and as each leap is a feet,

$(60 + 5x)a$ = the number of feet the hound runs.

$$\therefore 7ax = (60 + 5x)a.$$

Divide by a, $7x = 60 + 5x.$

Transpose, $7x - 5x = 60.$

Combine, $2x = 60.$

$$\therefore x = 30.$$

$$3x = 90.$$

Therefore, the hound must take 90 leaps.

62. A hound takes 4 leaps while a hare takes 5; but 5 of the hound's leaps are equivalent to 8 of the hare's. The hare has a start of 84 of her own leaps. How many leaps will the hare take before she is caught?

Let $4x =$ the number of leaps taken by the hound.
Then $5x =$ the number of leaps of the hare in the same time.
Also let $a =$ the number of feet in one leap of the hare.
Then $\frac{8a}{5} =$ the number of feet in one leap of the hound,
and $4x \times \frac{8a}{5} = \frac{32ax}{5} =$ the number of feet the hound runs.

As the hare has a start of 84 leaps, and takes $5x$ leaps more before being caught, and as each leap is a feet,

$(84 + 5x)a =$ the number of feet the hound runs.

$$\therefore \frac{32ax}{5} = (84 + 5x)a.$$

Multiply by 5, $32ax = 420a + 25ax.$
Divide by a, $32x = 420 + 25x.$
Transpose, $32x - 25x = 420.$
Combine, $7x = 420.$
$\therefore x = 60.$
$5x = 300.$

Therefore, the hare will take 300 leaps before she is caught.

63. A hound takes 3 leaps while a hare takes 4; but 5 of the hound's leaps are equivalent to 9 of the hare's. The hare has a start of 189 of *the hound's leaps.* How many leaps will each take before the hare is caught?

Let $3x =$ the number of leaps taken by the hound.
Then $4x =$ the number of leaps of the hare in the same time.
Also let $a =$ the number of feet in one leap of the hound.
Then $\frac{5a}{9} =$ the number of feet in one leap of the hare,
and $3ax =$ the number of feet the hound runs.

As the hare has a start of 189 of the hound's leaps, and each of these leaps is a feet, she has a start of $189a$ feet.

As the hare takes $4x$ leaps of $\frac{5a}{9}$ feet each, she must run $4x \times \frac{5a}{9}$ feet, or $\frac{20ax}{9}$ feet.

Therefore, $189a + \frac{20ax}{9} =$ the number of feet the hound runs.

$$\therefore\ 3ax = 189a + \frac{20ax}{9}.$$

Divide by a, $$3x = 189 + \frac{20x}{9}.$$

Multiply by 9, $$27x = 1701 + 20x.$$

Transpose, $$27x - 20x = 1701.$$

Combine, $$7x = 1701.$$

$$\therefore\ x = 243.$$

$$3x = 729.$$

$$4x = 972.$$

Therefore, the hound will take 729 leaps and the hare 972 leaps before the hare is caught.

64. A merchant adds each year to his capital one fifth of it, and deducts, at the end of each year, $1000 for expenses. In four years, after deducting the last $1000, he has $3160 less than twice his original capital. How much had he at first?

Let $x =$ the number of dollars the merchant had at first.

Then $\frac{6x}{5} - 1000$, or $\frac{6x - 5000}{5}$

= the number of dollars he had at the end of the first year,

$\frac{6}{5}\left(\frac{6x - 5000}{5}\right) - 1000$, or $\frac{36x - 55000}{25}$

= the number of dollars he had at the end of the second year,

$\frac{6}{5}\left(\frac{36x - 55000}{25}\right) - 1000$, or $\frac{216x - 455000}{125}$

= the number of dollars he had at the end of the third year,

and $\frac{6}{5}\left(\frac{216x - 455000}{125}\right) - 1000$, or $\frac{1296x - 3355000}{625}$

= the number of dollars he had at the end of the fourth year.

But $2x - 3160 =$ the number of dollars he had at the end of the fourth year.

$$\therefore\ \frac{1296x - 3355000}{625} = 2x - 3160.$$

Multiply by 625, $$1296x - 3355000 = 1250x - 1975000.$$

Transpose, $$1296x - 1250x = 3355000 - 1975000.$$

Combine, $$46x = 1380000.$$

$$\therefore\ x = 30000.$$

Therefore, the merchant had $30,000 at first.

65. A merchant adds each year to his capital one fourth of it, but takes from it, at the end of each year, $1200 for expenses. At the end of the third year, after deducting the last $1200, he has $950 less than $1\frac{1}{4}$ times his original capital. How much had he at first?

Let $x =$ the number of dollars the merchant had at first.

Then $\frac{5x}{4} - 1200$, or $\frac{5x - 4800}{4}$

$=$ the number of dollars he had at the end of the first year,

$\frac{5}{4}\left(\frac{5x - 4800}{4}\right) - 1200$, or $\frac{25x - 43200}{16}$

$=$ the number of dollars he had at the end of the second year,

and $\frac{5}{4}\left(\frac{25x - 43200}{16}\right) - 1200$, or $\frac{125x - 292800}{64}$

$=$ the number of dollars he had at the end of the third year.

But $\frac{3x}{2} - 950 =$ the number of dollars he had at the end of the third year.

$$\therefore \frac{125x - 292800}{64} = \frac{3x}{2} - 950.$$

Multiply by 64, $125x - 292800 = 96x - 60800.$

Transpose, $125x - 96x = 292800 - 60800.$

Combine, $29x = 232000.$

$\therefore x = 8000.$

Therefore, the merchant had $8000 at first.

66. A trader maintained himself for 3 years at an expense of $2000 a year, and each year increased that part of his stock that was not so expended by one third of it. At the end of three years his original stock was doubled. How much did he have at first?

Let $x =$ the number of dollars the trader had at first.

Then $\frac{4}{3}(x - 2000)$, or $\frac{4x - 8000}{3}$

$=$ the number of dollars he had at the end of the first year,

$\frac{4}{3}\left(\frac{4x - 8000}{3} - 2000\right)$, or $\frac{16x - 56000}{9}$

$=$ the number of dollars he had at the end of the second year,

and $\frac{4}{3}\left(\frac{16x - 56000}{9} - 2000\right)$, or $\frac{64x - 296000}{27}$

$=$ the number of dollars he had at the end of the third year.

But $2x =$ the number of dollars he had at the end of the third year.

$$\therefore \frac{64x - 296000}{27} = 2x.$$

Multiply by 27, $64x - 296000 = 54x.$

Transpose, $64x - 54x = 296000.$

Combine, $10x = 296000.$

$\therefore x = 29600.$

Therefore, the trader had $29,600 at first.

67. A merchant adds each year to his capital one third of it, but takes from it, at the end of each year, $1250 for expenses. At the end of the third year, after he has deducted the last $1250, he finds that he has twice his original capital. What was his original capital?

Let $x =$ the number of dollars the merchant had at first.

Then $\frac{4x}{3} - 1250$, or $\frac{4x - 3750}{3}$

= the number of dollars he had at the end of the first year,

$\frac{4}{3}\left(\frac{4x - 3750}{3}\right) - 1250$, or $\frac{16x - 26250}{9}$

= the number of dollars he had at the end of the second year,

and $\frac{4}{3}\left(\frac{16x - 26250}{9}\right) - 1250$, or $\frac{64x - 138750}{27}$

= the number of dollars he had at the end of the third year.

But $2x =$ the number of dollars he had at the end of the third year.

$$\therefore \frac{64x - 138750}{27} = 2x.$$

Multiply by 27, $64x - 138750 = 54x.$

Transpose, $64x - 54x = 138750.$

Combine, $10x = 138750.$

$\therefore x = 13875.$

Therefore, the merchant's original capital was $13,875.

68. The sum of the ages of three brothers is 24 years, and there is a difference of two years between their successive ages. Find the age of each.

Let $x =$ the number of years in the age of the youngest.

Then $x + 2 =$ the number of years in the age of the second,

$x + 2 + 2 =$ the number of years in the age of the eldest,

and $x + x + 2 + x + 2 + 2 = 3x + 6$

= the number of years in the sum of their ages.

But $24 =$ the number of years in the sum of their ages.

$$\therefore 3x + 6 = 24.$$

Transpose, $3x = 24 - 6.$

Combine, $3x = 18.$

$$\therefore x = 6.$$

$$x + 2 = 8.$$

$$x + 2 + 2 = 10.$$

Therefore, the ages of the three brothers are 10 years, 8 years, and 6 years,

69. An uncle is 10 years older than his nephew, and 15 years ago the uncle's age was twice that of the nephew. Find the age of each.

Let x = the number of years in the age of the nephew.

Then $x + 10$ = the number of years in the age of the uncle,

$x + 10 - 15$ = the number of years in the age of the uncle 15 years ago,

and $x - 15$ = the number of years in the age of the nephew 15 years ago.

$$\therefore x + 10 - 15 = 2(x - 15).$$

Expand, $x + 10 - 15 = 2x - 30.$

Transpose, $x - 2x = 15 - 10 - 30.$

Combine, $-x = -25.$

$$\therefore x = 25.$$

$$x + 10 = 35.$$

Therefore, the uncle is 35 years old, and the nephew 25 years old.

70. In an election 1240 votes were cast for two candidates. Of these votes 47 were defective and were thrown out. The successful candidate received 153 votes more than his opponent. How many votes did each candidate receive?

Let x = the number of votes the defeated candidate received.

Then $x + 153$ = the number of votes the successful candidate received,

and $x + x + 153$ = the number of votes the two candidates received.

But $1240 - 47$ = the number of votes the two candidates received.

$$\therefore x + x + 153 = 1240 - 47.$$

Transpose, $x + x = 1240 - 47 - 153.$

Combine, $2x = 1040.$

$$\therefore x = 520.$$

$$x + 153 = 673.$$

Therefore, the two candidates received 673 and 520 votes respectively.

[illegible]. A man and a woman together can do a piece of work in 12 hours and the woman alone can do it in 30 hours. How long will it take the man alone?

Let $x =$ the number of hours it will take the man alone.

Then $\frac{1}{x} =$ the part the man can do in one hour,

$\frac{1}{30} =$ the part the woman can do in one hour,

and $\frac{1}{x} + \frac{1}{30} =$ the part they can do in one hour working together.

But $\frac{1}{12} =$ the part they can do in one hour working together.

$$\therefore \frac{1}{12} = \frac{1}{x} + \frac{1}{30}.$$

Multiply by 60 $\quad 5x = 60 + 2x.$

Transpose, $\quad 5x - 2x = 60.$

Combine, $\quad 3x = 60.$

$$\therefore x = 20.$$

Therefore, the man can do the work in 20 hours.

72. A can do a piece of work in 50 days, B in 60 days, and C in 75 days. How long will it take the three working together?

Let $x =$ the number of days it will take the three working together.

Then $\frac{1}{x} =$ the part the three working together can do in one day,

$\frac{1}{50} =$ the part A can do in one day,

$\frac{1}{60} =$ the part B can do in one day,

$\frac{1}{75} =$ the part C can do in one day,

and $\frac{1}{50} + \frac{1}{60} + \frac{1}{75} =$ the part the three working together can do in one day.

$$\therefore \frac{1}{50} + \frac{1}{60} + \frac{1}{75} = \frac{1}{x}.$$

Multiply by $300x$, $\quad 6x + 5x + 4x = 300.$

Combine, $\quad 15x = 300.$

$$\therefore x = 20.$$

Therefore, the three together can do the work in 20 days.

73. A can dig a ditch in half the time it takes B, and B can dig it in two thirds the time it takes C. If the three work together they can dig the ditch in 6 days. How long will it take each alone?

Let $x =$ the number of days it will take C to dig the ditch.

Then $\frac{2x}{3} =$ the number of days it will take B to dig the ditch,

$\frac{x}{3} =$ the number of days it will take A to dig the ditch,

$\frac{1}{x} =$ the part C can dig in one day,

$\frac{3}{2x} =$ the part B can dig in one day,

$\frac{3}{x} =$ the part A can dig in one day,

$\frac{1}{x} + \frac{3}{2x} + \frac{3}{x} =$ the part the three working together can dig in one day,

and $\frac{1}{6} =$ the part the three working together can dig in one day.

$$\therefore \frac{1}{6} = \frac{1}{x} + \frac{3}{2x} + \frac{3}{x}.$$

Multiply by $6x$, $x = 6 + 9 + 18.$

Combine, $x = 33.$

$$\frac{2x}{3} = 22.$$

$$\frac{x}{3} = 11.$$

Therefore, A can dig the ditch in 11 days, B in 22 days, and C in 33 days.

74. Divide 451 into two parts such that if 14 is added to the first part and the second part is divided by 14 the results are the same.

Let $x =$ one part.

Then $451 - x =$ the other part.

$$\therefore x + 14 = \frac{451 - x}{14}.$$

Multiply by 14, $14x + 196 = 451 - x.$

Transpose, $14x + x = 451 - 196.$

Combine, $15x = 255.$

$$\therefore x = 17.$$

$$451 - x = 434.$$

Therefore, the parts required are 17 and 434.

75. A farmer agreed to give a farm hand in payment for a year's work \$200 in money and a cow. The farm hand was obliged to leave after working 10 months and received \$160 in money and the cow. At what price was the value of the cow estimated?

Let x = the number of dollars in the value of the cow.

Then $200 + x$ = the number of dollars the farm hand was to receive for 12 months' work,

and $160 + x$ = the number of dollars the farm hand did receive for 10 months' work.

Since 10 months is $\frac{5}{6}$ of a year, the farm hand should receive $\frac{5}{6}$ of his year's wages.

$$\therefore\ 160 + x = \tfrac{5}{6}(200 + x).$$

Multiply by 6, $960 + 6x = 1000 + 5x.$

Transpose, $6x - 5x = 1000 - 960.$

Combine, $x = 40.$

Therefore, the value of the cow was estimated at \$40.

76. If three times the difference between a third and a fourth of a certain number is subtracted from three fourths of the number, the remainder is equal to 6 more than one fifth of the number. Find the number.

Let x = the number.

From the conditions of the problem,

$$\tfrac{3}{4}x - 3\left(\frac{x}{3} - \frac{x}{4}\right) = \frac{x}{5} + 6.$$

Multiply by 20, $15x - 20x + 15x = 4x + 120.$

Transpose, $15x - 20x + 15x - 4x = 120.$

Combine, $6x = 120.$

$\therefore\ x = 20.$

Therefore, the required number is 20.

77. The denominator of a certain fraction is 5 greater than the numerator. If 8 is added to each term, the resulting fraction is equal to $\frac{3}{4}$. Find the fraction.

Let x = the numerator.

Then $x + 5$ = the denominator,

and $\dfrac{x}{x+5}$ = the fraction.

$$\therefore \frac{x+8}{x+13}=\frac{3}{4}.$$

Multiply by $4(x+13)$, $4x+32=3x+39.$

Transpose, $4x-3x=39-32.$

Combine, $x=7.$

$$x+5=12.$$

$$\frac{x}{x+5}=\frac{7}{12}.$$

Therefore, the required fraction is $\frac{7}{12}$.

78. A number is composed of two digits whose difference is 5. If the digits are interchanged, the resulting number is three eighths of the original number. Find the original number.

Let $x=$ the first digit.

Then $x-5=$ the second digit,

$10x+x-5=$ the number,

and $10(x-5)+x=$ the number with digits reversed.

$$\therefore 10(x-5)+x=\tfrac{3}{8}(10x+x-5).$$

Multiply by 8, $80x-400+8x=30x+3x-15.$

Transpose, $80x+8x-30x-3x=400-15.$

Combine, $55x=385.$

$$\therefore x=7.$$

$$x-5=2.$$

$$10x+x-5=72.$$

Therefore, the required number is 72.

79. A does $\frac{7}{30}$ of a piece of work in 5 hours, B does $\frac{5}{16}$ of the remainder in $2\frac{1}{2}$ hours, and C completes the work in 8 hours. How many hours would it take the three working together to do the work?

Let $x=$ the number of hours it would take the three together.

Then $\frac{1}{x}=$ the part the three together can do in one hour,

$\frac{1}{5}$ of $\frac{7}{30}=\frac{7}{150}=$ the part A can do in one hour,

$\frac{5}{16}$ of $\frac{23}{30}=\frac{23}{96}=$ the part B does in $2\frac{1}{2}$ hours,

$\frac{2}{5}$ of $\frac{23}{96}=\frac{23}{240}=$ the part B can do in one hour,

$1-(\frac{7}{30}+\frac{23}{96})=\frac{253}{480}=$ the part C does in 8 hours,

$\frac{1}{8}$ of $\frac{253}{480}=\frac{253}{3840}=$ the part C can do in one hour,

and $\frac{7}{150}+\frac{23}{240}+\frac{253}{3840}=$ the part the three together can do in one hour.

$$\therefore \frac{7}{150}+\frac{23}{240}+\frac{253}{3840}=\frac{1}{x}.$$

Multiply by $19200x$, $896x+1840x+1265x=19200.$

Combine, $4001x=19200.$

$$\therefore x=4\tfrac{3196}{4001}.$$

Therefore, the three working together can do the work in $4\frac{3196}{4001}$ hours.

80. If a certain number is divided by 4 and by 9 and the two quotients multiplied together, the product is equal to the product obtained by multiplying the number by the difference between the number and 35. What is the number?

Let $x=$ the number.

Then $\frac{x}{4}\times\frac{x}{9}=\frac{x^2}{36}=$ the product of the quotients obtained by dividing the number by 4 and by 9,

and $x(x-35)=$ the product obtained by multiplying the number by the difference between the number and 35.

$$\therefore x(x-35)=\frac{x^2}{36}.$$

Multiply by 36, $36x^2-36\times 35x=x^2.$

Transpose, $36x^2-x^2=36\times 35x.$

Combine, $35x^2=36\times 35x.$

Divide by $35x$, $x=36.$

Therefore, the required number is 36.

81. The length of a rectangular garden exceeds the breadth by 4 yards. If each dimension is increased by 1 yard, the area is increased by 27 square yards. Find the dimensions.

Let $x=$ the number of yards in the breadth of the garden.

Then $x+4=$ the number of yards in the length of the garden,

$x(x+4)=$ the number of square yards in the area of the garden,

$x+1=$ the number of yards in the breadth increased by 1 yard,

$x+5=$ the number of yards in the length increased by 1 yard,

and $(x+1)(x+5)=$ the number of square yards in the area of the increased garden.

$$\therefore (x+1)(x+5)=x(x+4)+27.$$

Expand, $x^2+6x+5=x^2+4x+27.$

Transpose, $x^2-x^2+6x-4x=27-5.$

Combine, $2x=22.$

$$\therefore x = 11.$$
$$x + 4 = 15.$$

Therefore, the length of the garden is 15 yards and the width of the garden is 11 yards.

82. A and B have the same income. A saves 5 per cent of his income, while B, who spends each year $175 more than A, finds at the end of 4 years that he is $200 in debt. What income does each receive?

Let $x =$ the number of dollars in the income of each.

Since A saves 5 per cent, or $\frac{1}{20}$, of his income, he must spend $\frac{19}{20}$ of his income; and since in 4 years B gets $200 in debt, in one year B spends $\frac{1}{4}$ of $200, or $50, more than his income.

Therefore, $x+50=$ the number of dollars B spends in one year,

$\frac{19x}{20} =$ the number of dollars A spends in one year,

and $\frac{19x}{20} + 175 =$ the number of dollars B spends in one year.

$$\therefore x + 50 = \frac{19x}{20} + 175.$$

Multiply by 20, $20x + 1000 = 19x + 3500.$

Transpose, $20x - 19x = 3500 - 1000.$

Combine, $x = 2500.$

Therefore, each receives an income of $2500.

83. **Find three numbers whose sum is 70 such that the second number divided by the first gives 2 for a quotient and 1 for a remainder, while the third number divided by the second gives 3 for a quotient and 3 for a remainder.**

Let $x =$ the first number.

Then $2x + 1 =$ the second number,

and $70 - (x + 2x + 1) = 69 - 3x =$ the third number.

$$\therefore \frac{69 - 3x}{2x+1} = 3 + \frac{3}{2x+1}.$$

Multiply by $2x + 1$, $69 - 3x = 6x + 3 + 3.$

Transpose, $-3x - 6x = -69 + 3 + 3.$

Combine, $-9x = -63.$

$$\therefore x = 7.$$
$$2x + 1 = 15.$$
$$69 - 3x = 48.$$

Therefore, the three numbers required are 7, 15, and 48.

84. A railway train which travels at the rate of 32 miles an hour is 45 minutes in advance of a train which travels at the rate of 42 miles an hour. In how long a time will the second train overtake the first?

Let $x =$ the number of hours in which the second train will overtake the first.

Then $x + \frac{3}{4} =$ the number of hours the first train takes to make the distance,

$32(x + \frac{3}{4}) =$ the number of miles the first train travels,

and $42x =$ the number of miles the second train travels.

But these distances are equal.

$$\therefore\ 42x = 32(x + \tfrac{3}{4}).$$

Expand, $42x = 32x + 24.$

Transpose, $42x - 32x = 24.$

Combine, $10x = 24.$

$$\therefore\ x = 2\tfrac{2}{5}.$$

Therefore, the second train will overtake the first in $2\frac{2}{5}$ hours, or 2 hours 24 minutes.

85. At 12 o'clock the hands of a watch point in the same direction. At what time do the hands next point in the same direction?

The hands next point in the same direction between 1 and 2 o'clock. At 1 o'clock the hour-hand is 5 minute-spaces ahead of the minute-hand.

Let $x =$ the number of spaces the minute-hand moves over.

Then $x - 5 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves 12 times as fast as the hour-hand,

$12(x - 5) =$ the number of spaces the minute-hand moves over.

$$\therefore\ 12(x - 5) = x.$$

Expand, $12x - 60 = x.$

Transpose, $12x - x = 60.$

Combine, $11x = 60.$

$$\therefore\ x = 5\tfrac{5}{11}.$$

Therefore, the hands of the watch next point in the same direction at $5\frac{5}{11}$ minutes past 1 o'clock.

86. An express train which makes 42 miles an hour starts 50 minutes after an accommodation train, and overtakes it in 2 hours 5 minutes. How many miles an hour does the accommodation train make?

The express train is traveling 2 hours 5 minutes, or $2\frac{1}{12}$ hours; and the accommodation train 2 hours 55 minutes, or $2\frac{11}{12}$ hours.

Let x = the number of miles the accommodation train goes in one hour.

Then $2\frac{11}{12}x$ = the number of miles the accommodation train goes,

and $2\frac{1}{12} \times 42 = 87\frac{1}{2}$ = the number of miles the express train goes.

But these distances are equal.

$$\therefore 2\tfrac{11}{12}x = 87\tfrac{1}{2}.$$

Multiply by 12, $$35x = 1050.$$

$$\therefore x = 30.$$

Therefore, the accommodation train goes 30 miles an hour.

87. A starts on a trip, making 4 miles an hour, and 15 minutes afterwards B sets out from the same place at the rate of $4\frac{3}{4}$ miles per hour. How many miles will B walk to overtake A and how long will it take him?

Let x = the number of hours B walks.

Then $x + \frac{1}{4}$ = the number of hours A walks,

$4\frac{3}{4}x$ = the number of miles B walks,

and $4(x + \frac{1}{4})$ = the number of miles A walks.

But these distances are equal.

$$\therefore 4\tfrac{3}{4}x = 4(x + \tfrac{1}{4}).$$

Expand, $$\frac{19x}{4} = 4x + 1.$$

Multiply by 4, $$19x = 16x + 4.$$

Transpose, $$19x - 16x = 4.$$

Combine, $$3x = 4.$$

$$\therefore x = 1\tfrac{1}{3}.$$

$$4\tfrac{3}{4}x = 6\tfrac{1}{3}.$$

Therefore, to overtake A, B walks $6\frac{1}{3}$ miles, and requires $1\frac{1}{3}$ hours, or 1 hour 20 minutes.

88. Three men, A, B, and C, set out from the same place and travel over the same route at the rates of 4, 5, and 6 miles respectively per hour. B starts two hours after A. How long after B should C start that B and C may overtake A at the same moment?

Let x = the number of hours A travels.

Then $x - 2$ = the number of hours B travels,

$4x$ = the number of miles A travels,

and $5(x - 2)$ = the number of miles B travels.

But these distances are equal.

$$\therefore\ 5(x-2)=4x.$$

Expand, $5x-10=4x.$

Transpose, $5x-4x=10.$

Combine, $x=10.$

$$4x=40.$$

Hence, each man travels 40 miles; and since C travels 6 miles per hour, he requires $6\frac{2}{3}$ hours to travel the distance. Since A travels 10 hours, B travels 2 hours less, or 8 hours.

Therefore, C should start 8 hours $-$ $6\frac{2}{3}$ hours, or $1\frac{1}{3}$ hours, after B.

89. At what times between 6 and 7 o'clock do the hands of a watch make right angles with each other?

At 6 o'clock the hour-hand is 30 minute-spaces ahead of the minute-hand, and must be 15 minute-spaces ahead or 15 minute-spaces behind to be at right angles with the minute-hand.

Let x = the number of spaces the minute-hand moves over.

Then $x-30+15$ or $x-30-15$ = the number of spaces the hour-hand moves over.

As the minute-hand moves 12 times as fast as the hour-hand,

$12(x-30+15)$ or $12(x-30-15)$ = the number of spaces the minute-hand moves over.

	$\therefore\ 12(x-30+15)=x.$	$\therefore\ 12(x-30-15)=x.$
Expand,	$12x-360+180=x.$	$12x-360-180=x.$
Transpose,	$12x-x=360-180.$	$12x-x=360+180.$
Combine,	$11x=180.$	$11x=540.$
	$\therefore\ x=16\frac{4}{11}.$	$\therefore\ x=49\frac{1}{11}.$

Therefore, the required times are $16\frac{4}{11}$ minutes past 6 o'clock and $49\frac{1}{11}$ minutes past 6 o'clock.

90. At what time between 3 and 4 o'clock do the hands of a watch point in opposite directions?

At 3 o'clock the hour-hand is 15 minute-spaces ahead of the minute-hand, and must be 30 minute-spaces behind that the hands may point in opposite directions.

Let x = the number of spaces the minute-hand moves over.

Then $x-15-30$ = the number of spaces the hour-hand moves over.

As the minute-hand moves 12 times as fast as the hour-hand,

$12(x-15-30)$ = the number of spaces the minute-hand moves over.

$$\therefore\ 12(x - 15 - 30) = x.$$

Expand, $12x - 180 - 360 = x.$

Transpose, $12x - x = 180 + 360.$

Combine, $11x = 540.$

$$\therefore\ x = 49\tfrac{1}{11}.$$

Therefore, the required time is $49\frac{1}{11}$ minutes past 3 o'clock.

91. A man is 565 of his own steps ahead of a horse. If 4 of the man's steps are equal to 3 of the horse's, and the man takes 5 steps while the horse takes 4, how many steps must the horse take to overtake the man?

Let $4x =$ the number of steps taken by the horse.

Then $5x =$ the number of steps taken by the man.

Also let $a =$ the number of feet in one step of the man.

Then $\dfrac{4a}{3} =$ the number of feet in one step of the horse,

and $4x \times \dfrac{4a}{3} = \dfrac{16ax}{3} =$ the number of feet the horse goes.

As the man has a start of 565 steps, and takes $5x$ steps more before being overtaken, and as each step is a feet,

$(565 + 5x)a =$ the number of feet the horse goes.

$$\therefore\ \frac{16ax}{3} = (565 + 5x)a.$$

Multiply by 3, $16ax = 1695a + 15ax.$

Transpose, $16ax - 15ax = 1695a.$

Combine, $ax = 1695a.$

$$\therefore\ x = 1695.$$

$$4x = 6780.$$

Therefore, the horse must take 6780 steps to overtake the man.

92. A father 50 years old has two sons whose ages respectively are 14 years and 11 years. In how many years will the sum of the ages of the sons equal the age of the father?

Let $x =$ the required number of years.

Then $50 + x =$ the number of years in the age of the father x years hence,

$14 + x =$ the number of years in the age of the elder son x years hence,

and $11 + x =$ the number of years in the age of the younger son x years hence.

$$\therefore\ 14 + x + 11 + x = 50 + x.$$

Transpose, $x + x - x = 50 - 14 - 11.$

Combine, $x = 25.$

Therefore, the required number of years is 25.

93. Four workmen do a piece of work together. A could do the work in 8 days, B in 10 days, C in 12 days, and D in 15 days. In how many days do they finish the work?

Let x = the number of days it will take the four working together.

Then $\frac{1}{x}$ = the part the four together can do in one day,

$\frac{1}{8}$ = the part A can do in one day,

$\frac{1}{10}$ = the part B can do in one day,

$\frac{1}{12}$ = the part C can do in one day,

$\frac{1}{15}$ = the part D can do in one day,

and $\frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15}$ = the part the four together can do in one day.

$$\therefore\ \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} = \frac{1}{x}.$$

Multiply by $120x$, $15x + 12x + 10x + 8x = 120.$

Combine, $45x = 120.$

$$\therefore\ x = 2\tfrac{2}{3}.$$

Therefore, the four working together can do the work in $2\frac{2}{3}$ days.

94. A messenger starts to carry a despatch; $4\frac{1}{2}$ hours after his departure a second messenger is sent, who ought to overtake the first in 9 hours, and to do this he is obliged to travel 3 more miles per hour. How many miles per hour does the first messenger travel?

Let x = the number of miles the first messenger travels in one hour.

Then $x + 3$ = the number of miles the second messenger travels in one hour,

$9(x + 3)$ = the number of miles the second messenger travels,

and $13\frac{1}{2}x$ = the number of miles the first messenger travels.

But these distances are equal.

$$\therefore\ 13\tfrac{1}{2}x = 9(x + 3).$$

Multiply by 2, $27x = 18x + 54.$

Transpose, $27x - 18x = 54.$

Combine, $9x = 54.$

$$\therefore\ x = 6.$$

Therefore, the first messenger travels 6 miles per hour.

95. A, B, and C have together \$1000. A has \$70 less than a third more than B, and C has \$20 more than three sevenths of the amount A and B together have. How many dollars has each?

Let $\qquad x =$ the number of dollars B has.

Then $\qquad \frac{4x}{3} - 70 =$ the number of dollars A has,

and $\quad \frac{3}{7}\left(x + \frac{4x}{3} - 70\right) + 20 =$ the number of dollars C has.

$$\therefore x + \frac{4x}{3} - 70 + \frac{3}{7}\left(x + \frac{4x}{3} - 70\right) + 20 = 1000.$$

Expand, $\qquad x + \frac{4x}{3} - 70 + x - 30 + 20 = 1000.$

Multiply by 3, $\quad 3x + 4x - 210 + 3x - 90 + 60 = 3000.$

Transpose, $\quad 3x + 4x + 3x = 3000 + 210 + 90 - 60.$

Combine, $\qquad 10x = 3240.$

$$\therefore x = 324.$$

$$\frac{4x}{3} - 70 = 362.$$

$$\frac{3}{7}\left(x + \frac{4x}{3} - 70\right) + 20 = 314.$$

Therefore, A has \$362; B, \$324; and C, \$314.

96. A man has 9 hours at his disposal. How far into the country may he ride in an automobile at $22\frac{1}{2}$ miles an hour that he may return in time, driving a horse at the rate of $6\frac{1}{2}$ miles an hour?

Let $\qquad x =$ the number of hours he may ride in the automobile.

Then $\qquad 9 - x =$ the number of hours he may drive the horse,

$\qquad 22\frac{1}{2}x =$ the number of miles he rides in the automobile,

and $\qquad 6\frac{1}{2}(9 - x) =$ the number of miles he drives the horse.

But these distances are equal.

$$\therefore 22\tfrac{1}{2}x = 6\tfrac{1}{2}(9 - x).$$

Multiply by 2, $\qquad 45x = 117 - 13x.$

Transpose, $\qquad 45x + 13x = 117.$

Combine, $\qquad 58x = 117.$

$$\therefore x = 2\tfrac{1}{58}.$$

$$22\tfrac{1}{2}x = 45\tfrac{45}{116}.$$

Therefore, the man may ride into the country $45\frac{45}{116}$ miles.

97. In going from New York to Buffalo, a fast express train at 55 miles an hour takes 3 hours less than a passenger train at 40 miles an hour. Find the distance from New York to Buffalo.

Let $x =$ the number of hours it takes the express train.
Then $x + 3 =$ the number of hours it takes the passenger train,
$55x =$ the number of miles the express train runs,
and $40(x + 3) =$ the number of miles the passenger train runs.

But these distances are equal.

$$\therefore\ 55x = 40(x + 3).$$

Expand, $55x = 40x + 120.$
Transpose, $55x - 40x = 120.$
Combine, $15x = 120.$

$$\therefore\ x = 8.$$
$$55x = 440.$$

Therefore, the distance from New York to Buffalo is 440 miles.

98. A boy buys apples at the rate of 5 for 2 cents. He sells half of them 2 for a cent and the other half 3 for a cent, and gains 10 cents by the transaction. How many apples does he buy?

Five apples for 2 cents is $\frac{2}{5}$ of a cent each; 2 apples for a cent is $\frac{1}{2}$ of a cent each; 3 apples for a cent is $\frac{1}{3}$ of a cent each.

Let $x =$ the number of apples the boy buys.
Then $\frac{x}{2} =$ the number of apples he sells 2 for a cent,
$\frac{x}{2} =$ the number of apples he sells 3 for a cent,
$\frac{2x}{5} =$ the number of cents he pays for the apples,
$\frac{x}{4} =$ the number of cents he receives for the apples he sells 2 for a cent,
$\frac{x}{6} =$ the number of cents he receives for the apples he sells 3 for a cent,
and $\frac{x}{4} + \frac{x}{6} =$ the number of cents he receives for the apples.

$$\therefore\ \frac{x}{4} + \frac{x}{6} = \frac{2x}{5} + 10.$$

Multiply by 60, $15x + 10x = 24x + 600.$
Transpose, $15x + 10x - 24x = 600.$
Combine, $x = 600.$

Therefore, the boy buys 600 apples.

99. When a body of troops is formed into a solid square, there are 54 men over; but when formed in a column with 5 men more in front and 4 men fewer in depth, there are lacking 4 men to complete it. How many men are there?

Let $x =$ the number of men on a side of the solid square.

Then $x^2 + 54 =$ the whole number of men,

$x + 5 =$ the number of men on the front of the column,

$x - 4 =$ the number of men on the depth of the column,

and $(x + 5)(x - 4) - 4 =$ the whole number of men.

$$\therefore (x + 5)(x - 4) - 4 = x^2 + 54.$$

Expand, $x^2 + x - 20 - 4 = x^2 + 54.$

Transpose, $x^2 - x^2 + x = 54 + 20 + 4.$

Combine, $x = 78.$

$$x^2 + 54 = 6138.$$

Therefore, there are 6138 men.

100. Divide the number 628 into two parts such that the smaller part is contained in the larger part twice, with a remainder of 7.

Let $x =$ the larger part.

Then $628 - x =$ the smaller part.

$$\therefore \frac{x - 7}{628 - x} = 2.$$

Multiply by $628 - x$, $x - 7 = 1256 - 2x.$

Transpose, $x + 2x = 1256 + 7.$

Combine, $3x = 1263.$

$$\therefore x = 421.$$

$$628 - x = 207.$$

Therefore, the parts required are 421 and 207.

101. The difference between two numbers is 154, and if the greater number is divided by the less the quotient is 3 and the remainder 10. Find the numbers.

Let $x =$ the greater number.

Then $x - 154 =$ the smaller number.

$$\therefore \frac{x-10}{x-154} = 3.$$

Multiply by $x - 154$, $x - 10 = 3x - 462.$

Transpose, $x - 3x = 10 - 462.$

Combine, $-2x = -452.$

$$\therefore x = 226.$$

$$x - 154 = 72.$$

Therefore, the numbers required are 226 and 72.

102. The width of a room is two thirds the length. If the width was 4 feet more and the length 4 feet less, the room would be square. Find the dimensions.

Let $3x =$ the number of feet in the length of the room.

Then $2x =$ the number of feet in the width of the room.

$$\therefore 3x - 4 = 2x + 4.$$

Transpose, $3x - 2x = 4 + 4.$

Combine, $x = 8.$

$$3x = 24.$$

$$2x = 16.$$

Therefore, the length of the room is 24 feet, and the width 16 feet.

103. A walks to the top of a mountain at the rate of $1\frac{3}{4}$ miles an hour, and back again at the rate of $2\frac{1}{3}$ miles an hour, and is out 7 hours. How far is it to the top of the mountain?

Let $x =$ the number of miles to the top of the mountain.

Then $\dfrac{x}{1\frac{3}{4}} = \dfrac{4x}{7} =$ the number of hours A is ascending the mountain,

and $\dfrac{x}{2\frac{1}{3}} = \dfrac{3x}{7} =$ the number of hours A is descending the mountain.

$$\therefore \frac{4x}{7} + \frac{3x}{7} = 7.$$

Multiply by 7, $4x + 3x = 49.$

Combine, $7x = 49.$

$$\therefore x = 7.$$

Therefore, it is 7 miles to the top of the mountain.

Exercise 68. Page 165

1. A and B together can do a piece of work in m days, A and C together in n days, and B and C together in p days. How many days will it take each alone to do the work, and how many days if the three work together?

Let $x =$ the number of days required for the three together to do the work.

Then $\frac{1}{x} =$ the part the three together can do in one day,

$\frac{2}{x} =$ the part the three together can do in two days,

$\frac{1}{m} =$ the part A and B together can do in one day,

$\frac{1}{n} =$ the part A and C together can do in one day,

$\frac{1}{p} =$ the part B and C together can do in one day,

and $\frac{1}{m}+\frac{1}{n}+\frac{1}{p} =$ the part the three together can do in two days.

$$\therefore \frac{1}{m}+\frac{1}{n}+\frac{1}{p}=\frac{2}{x}.$$

Multiply by $mnpx$, $npx + mpx + mnx = 2\,mnp.$

Factor, $(np + mp + mn)x = 2\,mnp.$

$$\therefore x = \frac{2\,mnp}{mn + mp + np}.$$

Again, $\frac{1}{x}-\frac{1}{m} =$ the part C can do in one day,

$\frac{1}{x}-\frac{1}{n} =$ the part B can do in one day,

and $\frac{1}{x}-\frac{1}{p} =$ the part A can do in one day.

$$\frac{1}{x}-\frac{1}{m}=\frac{mn+mp+np}{2\,mnp}-\frac{1}{m}=\frac{mn+mp+np-2\,np}{2\,mnp}=\frac{mn+mp-np}{2\,mnp}.$$

$$\frac{1}{x}-\frac{1}{n}=\frac{mn+mp+np}{2\,mnp}-\frac{1}{n}=\frac{mn+mp+np-2\,mp}{2\,mnp}=\frac{mn-mp+np}{2\,mnp}.$$

$$\frac{1}{x}-\frac{1}{p}=\frac{mn+mp+np}{2\,mnp}-\frac{1}{p}=\frac{mn+mp+np-2\,mn}{2\,mnp}=\frac{mp+np-mn}{2\,mnp}.$$

Therefore, the three together can do the work in $\frac{2\,mnp}{mn+mp+np}$ days;

A alone in $\frac{2\,mnp}{mp+np-mn}$ days; B in $\frac{2\,mnp}{mn+np-mp}$ days;

C in $\frac{2\,mnp}{mn+mp-np}$ days.

2. Three partners, A, B, and C, are to divide a profit of d dollars made in a partnership. A furnished a dollars for m months; B, b dollars for n months; and C, c dollars for p months. What will be the share of each?

Let x = the number of dollars in A's share,
y = the number of dollars in B's share,
and z = the number of dollars in C's share.

The share of each is proportional to the product of the number of dollars furnished by the number of months. A furnished a dollars for m months; B, b dollars for n months; and C, c dollars for p months.

$$\therefore\ x = \frac{am}{am+bn+cp} \text{ of } d = \frac{amd}{am+bn+cp},$$

$$y = \frac{bn}{am+bn+cp} \text{ of } d = \frac{bnd}{am+bn+cp},$$

and

$$z = \frac{cp}{am+bn+cp} \text{ of } d = \frac{cpd}{am+bn+cp}.$$

Therefore, A's share is $\frac{amd}{am+bn+cp}$ dollars; B's share, $\frac{bnd}{am+bn+cp}$ dollars; and C's share, $\frac{cpd}{am+bn+cp}$ dollars.

3. The population of a city increases each year t per cent of that of the preceding year. The population is a inhabitants. What was the population n years ago?

Let x = the population n years ago.

Then $\left(\frac{100+t}{100}\right)x$ = the population $n-1$ years ago,

$\left(\frac{100+t}{100}\right)^2 x$ = the population $n-2$ years ago,

$\left(\frac{100+t}{100}\right)^3 x$ = the population $n-3$ years ago,

. . . .

and $\left(\frac{100+t}{100}\right)^n x$ = the population $n-n$ years ago; that is, the population now.

But a = the population now.

$$\therefore\ \left(\frac{100+t}{100}\right)^n x = a.$$

$$\frac{(100+t)^n}{100^n} x = a.$$

Multiply by 100^n, $(100+t)^n x = 100^n a$.

$$\therefore x = \frac{100^n a}{(100+t)^n}.$$

Therefore, the population of the city n years ago was $\frac{100^n a}{(100+t)^n}$.

4. A number is divided by d, and the sum of the quotient, dividend, and divisor is b. What is the number?

Let $x =$ the number.

Then $\frac{x}{d} =$ the quotient,

$x =$ the dividend,

$d =$ the divisor,

and $\frac{x}{d} + x + d =$ the sum of the quotient, dividend, and divisor.

But $b =$ the sum of the quotient, dividend, and divisor.

$$\therefore \frac{x}{d} + x + d = b.$$

Multiply by d, $x + dx + d^2 = bd$.

Transpose, $dx + x = bd - d^2$.

Factor, $x(d+1) = d(b-d)$.

$$\therefore x = \frac{d(b-d)}{d+1}.$$

Therefore, the required number is $\frac{d(b-d)}{d+1}$.

5. A traveler wishes to accomplish a journey of k miles in n days. At the end of d days he receives news which obliges him to reach the end of his journey a days sooner. How many more miles must he travel each day?

Let $x =$ the number of extra miles he must travel per day.

Then $\frac{k}{n} =$ the number of miles per day he travels at first,

$\frac{dk}{n} =$ the number of miles he travels at the first rate,

$k - \frac{dk}{n} =$ the number of miles he then has to travel,

$n - d - a =$ the number of days in which he is to finish his journey,

and $\frac{k - \frac{dk}{n}}{n - d - a} =$ the number of miles per day he must travel.

But $\frac{k}{n} + x =$ the number of miles per day he must travel.

$$\therefore \frac{k}{n} + x = \frac{k - \frac{dk}{n}}{n - d - a}.$$

Multiply by $n(n - d - a)$,

$nk - dk - ak + (n - d - a)nx = nk - dk.$

Transpose, $(n - d - a)nx = nk - dk - nk + dk + ak.$

Combine, $(n - d - a)nx = ak.$

$$\therefore x = \frac{ak}{n(n - d - a)}.$$

Therefore, the traveler must travel $\frac{ak}{n(n - d - a)}$ more miles each day.

6. The circumference of a fore wheel of a carriage is a feet, and that of a hind wheel is b feet. Find the distance the carriage has traveled when the fore wheel has made n revolutions more than the hind wheel.

Let x = the number of feet the carriage has then traveled.

Then $\frac{x}{a}$ = the number of revolutions the fore wheel has made,

and $\frac{x}{b}$ = the number of revolutions the hind wheel has made.

$$\therefore \frac{x}{a} - \frac{x}{b} = n.$$

Multiply by ab, $bx - ax = abn.$

$$\therefore x = \frac{abn}{b - a}.$$

Therefore, the required distance is $\frac{abn}{b - a}$ feet.

7. A boatman, rowing with the tide, makes n miles in t hours. In returning he rows against a tide m times as strong as the first and makes the distance in t' hours. Find the velocity of the second tide.

Let x = the number of miles per hour the first tide flows.

Then mx = the number of miles per hour the second tide flows,

$\frac{n}{t}$ = the number of miles per hour he makes with the first tide,

$\frac{n}{t} - x$ = the number of miles per hour he actually rows,

$\frac{n}{t'}$ = the number of miles per hour he makes against the second tide,

and $\frac{n}{t'} + mx$ = the number of miles per hour he actually rows,

$$\therefore \frac{n}{t'} + mx = \frac{n}{t} - x.$$

Multiply by tt', $nt + mtt'x = nt' - tt'x.$

Transpose, $mtt'x + tt'x = nt' - nt.$

Factor, $tt'(m+1)x = n(t'-t).$

$$\therefore x = \frac{n(t'-t)}{tt'(m+1)}.$$

$$mx = \frac{mn(t'-t)}{tt'(m+1)}.$$

Therefore, the velocity of the second tide is $\frac{mn(t'-t)}{tt'(m+1)}$ miles per hour.

Exercise 69. Page 167

1. Find the interest on $1100 for 3 years 4 months at 5 per cent.

By Formula 1, $i = prt.$

Here $p = 1100,\ r = \frac{5}{100},\ t = 3\frac{1}{3}.$

$$\therefore i = 1100 \times \frac{5}{100} \times \frac{10}{3} = 183\frac{1}{3}.$$

Therefore, the required interest is $183.33.

2. Find the interest on $1275 for 3 years 2 months 15 days at 8 per cent.

By Formula 1, $i = prt.$

Here $p = 1275,\ r = \frac{8}{100},\ t = 3\frac{5}{24}.$

$$\therefore i = 1275 \times \frac{8}{100} \times \frac{77}{24} = 327\frac{1}{4}.$$

Therefore, the required interest is $327.25.

3. Find the amount of $6460 for 1 year 10 months 22 days at $4\frac{1}{4}$ per cent.

By Formula 1, $a = p + prt.$

Here $p = 6460,\ r = \frac{425}{10000},\ t = 1\frac{161}{180}.$

$$\therefore a = 6460 + 6460 \times \frac{425}{10000} \times \frac{341}{180} = 6460 + 520.12 = 6980.12.$$

Therefore, the required amount is $6980.12.

4. Find the amount of $1250 for 3 months 16 days at 5 per cent.

By Formula 1, $a = p + prt.$

Here $p = 1250,\ r = \frac{5}{100},\ t = \frac{53}{180}.$

$$\therefore a = 1250 + 1250 \times \frac{5}{100} \times \frac{53}{180} = 1250 + 18.40 = 1268.40.$$

Therefore, the required amount is $1268.40.

5. Find the rate per cent when the interest on \$997.75 is \$199.55 for 5 years 4 months.

By Formula 5, $r = \dfrac{a-p}{pt}$.

Here $a - p = i = 199.55$, $p = 997.75$, $t = 5\frac{1}{3}$.

$$\therefore\ r = \frac{199.55}{997.75 \times \frac{16}{3}} = \frac{199.55 \times 3}{997.75 \times 16} = \frac{3}{80}.$$

$$\tfrac{3}{80} \text{ of } 100 = 3\tfrac{3}{4}.$$

Therefore, the required rate is $3\frac{3}{4}$ per cent.

6. Find the rate per cent when \$350 amounts to \$406.70 in 3 years 7 months 6 days.

By Formula 5, $r = \dfrac{a-p}{pt}$.

Here $a = 406.70$, $p = 350$, $t = 3\frac{3}{5}$.

$$\therefore\ r = \frac{406.70 - 350}{350 \times 3\frac{3}{5}} = \frac{56.70 \times 5}{350 \times 18} = \frac{9}{200}.$$

$$\tfrac{9}{200} \text{ of } 100 = 4\tfrac{1}{2}.$$

Therefore, the required rate is $4\frac{1}{2}$ per cent.

7. Find the time in which the interest on \$487.50 will amount to \$39 at 4 per cent.

By Formula 4, $t = \dfrac{a-p}{pr}$.

Here $a - p = i = 39$, $p = 487.50$, $r = \frac{4}{100}$.

$$\therefore\ t = \frac{39}{487.50 \times \frac{4}{100}} = \frac{39 \times 100}{487.50 \times 4} = 2.$$

Therefore, the required time is 2 years.

8. Find the time in which the sum of \$1587.75 will amount to \$1611.68 at $5\frac{1}{2}$ per cent.

By Formula 4, $t = \dfrac{a-p}{pr}$.

Here $a = 1611.68$, $p = 1587.75$, $r = \frac{55}{1000}$.

$$\therefore\ t = \frac{1611.68 - 1587.75}{1587.75 \times \frac{55}{1000}} = \frac{23.93 \times 1000}{1587.75 \times 55} = \frac{19144}{69861}.$$

$$\tfrac{19144}{69861} \text{ years} = 3 \text{ months } 9 \text{ days}.$$

Therefore, the required time is 3 months 9 days.

9. Find the principal that will produce $1746.60 interest in 3 years 5 months at 6 per cent.

By Formula 2, $p = \frac{i}{rt}$.

Here $i = 1746.60$, $r = \frac{6}{100}$, $t = 3\frac{5}{12}$.

$$\therefore\ p = \frac{1746.60}{\frac{6}{100} \times \frac{41}{12}} = \frac{1746.60 \times 100 \times 12}{6 \times 41} = 8520.$$

Therefore, the required principal is $8520.

10. Find the principal that will produce $1339.28 interest in 2 years 7 months 24 days at 6 per cent.

By Formula 2, $p = \frac{i}{rt}$.

Here $i = 1339.28$, $r = \frac{6}{100}$, $t = 2\frac{13}{20}$.

$$\therefore\ p = \frac{1339.28}{\frac{6}{100} \times \frac{53}{20}} = \frac{1339.28 \times 100 \times 20}{6 \times 53} = 8423.14.$$

Therefore, the required principal is $8423.14.

11. Find the principal that will amount to $6378.75 in 1 year 1 month at 5 per cent.

By Formula 3, $p = \frac{a}{1 + rt}$.

Here $a = 6378.75$, $r = \frac{5}{100}$, $t = 1\frac{1}{12}$.

$$\therefore\ p = \frac{6378.75}{1 + \frac{5}{100} \times \frac{13}{12}} = \frac{6378.75}{1 + \frac{13}{240}} = \frac{6378.75}{\frac{253}{240}} = \frac{6378.75 \times 240}{253} = 6050.99.$$

Therefore, the required principal is $6050.99.

12. Find the principal that will amount to $21,047.95 in 1 year 7 months 21 days at $4\frac{1}{2}$ per cent.

By Formula 3, $p = \frac{a}{1 + rt}$.

Here $a = 21047.95$, $r = \frac{45}{1000}$, $t = 1\frac{77}{120}$.

$$\therefore\ p = \frac{21047.95}{1 + \frac{45}{1000} \times \frac{197}{120}} = \frac{21047.95}{1 + \frac{591}{8000}} = \frac{21047.95}{\frac{8591}{8000}} = \frac{21047.95 \times 8000}{8591} = 19600.$$

Therefore, the required principal is $19,600.

13. If A can do a piece of work in $3\frac{1}{2}$ days, and B can do the same work in $4\frac{1}{2}$ days, in how many days can both together do the work?

By Problem 2, page 164, $x = \frac{mn}{m + n}$.

Here $m = 3\frac{1}{2}$, $n = 4\frac{1}{2}$.

$$\therefore\ x = \frac{3\frac{1}{2} \times 4\frac{1}{2}}{3\frac{1}{2} + 4\frac{1}{2}} = \frac{\frac{63}{4}}{8} = \frac{63}{32} = 1\frac{31}{32}.$$

Therefore, the required time is $1\frac{31}{32}$ days.

14. A and B together can do a piece of work in 12 days, A and C together in 15 days, and B and C together in 16 days. How many days will it take each alone to do the work, and how many days if the three work together?

By Problem 1, Exercise 68, page 165,

$$x = \frac{2\,mnp}{mn + mp + np};\quad \text{A} = \frac{2\,mnp}{mp + np - mn};$$
$$\text{B} = \frac{2\,mnp}{mn + np - mp};\quad \text{C} = \frac{2\,mnp}{mn + mp - np}.$$

Here $m = 12$, $n = 15$, $p = 16$.

$$\therefore\ x = \frac{2 \times 12 \times 15 \times 16}{12 \times 15 + 12 \times 16 + 15 \times 16} = \frac{2 \times 12 \times 15 \times 16}{612} = \frac{160}{17} = 9\tfrac{7}{17};$$
$$\text{A} = \frac{2 \times 12 \times 15 \times 16}{12 \times 16 + 12 \times 16 - 15 \times 15} = \frac{2 \times 12 \times 15 \times 16}{252} = \frac{160}{7} = 22\tfrac{6}{7};$$
$$\text{B} = \frac{2 \times 12 \times 15 \times 16}{12 \times 15 + 15 \times 16 - 12 \times 16} = \frac{2 \times 12 \times 15 \times 16}{228} = \frac{480}{19} = 25\tfrac{5}{19};$$
$$\text{C} = \frac{2 \times 12 \times 15 \times 16}{12 \times 15 + 12 \times 16 - 15 \times 16} = \frac{2 \times 12 \times 15 \times 16}{132} = \frac{480}{11} = 43\tfrac{7}{11}.$$

Therefore, A will do the work in $22\frac{6}{7}$ days; B in $25\frac{5}{19}$ days; C in $43\frac{7}{11}$ days; and the three together in $9\frac{7}{17}$ days.

15. Three graziers, A, B, and C, hire a pasture for which they pay $132.50. A puts in 10 oxen for 3 months; B, 12 oxen for 4 months; and C, 14 oxen for 2 months. How much rent ought each to pay?

By Problem 2, Exercise 68, page 165,

$$x = \frac{amd}{am + bn + cp};\ y = \frac{bnd}{am + bn + cp};\ z = \frac{cpd}{am + bn + cp}.$$

Here $a = 10$, $b = 12$, $c = 14$, $d = 132.50$, $m = 3$, $n = 4$, $p = 2$.

$$\therefore\ x = \frac{10 \times 3 \times 132.50}{10 \times 3 + 12 \times 4 + 14 \times 2} = 37.50;$$
$$y = \frac{12 \times 4 \times 132.50}{10 \times 3 + 12 \times 4 + 14 \times 2} = 60.00;$$
$$z = \frac{14 \times 2 \times 132.50}{10 \times 3 + 12 \times 4 + 14 \times 2} = 35.00.$$

Therefore, A should pay $37.50; B, $60.00; and C, $35.00.

16. A traveler wishes to accomplish a journey of 240 miles in 8 days. At the end of 3 days he receives news which obliges him to reach the end of his journey 2 days sooner. How many more miles must he travel each day?

By Problem 5, Exercise 68, page 166, $x = \frac{ak}{n(n-d-a)}$.

Here $a = 2,\ k = 240,\ n = 8,\ d = 3.$

$$\therefore\ x = \frac{2 \times 240}{8(8-3-2)} = \frac{2 \times 240}{8 \times 3} = 20.$$

Therefore, the traveler must travel 20 miles more each day.

17. The circumference of a fore wheel of a carriage is 11 feet 3 inches and that of a hind wheel is 12 feet 2 inches. Find the distance the carriage has traveled when the fore wheel has made 150 more revolutions than the hind wheel.

By Problem 6, Exercise 68, page 166, $x = \frac{abn}{b-a}$.

Here $a = 11\frac{1}{4},\ b = 12\frac{1}{6},\ n = 150.$

$$\therefore\ x = \frac{\frac{45}{4} \times \frac{73}{6} \times 150}{12\frac{1}{6} - 11\frac{1}{4}} = \frac{45 \times 73 \times 150 \times 12}{4 \times 6 \times 11} = 22,397\tfrac{8}{11}.$$

Therefore, the required distance is $22,397\frac{8}{11}$ feet.

18. A boatman rowing with the tide makes 27 miles in 4 hours. In returning he rows against a tide twice as strong as the first and makes the distance in 9 hours. Find the velocity of the second tide.

By Problem 7, Exercise 68, page 166, $x = \frac{mn(t'-t)}{tt'(m+1)}$.

Here $m = 2,\ n = 27,\ t' = 9,\ t = 4.$

$$\therefore\ x = \frac{2 \times 27(9-4)}{9 \times 4(2+1)} = \frac{2 \times 27 \times 5}{9 \times 4 \times 3} = 2\tfrac{1}{2}.$$

Therefore, the velocity of the second tide is $2\frac{1}{2}$ miles an hour.

19. The population of a city increases each year 2 per cent of that of the preceding year. The population now is 54,121 inhabitants. What was the population 4 years ago?

By Problem 3, Exercise 68, page 166, $x = \frac{100^n a}{(100+t)^n}$.

Here $n = 4,\ a = 54121,\ t = 2.$

$$\therefore\ x = \frac{100^4 \times 54121}{(100+2)^4} = 50,000.$$

Therefore, the population 4 years ago was 50,000.

20. A number is divided by 16, and the sum of the quotient, dividend, and divisor is 50. What is the number?

By Problem 4, Exercise 68, page 166, $x = \frac{d(b-d)}{d+1}$.

Here $$d = 16, \; b = 50.$$

$$\therefore\ x = \frac{16(50-16)}{16+1} = \frac{16 \times 34}{17} = 32.$$

Therefore, the required number is 32.

***21.** Find the time required for the interest at 4 per cent on a sum of money to be equal to the principal.

By Formula 4, $$t = \frac{a-p}{pr}.$$

Here $$a - p = p, \; p = p, \; r = \tfrac{4}{100}.$$

$$\therefore\ t = \frac{p}{p \times \frac{4}{100}} = 25.$$

Therefore, the required time is 25 years.

22. The sum of two numbers is 325 and the difference between them is 93. Find the numbers.

By Problem 1, page 164, the required numbers are $\frac{s+d}{2}$ and $\frac{s-d}{2}$.

Here $$s = 325, \; d = 93.$$

$$\therefore\ \frac{s+d}{2} = \frac{325+93}{2} = 209, \text{ and } \frac{s-d}{2} = \frac{325-93}{2} = 116.$$

Therefore, the required numbers are 209 and 116.

23. Three partners in a restaurant furnish respectively \$500 for 7 months, \$600 for 8 months, and \$900 for 9 months. If they lose \$410, what is each one's share of the loss?

By Problem 2, Exercise 68, page 165,

$$\text{A's share} = \frac{amd}{am+bn+cp}, \; \text{B's} = \frac{bnd}{am+bn+cp}, \; \text{C's} = \frac{cpd}{am+bn+cp}.$$

Here $d = 410$, $a = 500$, $b = 600$, $c = 900$, $m = 7$, $n = 8$, $p = 9$.

$$\therefore\ \frac{amd}{am+bn+cp} = \frac{500 \times 7 \times 410}{500 \times 7 + 600 \times 8 + 900 \times 9} = 87.50,$$

$$\frac{bnd}{am+bn+cp} = \frac{600 \times 8 \times 410}{500 \times 7 + 600 \times 8 + 900 \times 9} = 120.00,$$

and $$\frac{cpd}{am+bn+cp} = \frac{900 \times 9 \times 410}{500 \times 7 + 600 \times 8 + 900 \times 9} = 202.50.$$

Therefore, A's share of the loss is \$87.50; B's, \$120.00; C's, \$202.50.

24. Find the time in which the interest on $8520 will amount to $1746.60 at 6 per cent.

By Formula 4, $t = \dfrac{a-p}{pr}$.

Here $a - p = 1746.60$, $p = 8520$, $r = \frac{6}{100}$.

$$\therefore\ t = \frac{1746.60}{8520 \times \frac{6}{100}} = \frac{1746.60}{511.20} = 3\tfrac{5}{12}.$$

$3\frac{5}{12}$ years = 3 years 5 months.

Therefore, the required time is 3 years 5 months.

25. Find the principal that will produce $1312.65 interest in 2 years 3 months at 6 per cent.

By Formula 2, $p = \dfrac{i}{rt}$.

Here $i = 1312.65$, $r = \frac{6}{100}$, $t = 2\frac{1}{4}$.

$$\therefore\ p = \frac{1312.65}{\frac{6}{100} \times \frac{9}{4}} = \frac{1312.65}{\frac{27}{200}} = \frac{1312.65 \times 200}{27} = 9723\tfrac{1}{3}.$$

Therefore, the principal required is $9723.33.

26. Find the time required for $4000 to amount to $4625 at $5\frac{1}{2}$ per cent.

By Formula 4, $t = \dfrac{a-p}{pr}$.

Here $a = 4625$, $p = 4000$, $r = \frac{55}{1000}$.

$$\therefore\ t = \frac{4625 - 4000}{4000 \times \frac{55}{1000}} = \frac{625}{220} = 2\tfrac{37}{44}.$$

$2\frac{37}{44}$ years = 2 years 10 months 3 days.

Therefore, the required time is 2 years 10 months 3 days.

27. Find the time required for $3904.92 to amount to $4568.76 at 5 per cent.

By Formula 4, $t = \dfrac{a-p}{pr}$.

Here $a = 4568.76$, $p = 3904.92$, $r = \frac{5}{100}$.

$$\therefore\ t = \frac{4568.76 - 3904.92}{3904.92 \times \frac{5}{100}} = \frac{663.84}{195.246} = 3\tfrac{4339}{10847}.$$

$3\frac{4339}{10847}$ years = 3 years 4 months 24 days.

Therefore, the required time is 3 years 4 months 24 days.

28. A and B go into partnership. A furnishes \$5000 for 13 months and B furnishes \$7000 for 9 months. Their profits are \$1700. What is the share of each?

By Problem 2, Exercise 68, page 165,

$$\text{A's share} = \frac{amd}{am+bn} \text{ and B's share} = \frac{bnd}{am+bn}.$$

Here $a = 5000$, $b = 7000$, $m = 13$, $n = 9$, $d = 1700$.

$$\therefore \frac{amd}{am+bn} = \frac{5000 \times 13 \times 1700}{5000 \times 13 + 7000 \times 9} = 863.28,$$

and $$\frac{bnd}{am+bn} = \frac{7000 \times 9 \times 1700}{5000 \times 13 + 7000 \times 9} = 836.72.$$

Therefore, A's share is \$863.28, and B's share is \$836.72.

Exercise 70. Page 172

Eliminate by addition or subtraction.

1. Solve $\left.\begin{aligned}4x + 9y &= 3\\ 3x + 7y &= 2\end{aligned}\right\}$ (1) (2)

Multiply (1) by 3, $12x + 27y = 9.$ (3)

Multiply (2) by 4, $12x + 28y = 8.$ (4)

Subtract (3) from (4), $y = -1.$

Substitute the value of y in (2), $3x - 7 = 2.$

Transpose and combine, $3x = 9.$

$\therefore x = 3.$

Therefore, $x = 3$, and $y = -1$.

2. Solve $\left.\begin{aligned}8x + 3y &= 38\\ 2x - 7y &= -6\end{aligned}\right\}$ (1) (2)

Multiply (2) by 4, $8x - 28y = -24.$ (3)

Subtract (3) from (1), $31y = 62.$

$\therefore y = 2.$

Substitute the value of y in (2), $2x - 14 = -6.$

Transpose and combine, $2x = 8.$

$\therefore x = 4.$

Therefore, $x = 4$, and $y = 2$.

3. Solve
$$\left.\begin{aligned} x + 4y &= 31 \\ 4x + y &= 19 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Add (1) and (2), $5x + 5y = 50.$ (3)

Divide (3) by 5, $x + y = 10.$ (4)

Subtract (4) from (1), $3y = 21.$

$\therefore\ y = 7.$

Subtract (4) from (2), $3x = 9.$

$\therefore\ x = 3.$

Therefore, $x = 3$, and $y = 7$.

4. Solve
$$\left.\begin{aligned} 5x - 3y &= 20 \\ 3x - 4y &= 1 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 4, $20x - 12y = 80.$ (3)

Multiply (2) by 3, $9x - 12y = 3.$ (4)

Subtract (4) from (3), $11x = 77.$

$\therefore\ x = 7.$

Substitute the value of x in (1), $35 - 3y = 20.$

Transpose and combine, $-3y = -15.$

$\therefore\ y = 5.$

Therefore, $x = 7$, and $y = 5$.

5. Solve
$$\left.\begin{aligned} 8x + 4y &= 100 \\ 9x - 5y &= 8 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Divide (1) by 4, $2x + y = 25.$ (3)

Multiply (3) by 5, $10x + 5y = 125.$ (4)

Add (2) and (4), $19x = 133.$

$\therefore\ x = 7.$

Substitute the value of x in (3), $14 + y = 25.$

Transpose and combine, $y = 11.$

Therefore, $x = 7$, and $y = 11$.

6. Solve
$$\left.\begin{aligned} 5x - 2y &= 23 \\ 13x - 3y &= 51 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 3, $15x - 6y = 69.$ (3)

Multiply (2) by 2, $26x - 6y = 102.$ (4)

Subtract (3) from (4), $11x = 33.$

$\therefore\ x = 3.$

Substitute the value of x in (1), $15 - 2y = 23.$

Transpose and combine, $-2y = 8.$

$\therefore\ y = -4.$

Therefore, $x = 3$, and $y = -4$.

7. Solve $\left.\begin{aligned} 6x + 7y &= 31 \\ 3x + 2y &= 5 \end{aligned}\right\}$ $\begin{aligned}&(1)\\&(2)\end{aligned}$

Multiply (2) by 2, $6x + 4y = 10.$ (3)

Subtract (3) from (1), $3y = 21.$

$\therefore y = 7.$

Substitute the value of y in (2), $3x + 14 = 5.$

Transpose and combine, $3x = -9.$

$\therefore x = -3.$

Therefore, $x = -3$, and $y = 7$.

8. Solve $\left.\begin{aligned} 11x - 12y &= 31 \\ 10x + 7y &= 64 \end{aligned}\right\}$ $\begin{aligned}&(1)\\&(2)\end{aligned}$

Multiply (1) by 7, $77x - 84y = 217.$ (3)

Multiply (2) by 12, $120x + 84y = 768.$ (4)

Add (3) and (4), $197x = 985.$

$\therefore x = 5.$

Substitute the value of x in (2), $50 + 7y = 64.$

Transpose and combine, $7y = 14.$

$\therefore y = 2.$

Therefore, $x = 5$, and $y = 2$.

9. Solve $\left.\begin{aligned} 4x - 3y &= 29 \\ 2x + y &= 47 \end{aligned}\right\}$ $\begin{aligned}&(1)\\&(2)\end{aligned}$

Multiply (2) by 3, $6x + 3y = 141.$ (3)

Add (1) and (3), $10x = 170.$

$\therefore x = 17.$

Substitute the value of x in (2), $34 + y = 47.$

Transpose and combine, $y = 13.$

Therefore, $x = 17$, and $y = 13$.

10. Solve $\left.\begin{aligned} 2x - 11y &= 67 \\ 5x - 6y &= 60 \end{aligned}\right\}$ $\begin{aligned}&(1)\\&(2)\end{aligned}$

Multiply (1) by 5, $10x - 55y = 335.$ (3)

Multiply (2) by 2, $10x - 12y = 120.$ (4)

Subtract (3) from (4), $43y = -215.$

$\therefore y = -5.$

Substitute the value of y in (1), $2x + 55 = 67.$

Transpose and combine, $2x = 12.$

$\therefore x = 6.$

Therefore, $x = 6$, and $y = -5$.

11. Solve $$\left.\begin{aligned}7x-3y&=22\\3x+5y&=66\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 5, $35x - 15y = 110.$ (3)

Multiply (2) by 3, $9x + 15y = 198.$ (4)

Add (3) and (4), $44x = 308.$

$\therefore\ x = 7.$

Substitute the value of x in (2), $21 + 5y = 66.$

Transpose and combine, $5y = 45.$

$\therefore\ y = 9.$

Therefore, $x = 7$, and $y = 9$.

12. Solve $$\left.\begin{aligned}2x-3y&=3\\5x+7y&=109\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 5, $10x - 15y = 15.$ (3)

Multiply (2) by 2, $10x + 14y = 218.$ (4)

Subtract (3) from (4), $29y = 203.$

$\therefore\ y = 7.$

Substitute the value of y in (1), $2x - 21 = 3.$

Transpose and combine, $2x = 24.$

$\therefore\ x = 12.$

Therefore, $x = 12$, and $y = 7$.

13. Solve $$\left.\begin{aligned}10x+7y&=75\\12x-5y&=23\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 5, $50x + 35y = 375.$ (3)

Multiply (2) by 7, $84x - 35y = 161.$ (4)

Add (3) and (4), $134x = 536.$

$\therefore\ x = 4.$

Substitute the value of x in (1), $40 + 7y = 75.$

Transpose and combine, $7y = 35.$

$\therefore\ y = 5.$

Therefore, $x = 4$, and $y = 5$.

14. Solve $$\left.\begin{aligned}7x-5y&=1\\9x-10y&=12\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 2, $14x - 10y = 2.$ (3)

Subtract (2) from (3), $5x = -10.$

$\therefore\ x = -2.$

Substitute the value of x in (1), $-14 - 5y = 1.$

Transpose and combine, $-5y = 15.$

$\therefore\ y = -3.$

Therefore, $x = -2$, and $y = -3$.

Exercise 71. Page 174

Eliminate by substitution.

1. Solve
$$\left.\begin{aligned} x+5y&=25\\ 7x+3y&=47\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $x = 25 - 5y.$ (3)

Substitute the value of x in (2), $175 - 35y + 3y = 47.$

Transpose and combine, $-32y = -128.$

$\therefore\ y = 4.$

Substitute the value of y in (3), $x = 25 - 20.$

$\therefore\ x = 5.$

Therefore, $x = 5$, and $y = 4$.

2. Solve
$$\left.\begin{aligned} 3x-2y&=15\\ 5x-9y&=8\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $3x = 15 + 2y.$ (3)

Divide (3) by 3, $x = \dfrac{15+2y}{3}.$ (4)

Substitute the value of x in (2), $\dfrac{75+10y}{3} - 9y = 8.$

Multiply by 3, $75 + 10y - 27y = 24.$

Transpose and combine, $-17y = -51.$

$\therefore\ y = 3.$

Substitute the value of y in (4), $x = \dfrac{15+6}{3}.$

$\therefore\ x = 7.$

Therefore, $x = 7$, and $y = 3$.

3. Solve
$$\left.\begin{aligned} 10x-3y&=116\\ 3x-10y&=53\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (2), $3x = 53 + 10y.$ (3)

Divide (3) by 3, $x = \dfrac{53+10y}{3}.$ (4)

Substitute the value of x in (1), $\dfrac{530+100y}{3} - 3y = 116.$

Multiply by 3, $530 + 100y - 9y = 348.$

Transpose and combine, $91y = -182.$

$\therefore\ y = -2.$

Substitute the value of y in (4), $x = \dfrac{53-20}{3}.$

$\therefore\ x = 11.$

Therefore, $x = 11$, and $y = -2$.

4. Solve
$$\left.\begin{array}{r}2x+y=11\\4x-9y=99\end{array}\right\}\qquad\begin{array}{r}(1)\\(2)\end{array}$$

Transpose (1), $y=11-2x.$ (3)

Substitute the value of y in (2), $4x-99+18x=99.$

Transpose and combine, $22x=198.$

$\therefore x=9.$

Substitute the value of x in (3), $y=11-18.$

$\therefore y=-7.$

Therefore, $x=9$, and $y=-7$.

5. Solve
$$\left.\begin{array}{r}3x-y=16\\5x-12y=-25\end{array}\right\}\qquad\begin{array}{r}(1)\\(2)\end{array}$$

Transpose (1), $y=3x-16.$ (3)

Substitute the value of y in (2), $5x-36x+192=-25.$

Transpose and combine, $-31x=-217.$

$\therefore x=7.$

Substitute the value of x in (3), $y=21-16.$

$\therefore y=5.$

Therefore, $x=7$, and $y=5$.

6. Solve
$$\left.\begin{array}{r}7x-2y=91\\2x+5y=143\end{array}\right\}\qquad\begin{array}{r}(1)\\(2)\end{array}$$

Transpose (1), $2y=7x-91.$ (3)

Divide (3) by 2, $y=\dfrac{7x-91}{2}.$ (4)

Substitute the value of y in (2), $2x+\dfrac{35x-455}{2}=143.$

Multiply by 2, $4x+35x-455=286.$

Transpose and combine, $39x=741.$

$\therefore x=19.$

Substitute the value of x in (4), $y=\dfrac{133-91}{2}.$

$\therefore y=21.$

Therefore, $x=19$, and $y=21$.

7. Solve
$$\left.\begin{array}{r}2x+17y=61\\8x-y=37\end{array}\right\}\qquad\begin{array}{r}(1)\\(2)\end{array}$$

Transpose (1), $2x=61-17y.$ (3)

Divide (3) by 2, $x=\dfrac{61-17y}{2}.$ (4)

Substitute the value of x in (2), $244-68y-y=37.$

Transpose and combine, $-69y=-207.$

$\therefore y=3.$

Substitute the value of y in (4), $x=\dfrac{61-51}{2}.$

$\therefore x=5.$

Therefore, $x=5$, and $y=3$.

8. Solve $$\left.\begin{aligned}9x-14y&=14\\11x+12y&=119\end{aligned}\right\}\quad\begin{matrix}(1)\\(2)\end{matrix}$$

Transpose (1), $$9x = 14 + 14y. \quad (3)$$

Divide (3) by 9, $$x = \frac{14+14y}{9}. \quad (4)$$

Substitute the value of x in (2), $\frac{154+154y}{9} + 12y = 119.$

Multiply by 9, $154 + 154y + 108y = 1071.$

Transpose and combine, $262y = 917.$

$\therefore y = 3\frac{1}{2}.$

Substitute the value of y in (4), $x = \frac{14+49}{9}$

$\therefore x = 7.$

Therefore, $x = 7$, and $y = 3\frac{1}{2}$.

9. Solve $$\left.\begin{aligned}8x+3y&=113\\4x+7y&=29\end{aligned}\right\}\quad\begin{matrix}(1)\\(2)\end{matrix}$$

Transpose (2), $$4x = 29 - 7y. \quad (3)$$

Divide (3) by 4, $$x = \frac{29-7y}{4}. \quad (4)$$

Substitute the value of x in (1), $58 - 14y + 3y = 113.$

Transpose and combine, $-11y = 55.$

$\therefore y = -5.$

Substitute the value of y in (4), $x = \frac{29+35}{4}.$

$\therefore x = 16.$

Therefore, $x = 16$, and $y = -5$.

10. Solve $$\left.\begin{aligned}2x-9y&=6\\5x+4y&=68\end{aligned}\right\}\quad\begin{matrix}(1)\\(2)\end{matrix}$$

Transpose (1), $$2x = 6 + 9y. \quad (3)$$

Divide (3) by 2, $$x = \frac{6+9y}{2}. \quad (4)$$

Substitute the value of x in (2), $\frac{30+45y}{2} + 4y = 68.$

Multiply by 2, $30 + 45y + 8y = 136.$

Transpose and combine, $53y = 106.$

$\therefore y = 2.$

Substitute the value of y in (4), $x = \frac{6+18}{2}.$

$\therefore x = 12.$

Therefore, $x = 12$, and $y = 2$.

11. Solve $\left.\begin{aligned} 3y - 2x &= 30 \\ 2y - 5x &= 42 \end{aligned}\right\}$ (1) (2)

Transpose (2), $2y = 42 + 5x.$ (3)

Divide (3) by 2, $y = \frac{42 + 5x}{2}.$ (4)

Substitute the value of y in (1), $\frac{126 + 15x}{2} - 2x = 30.$

Multiply by 2, $126 + 15x - 4x = 60.$

Transpose and combine, $11x = -66.$

$\therefore\ x = -6.$

Substitute the value of x in (4), $y = \frac{42 - 30}{2}.$

$\therefore\ y = 6.$

Therefore, $x = -6$, and $y = 6$.

12. Solve $\left.\begin{aligned} 5x + 7y &= 125 \\ 7x - \ \ y &= \ \ 13 \end{aligned}\right\}$ (1) (2)

Transpose (2), $y = 7x - 13.$ (3)

Substitute the value of y in (1), $5x + 49x - 91 = 125.$

Transpose and combine, $54x = 216.$

$\therefore\ x = 4.$

Substitute the value of x in (3), $y = 28 - 13.$

$\therefore\ y = 15.$

Therefore, $x = 4$, and $y = 15$.

13. Solve $\left.\begin{aligned} 3x + 8y &= 71 \\ 2x + 3y &= 31 \end{aligned}\right\}$ (1) (2)

Transpose (2), $2x = 31 - 3y.$ (3)

Divide (3) by 2, $x = \frac{31 - 3y}{2}.$ (4)

Substitute the value of x in (1), $\frac{93 - 9y}{2} + 8y = 71.$

Multiply by 2, $93 - 9y + 16y = 142.$

Transpose and combine, $7y = 49.$

$\therefore\ y = 7.$

Substitute the value of y in (4), $x = \frac{31 - 21}{2}.$

$\therefore\ x = 5.$

Therefore, $x = 5$, and $y = 7$.

14. Solve
$$\left.\begin{aligned}5x-3y&=27\\7y-3x&=15\end{aligned}\right\}\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $5x = 27 + 3y.$ (3)

Divide (3) by 5, $x = \frac{27+3y}{5}.$ (4)

Substitute the value of x in (2), $7y - \frac{81+9y}{5} = 15.$

Multiply by 5, $35y - 81 - 9y = 75.$

Transpose and combine, $26y = 156.$

$\therefore\ y = 6.$

Substitute the value of y in (4), $x = \frac{27+18}{5}.$

$\therefore\ x = 9.$

Therefore, $x = 9$, and $y = 6$.

15. Solve
$$\left.\begin{aligned}7x-4y&=26\\12x+7y&=197\end{aligned}\right\}\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $4y = 7x - 26.$ (3)

Divide (3) by 4, $y = \frac{7x-26}{4}.$ (4)

Substitute the value of y in (2), $12x + \frac{49x-182}{4} = 197.$

Multiply by 4, $48x + 49x - 182 = 788.$

Transpose and combine, $97x = 970.$

$\therefore\ x = 10.$

Substitute the value of x in (4), $y = \frac{70-26}{4}.$

$\therefore\ y = 11.$

Therefore, $x = 10$, and $y = 11$.

16. Solve
$$\left.\begin{aligned}19x-5y&=36\\12x+25y&=248\end{aligned}\right\}\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $5y = 19x - 36.$ (3)

Divide (3) by 5, $y = \frac{19x-36}{5}.$ (4)

Substitute the value of y in (2), $12x + 95x - 180 = 248.$

Transpose and combine, $107x = 428.$

$\therefore\ x = 4.$

Substitute the value of x in (4), $y = \frac{76-36}{5}$

$\therefore\ y = 8.$

Therefore, $x = 4$, and $y = 8$.

Exercise 72. Page 175

Eliminate by comparison.

1. Solve
$$\left.\begin{aligned} 2x - 5y &= 8 \\ 3x - 7y &= 13 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $2x = 8 + 5y.$ (3)

Transpose (2), $3x = 13 + 7y.$ (4)

Divide (3) by 2, $x = \dfrac{8 + 5y}{2}.$ (5)

Divide (4) by 3, $x = \dfrac{13 + 7y}{3}.$ (6)

Equate the values of x, $\dfrac{8 + 5y}{2} = \dfrac{13 + 7y}{3}.$

Multiply by 6, $24 + 15y = 26 + 14y.$

Transpose and combine, $y = 2.$

Substitute the value of y in (5), $x = \dfrac{8 + 10}{2}$

$\therefore x = 9.$

Therefore, $x = 9$, and $y = 2$.

2. Solve
$$\left.\begin{aligned} 9x - 2y &= 69 \\ 7x - 3y &= 32 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $2y = 9x - 69.$ (3)

Transpose (2), $3y = 7x - 32.$ (4)

Divide (3) by 2, $y = \dfrac{9x - 69}{2}.$ (5)

Divide (4) by 3, $y = \dfrac{7x - 32}{3}.$ (6)

Equate the values of y, $\dfrac{9x - 69}{2} = \dfrac{7x - 32}{3}.$

Multiply by 6, $27x - 207 = 14x - 64.$

Transpose and combine, $13x = 143.$

$\therefore x = 11.$

Substitute the value of x in (5), $y = \dfrac{99 - 69}{2}.$

$\therefore y = 15.$

Therefore, $x = 11$, and $y = 15$.

3. Solve
$$\left.\begin{aligned} 5x + 7y &= 70 \\ 3x + 8y &= 61 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $5x = 70 - 7y.$ (3)

Transpose (2), $3x = 61 - 8y.$ (4)

Divide (3) by 5, $x = \frac{70 - 7y}{5}.$ (5)

Divide (4) by 3, $x = \frac{61 - 8y}{3}.$ (6)

Equate the values of x, $\frac{70 - 7y}{5} = \frac{61 - 8y}{3}.$

Multiply by 15, $210 - 21y = 305 - 40y.$

Transpose and combine, $19y = 95.$

$\therefore y = 5.$

Substitute the value of y in (5), $x = \frac{70 - 35}{5}.$

$\therefore x = 7.$

Therefore, $x = 7$, and $y = 5$.

4. Solve

$$\left.\begin{aligned} 4x - 11y &= -32 \quad (1) \\ 8x + 9y &= 60 \quad (2) \end{aligned}\right\}$$

Transpose (1), $4x = 11y - 32.$ (3)

Transpose (2), $8x = 60 - 9y.$ (4)

Divide (3) by 4, $x = \frac{11y - 32}{4}.$ (5)

Divide (4) by 8, $x = \frac{60 - 9y}{8}.$ (6)

Equate the values of x, $\frac{11y - 32}{4} = \frac{60 - 9y}{8}.$

Multiply by 8, $22y - 64 = 60 - 9y.$

Transpose and combine, $31y = 124.$

$\therefore y = 4.$

Substitute the value of y in (5), $x = \frac{44 - 32}{4}.$

$\therefore x = 3.$

Therefore, $x = 3$, and $y = 4$.

5. Solve

$$\left.\begin{aligned} 15x + 4y &= 3 \quad (1) \\ 10x - 7y &= 31 \quad (2) \end{aligned}\right\}$$

Transpose (1), $15x = 3 - 4y.$ (3)

Transpose (2), $10x = 31 + 7y.$ (4)

Divide (3) by 15, $x = \frac{3 - 4y}{15}.$ (5)

Divide (4) by 10, $x = \frac{31 + 7y}{10}.$ (6)

Equate the values of x, $\frac{31+7y}{10}=\frac{3-4y}{15}.$

Multiply by 30, $93+21y=6-8y.$

Transpose and combine, $29y=-87.$

$\therefore\ y=-3.$

Substitute the value of y in (6), $x=\frac{31-21}{10}.$

$\therefore\ x=1.$

Therefore, $x=1$, and $y=-3$.

6. Solve
$$\left.\begin{aligned}6x+7y&=-80\\9x-5y&=4\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $6x=-80-7y.$ (3)

Transpose (2), $9x=5y+4.$ (4)

Divide (3) by 6, $x=\frac{-80-7y}{6}.$ (5)

Divide (4) by 9, $x=\frac{5y+4}{9}.$ (6)

Equate the values of x, $\frac{5y+4}{9}=\frac{-80-7y}{6}.$

Multiply by 18, $10y+8=-240-21y.$

Transpose and combine, $31y=-248.$

$\therefore\ y=-8.$

Substitute the value of y in (6), $x=\frac{-40+4}{9}.$

$\therefore\ x=-4.$

Therefore, $x=-4$, and $y=-8$.

7. Solve
$$\left.\begin{aligned}17x-6y&=43\\13x-9y&=2\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $6y=17x-43.$ (3)

Transpose (2), $9y=13x-2.$ (4)

Divide (3) by 6, $y=\frac{17x-43}{6}.$ (5)

Divide (4) by 9, $y=\frac{13x-2}{9}.$ (6)

Equate the values of y, $\frac{17x-43}{6}=\frac{13x-2}{9}.$

Multiply by 18, $51x-129=26x-4.$

Transpose and combine, $25x=125.$

$\therefore\ x=5.$

Substitute the value of x in (6), $y = \dfrac{65 - 2}{9}.$

$\therefore\ y = 7.$

Therefore, $x = 5$, and $y = 7$.

8. Solve $\left.\begin{aligned} 11x + 4y &= 56 \\ 5x - 6y &= 2 \end{aligned}\right\}$ (1) (2)

Transpose (1), $4y = 56 - 11x.$ (3)

Transpose (2), $6y = 5x - 2.$ (4)

Divide (3) by 4, $y = \dfrac{56 - 11x}{4}.$ (5)

Divide (4) by 6, $y = \dfrac{5x - 2}{6}.$ (6)

Equate the values of y, $\dfrac{5x - 2}{6} = \dfrac{56 - 11x}{4}.$

Multiply by 12, $10x - 4 = 168 - 33x.$

Transpose and combine, $43x = 172.$

$\therefore\ x = 4.$

Substitute the value of x in (6), $y = \dfrac{20 - 2}{6}.$

$\therefore\ y = 3.$

Therefore, $x = 4$, and $y = 3$.

9. Solve $\left.\begin{aligned} 14x - 17y &= 38 \\ 21x - 19y &= 96 \end{aligned}\right\}$ (1) (2)

Transpose (1), $14x = 17y + 38.$ (3)

Transpose (2), $21x = 19y + 96.$ (4)

Divide (3) by 14, $x = \dfrac{17y + 38}{14}.$ (5)

Divide (4) by 21, $x = \dfrac{19y + 96}{21}.$ (6)

Equate the values of x, $\dfrac{17y + 38}{14} = \dfrac{19y + 96}{21}.$

Multiply by 42, $51y + 114 = 38y + 192.$

Transpose and combine, $13y = 78.$

$\therefore\ y = 6.$

Substitute the value of y in (5), $x = \dfrac{102 + 38}{14}.$

$\therefore\ x = 10.$

Therefore, $x = 10$, and $y = 6$.

10. Solve $$\left.\begin{aligned}9x-15y&=30\\14x+25y&=385\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Transpose (1), $15y=9x-30.$ (3)

Transpose (2), $25y=385-14x.$ (4)

Divide (3) by 15, $y=\dfrac{3x-10}{5}.$ (5)

Divide (4) by 25, $y=\dfrac{385-14x}{25}.$ (6)

Equate the values of y, $\dfrac{3x-10}{5}=\dfrac{385-14x}{25}.$

Multiply by 25, $15x-50=385-14x.$

Transpose and combine, $29x=435.$

$\therefore x=15.$

Substitute the value of x in (5), $y=\dfrac{45-10}{5}.$

$\therefore y=7.$

Therefore, $x=15$, and $y=7$.

11. Solve $$\left.\begin{aligned}12x-7y&=59\\9x-13y&=83\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Transpose (1), $12x=59+7y.$ (3)

Transpose (2), $9x=83+13y.$ (4)

Divide (3) by 12, $x=\dfrac{59+7y}{12}.$ (5)

Divide (4) by 9, $x=\dfrac{83+13y}{9}.$ (6)

Equate the values of x, $\dfrac{83+13y}{9}=\dfrac{59+7y}{12}.$

Multiply by 36, $332+52y=177+21y.$

Transpose and combine, $31y=-155.$

$\therefore y=-5.$

Substitute the value of y in (6), $x=\dfrac{83-65}{9}.$

$\therefore x=2.$

Therefore, $x=2$, and $y=-5$.

12. Solve $$\left.\begin{aligned}4x+9y&=56\\6x+5y&=50\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Transpose (1), $4x=56-9y.$ (3)

Transpose (2), $6x=50-5y.$ (4)

Divide (3) by 4, $x = \frac{56 - 9y}{4}$. (5)

Divide (4) by 6, $x = \frac{50 - 5y}{6}$. (6)

Equate the values of x, $\frac{50 - 5y}{6} = \frac{56 - 9y}{4}$.

Multiply by 12, $100 - 10y = 168 - 27y$.

Transpose and combine, $17y = 68$.

$\therefore y = 4$.

Substitute the value of y in (5), $x = \frac{56 - 36}{4}$.

$\therefore x = 5$.

Therefore, $x = 5$, and $y = 4$.

13. Solve

$$\left.\begin{aligned} 18x + 13y &= 74 \quad (1)\\ 12x - 7y &= 112 \quad (2) \end{aligned}\right\}$$

Transpose (1), $18x = 74 - 13y$. (3)

Transpose (2), $12x = 112 + 7y$. (4)

Divide (3) by 18, $x = \frac{74 - 13y}{18}$. (5)

Divide (4) by 12, $x = \frac{112 + 7y}{12}$. (6)

Equate the values of x, $\frac{112 + 7y}{12} = \frac{74 - 13y}{18}$.

Multiply by 36, $336 + 21y = 148 - 26y$.

Transpose and combine, $47y = -188$.

$\therefore y = -4$.

Substitute the value of y in (6), $x = \frac{112 - 28}{12}$.

$\therefore x = 7$.

Therefore, $x = 7$, and $y = -4$.

14. Solve

$$\left.\begin{aligned} 16x - 5y &= 40 \quad (1)\\ 12x + 7y &= 116 \quad (2) \end{aligned}\right\}$$

Transpose (1), $16x = 40 + 5y$. (3)

Transpose (2), $12x = 116 - 7y$. (4)

Divide (3) by 16, $x = \frac{40 + 5y}{16}$. (5)

Divide (4) by 12, $x = \frac{116 - 7y}{12}$. (6)

Equate the values of x, $\frac{40 + 5y}{16} = \frac{116 - 7y}{12}$.

Multiply by 48, $120 + 15y = 464 - 28y.$

Transpose and combine, $43y = 344.$

$\therefore y = 8.$

Substitute the value of y in (6), $x = \frac{116 - 56}{12}.$

$\therefore x = 5.$

Therefore, $x = 5$, and $y = 8$.

15. Solve

$$\left.\begin{aligned} 3x + 14y &= 125 \\ 19x - 21y &= 24 \end{aligned}\right\}. \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Transpose (1), $14y = 125 - 3x.$ (3)

Transpose (2), $21y = 19x - 24.$ (4)

Divide (3) by 14, $y = \frac{125 - 3x}{14}.$ (5)

Divide (4) by 21, $y = \frac{19x - 24}{21}.$ (6)

Equate the values of y, $\frac{19x - 24}{21} = \frac{125 - 3x}{14}.$

Multiply by 42, $38x - 48 = 375 - 9x.$

Transpose and combine, $47x = 423.$

$\therefore x = 9.$

Substitute the value of x in (5), $y = \frac{125 - 27}{14}.$

$\therefore y = 7.$

Therefore, $x = 9$, and $y = 7$.

16. Solve

$$\left.\begin{aligned} 41x + 33y &= 42 \\ 31x + 22y &= 17 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Transpose (1), $33y = 42 - 41x.$ (3)

Transpose (2), $22y = 17 - 31x.$ (4)

Divide (3) by 33, $y = \frac{42 - 41x}{33}.$ (5)

Divide (4) by 22, $y = \frac{17 - 31x}{22}.$ (6)

Equate the values of y, $\frac{42 - 41x}{33} = \frac{17 - 31x}{22}.$

Multiply by 66, $84 - 82x = 51 - 93x.$

Transpose and combine, $11x = -33.$

$\therefore x = -3.$

Substitute the value of x in (6), $y = \frac{17 + 93}{22}.$

$\therefore y = 5.$

Therefore, $x = -3$, and $y = 5$.

Exercise 73. Page 176

1. Solve

$$\left.\begin{aligned} \frac{x}{4}+\frac{y}{6}&=\frac{7}{2} \qquad (1)\\ \frac{x}{3}-\frac{y}{8}&=\frac{1}{2} \qquad (2)\end{aligned}\right\}$$

Multiply (1) by 12, $3x + 2y = 42.$ (3)

Multiply (2) by 24, $8x - 3y = 12.$ (4)

Multiply (3) by 3, $9x + 6y = 126.$ (5)

Multiply (4) by 2, $16x - 6y = 24.$ (6)

Add (5) and (6), $25x = 150.$

$\therefore x = 6.$

Substitute the value of x in (3), $18 + 2y = 42.$

Transpose and combine, $2y = 24.$

$\therefore y = 12.$

Therefore, $x = 6$, and $y = 12$.

2. Solve

$$\left.\begin{aligned} \frac{x}{4}+\frac{y}{4}&=14 \qquad (1)\\ \frac{x}{3}-\frac{y}{6}&=\frac{7}{6} \qquad (2)\end{aligned}\right\}$$

Multiply (1) by 4, $x + y = 56.$ (3)

Multiply (2) by 6, $2x - y = 7.$ (4)

Add (3) and (4), $3x = 63.$

$\therefore x = 21.$

Substitute the value of x in (3), $21 + y = 56.$

Transpose and combine, $y = 35.$

Therefore, $x = 21$, and $y = 35$.

3. Solve

$$\left.\begin{aligned} \frac{x-4}{x-3}&=\frac{y+4}{y+7} \qquad (1)\\ \frac{x+5}{x+2}&=\frac{y-1}{y-2} \qquad (2)\end{aligned}\right\}$$

Multiply (1) by $(x-3)(y+7)$,

$$xy + 7x - 4y - 28 = xy + 4x - 3y - 12. \qquad (3)$$

Multiply (2) by $(x+2)(y-2)$,

$$xy - 2x + 5y - 10 = xy - x + 2y - 2. \qquad (4)$$

Simplify (3), $3x - y = 16.$ (5)

Simplify (4), $x - 3y = -8.$ (6)

Subtract (6) from (5), $2x + 2y = 24.$ (7)

Divide (7) by 2, $x + y = 12.$ (8)

Add (5) and (8), $4x = 28.$

$$\therefore\ x = 7.$$

Subtract (6) from (8), $4y = 20.$

$$\therefore\ y = 5.$$

Therefore, $x = 7$, and $y = 5$.

4. Solve

$$\left.\begin{aligned} \frac{2(5-11x)}{11(x-1)} + \frac{11-7y}{3-y} &= 5 \qquad (1)\\ \frac{7+2x}{3-x} - \frac{125-144y}{36(y+5)} &= 2 \qquad (2)\end{aligned}\right\}$$

Multiply (1) by $11(x-1)(3-y)$,

$$30 - 66x - 10y + 22xy + 121x - 77xy - 121 + 77y = 165x - 165 - 55xy + 55y. \tag{3}$$

Multiply (2) by $36(3-x)(y+5)$,

$$252y + 72xy + 1260 + 360x - 375 + 432y + 125x - 144xy = 216y + 1080 - 72xy - 360x. \tag{4}$$

Simplify (3), $55x - 6y = 37.$ (5)

Simplify (4), $65x + 36y = 15.$ (6)

Multiply (5) by 6, $330x - 36y = 222.$ (7)

Add (6) and (7), $395x = 237.$

$$\therefore\ x = \tfrac{237}{395} = \tfrac{3}{5}.$$

Substitute the value of x in (5), $33 - 6y = 37.$

Transpose and combine, $6y = -4.$

$$\therefore\ y = -\tfrac{2}{3}.$$

Therefore, $x = \frac{3}{5}$, and $y = -\frac{2}{3}$.

5. Solve

$$\left.\begin{aligned} x - \frac{2y-x}{23-x} &= 20 + \frac{2x-59}{2} \qquad (1)\\ y - \frac{y-3}{x-18} &= 30 - \frac{73-3y}{3} \qquad (2)\end{aligned}\right\}$$

Transpose (1), $x - 20 - \dfrac{2x-59}{2} = \dfrac{2y-x}{23-x}.$ (3)

Transpose (2), $y - 30 + \dfrac{73-3y}{3} = \dfrac{y-3}{x-18}.$ (4)

Simplify (3), $\dfrac{19}{2} = \dfrac{2y-x}{23-x}.$ (5)

Simplify (4), $-\dfrac{17}{3} = \dfrac{y-3}{x-18}.$ (6)

Multiply (5) by $2(23-x)$, $437 - 19x = 4y - 2x.$ (7)

Multiply (6) by $3(x-18)$, $-17x + 306 = 3y - 9.$ (8)

Simplify (7), $17x + 4y = 437.$ (9)

Simplify (8), $17x + 3y = 315.$ (10)

Subtract (10) from (9), $y = 122.$

Substitute the value of y in (10),

$$17x + 366 = 315.$$

Transpose and combine, $17x = -51.$

$$\therefore x = -3.$$

Therefore, $x = -3$, and $y = 122$.

6. Solve

$$\left.\begin{aligned} \frac{7-6x}{10y-19} &= \frac{4-3x}{5y-11} \qquad (1)\\ \frac{6x-10y-17}{3x-5y+2} &= \frac{4x-14y-5}{2x-7y+12} \qquad (2) \end{aligned}\right\}$$

Multiply (1) by $(10y-19)(5y-11)$,

$$35y - 77 - 30xy + 66x = 40y - 76 - 30xy + 57x. \qquad (3)$$

Multiply (2) by $(3x-5y+2)(2x-7y+12)$,

$$12x^2 - 62xy + 38x + 70y^2 - y - 204$$
$$= 12x^2 - 62xy - 7x + 70y^2 - 3y - 10. \qquad (4)$$

Simplify (3), $9x - 5y = 1.$ (5)

Simplify (4), $45x + 2y = 194.$ (6)

Multiply (5) by 5, $45x - 25y = 5.$ (7)

Subtract (7) from (6), $27y = 189.$

$$\therefore y = 7.$$

Substitute the value of y in (5), $9x - 35 = 1.$

Transpose and combine, $9x = 36.$

$$\therefore x = 4.$$

Therefore, $x = 4$, and $y = 7$.

7. Solve

$$\left.\begin{aligned} \frac{x+y}{3} + x &= 15 \qquad (1)\\ \frac{x-y}{5} + y &= 6 \qquad (2) \end{aligned}\right\}$$

Multiply (1) by 3, $x + y + 3x = 45.$ (3)

Multiply (2) by 5, $x - y + 5y = 30.$ (4)

Simplify (3), $4x + y = 45.$ (5)

Simplify (4), $x + 4y = 30.$ (6)

Add (5) and (6), $5x + 5y = 75.$ (7)

Divide (7) by 5, $x + y = 15.$ (8)

Subtract (8) from (5), $3x = 30.$

$$\therefore x = 10.$$

Subtract (8) from (6), $3y = 15.$

$$\therefore y = 5.$$

Therefore, $x = 10$, and $y = 5$.

8. Solve
$$\left.\begin{aligned}\frac{5}{x+2y}&=\frac{7}{2x+y}\\[4pt] \frac{7}{3x-2}&=\frac{5}{6-y}\end{aligned}\right\}\qquad\begin{aligned}&(1)\\[4pt]&(2)\end{aligned}$$

Multiply (1) by $(x+2y)(2x+y)$, $10x+5y=7x+14y$. (3)

Multiply (2) by $(3x-2)(6-y)$, $42-7y=15x-10$. (4)

Simplify (3), $x=3y$. (5)

Simplify (4), $15x+7y=52$. (6)

Substitute in (6) the value of x from (5),

$$45y+7y=52.$$

Combine, $52y=52$.

$$\therefore\ y=1.$$

Substitute the value of y in (5), $x=3$.

Therefore, $x=3$, and $y=1$.

9. Solve
$$\left.\begin{aligned}\frac{3x-2}{5x-1}&=\frac{3y+7}{5y+16}\\[4pt] \frac{3x-1}{x+5}&=\frac{6y-5}{2y+3}\end{aligned}\right\}\qquad\begin{aligned}&(1)\\[4pt]&(2)\end{aligned}$$

Multiply (1) by $(5x-1)(5y+16)$,

$$15xy-10y+48x-32=15xy+35x-3y-7. \qquad (3)$$

Multiply (2) by $(x+5)(2y+3)$,

$$6xy+9x-2y-3=6xy+30y-5x-25. \qquad (4)$$

Simplify (3), $13x-7y=25$. (5)

Simplify (4), $7x-16y=-11$. (6)

Multiply (5) by 7, $91x-49y=175$. (7)

Multiply (6) by 13, $91x-208y=-143$. (8)

Subtract (8) from (7), $159y=318$.

$$\therefore\ y=2.$$

Substitute the value of y in (6), $7x-32=-11$.

Transpose and combine, $7x=21$.

$$\therefore\ x=3.$$

Therefore, $x=3$, and $y=2$.

10. Solve
$$\left.\begin{aligned}\frac{x+y}{8}+\frac{x-y}{6}&=5\\[4pt] \frac{x+y}{4}-\frac{x-y}{3}&=10\end{aligned}\right\}\qquad\begin{aligned}&(1)\\[4pt]&(2)\end{aligned}$$

Multiply (1) by 24, $3x+3y+4x-4y=120$. (3)

Multiply (2) by 12, $3x+3y-4x+4y=120$. (4)

Simplify (3), $7x - y = 120.$ (5)
Simplify (4), $7y - x = 120.$ (6)
Add (5) and (6), $6x + 6y = 240.$ (7)
Divide (7) by 6, $x + y = 40.$ (8)
Add (5) and (8), $8x = 160.$
$\therefore x = 20.$
Add (6) and (8), $8y = 160.$
$\therefore y = 20.$

Therefore, $x = 20$, and $y = 20$.

11. Solve

$$\left.\begin{array}{l}\dfrac{x+3y+13}{4x+5y-25}=3 \qquad (1)\\[2ex] \dfrac{8x+y+6}{5x+3y-23}=5 \qquad (2)\end{array}\right\}$$

Multiply (1) by $4x + 5y - 25$, $x + 3y + 13 = 12x + 15y - 75.$ (3)
Multiply (2) by $5x + 3y - 23$, $8x + y + 6 = 25x + 15y - 115.$ (4)
Simplify (3), $11x + 12y = 88.$ (5)
Simplify (4), $17x + 14y = 121.$ (6)
Multiply (5) by 7, $77x + 84y = 616.$ (7)
Multiply (6) by 6, $102x + 84y = 726.$ (8)
Subtract (7) from (8), $25x = 110.$
$\therefore x = 4\frac{2}{5}.$
Substitute the value of x in (5), $\frac{242}{5} + 12y = 88.$
Multiply by 5, $242 + 60y = 440.$
Transpose and combine, $60y = 198.$
$\therefore y = 3\frac{3}{10}.$

Therefore, $x = 4\frac{2}{5}$, and $y = 3\frac{3}{10}$.

12. Solve

$$\left.\begin{array}{l}\dfrac{9x-7y+73}{13x-15y+17}=2 \qquad (1)\\[2ex] \dfrac{12x-2y+89}{13x-15y+17}=3 \qquad (2)\end{array}\right\}$$

Multiply (1) by $13x - 15y + 17$,
$9x - 7y + 73 = 26x - 30y + 34.$ (3)
Multiply (2) by $13x - 15y + 17$,
$12x - 2y + 89 = 39x - 45y + 51.$ (4)
Simplify (3), $17x - 23y = 39.$ (5)
Simplify (4), $27x - 43y = 38.$ (6)
Multiply (5) by 27, $459x - 621y = 1053.$ (7)
Multiply (6) by 17, $459x - 731y = 646.$ (8)
Subtract (8) from (7), $110y = 407.$

$$\therefore\ y = 3\tfrac{7}{10}.$$

Substitute the value of y in (5), $17x - \frac{851}{10} = 39.$

Multiply by 10, $170x - 851 = 390.$

Transpose and combine, $170x = 1241.$

$$\therefore\ x = 7\tfrac{3}{10}.$$

Therefore, $x = 7\frac{3}{10}$, and $y = 3\frac{7}{10}$.

13. Solve

$$\left.\begin{aligned} 2 + \frac{5x - 6y}{13} &= 4y - 3x \qquad (1)\\ 12 + \frac{5x - 6y}{6} &= 2y + \frac{3x - 2y}{4} \qquad (2)\end{aligned}\right\}$$

Multiply (1) by 13, $26 + 5x - 6y = 52y - 39x.$ (3)

Multiply (2) by 12, $144 + 10x - 12y = 24y + 9x - 6y.$ (4)

Simplify (3), $22x - 29y = -13.$ (5)

Simplify (4), $x - 30y = -144.$ (6)

Multiply (6) by 22, $22x - 660y = -3168.$ (7)

Subtract (7) from (5), $631y = 3155.$

$$\therefore\ y = 5.$$

Substitute the value of y in (6), $x - 150 = -144.$

Transpose and combine, $x = 6.$

Therefore, $x = 6$, and $y = 5$.

14. Solve

$$\left.\begin{aligned} \frac{3x - 4y + 2}{3} + \frac{7x - 2y + 7}{5} &= 3 \qquad (1)\\ \frac{6x + y - 7}{11} + \frac{5x + 3y + 4}{13} &= 5 \qquad (2)\end{aligned}\right\}$$

Multiply (1) by 15,

$$15x - 20y + 10 + 21x - 6y + 21 = 45. \qquad (3)$$

Multiply (2) by 143,

$$78x + 13y - 91 + 55x + 33y + 44 = 715. \qquad (4)$$

Simplify (3), $18x - 13y = 7.$ (5)

Simplify (4), $133x + 46y = 762.$ (6)

Multiply (5) by 46, $828x - 598y = 322.$ (7)

Multiply (6) by 13, $1729x + 598y = 9906.$ (8)

Add (7) and (8), $2557x = 10228.$

$$\therefore\ x = 4.$$

Substitute the value of x in (5), $72 - 13y = 7.$

Transpose and combine, $13y = 65.$

$$\therefore\ y = 5.$$

Therefore, $x = 4$, and $y = 5$.

15. Solve
$$\left.\begin{aligned}\frac{6y+5}{8}-\frac{4x-5y+5}{4x-2y}&=\frac{9y-4}{12}\\ \frac{8x+3}{4}-\frac{x-9y}{3-x}&=2x-2\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 24,
$$18y+15-\frac{48x-60y+60}{2x-y}=18y-8. \qquad (3)$$

Multiply (2) by 4, $8x+3-\dfrac{4x-36y}{3-x}=8x-8.$ (4)

Simplify (3), $23=\dfrac{48x-60y+60}{2x-y}.$ (5)

Simplify (4), $11=\dfrac{4x-36y}{3-x}.$ (6)

Multiply (5) by $2x-y$, $46x-23y=48x-60y+60.$ (7)

Multiply (6) by $3-x$, $33-11x=4x-36y.$ (8)

Simplify (7), $2x-37y=-60.$ (9)

Simplify (8), $5x-12y=11.$ (10)

Multiply (9) by 5, $10x-185y=-300.$ (11)

Multiply (10) by 2, $10x-24y=22.$ (12)

Subtract (11) from (12), $161y=322.$

$\therefore\ y=2.$

Substitute the value of y in (9), $2x-74=-60.$

Transpose and combine, $2x=14.$

$\therefore\ x=7.$

Therefore, $x=7$, and $y=2$.

16. Solve
$$\left.\begin{aligned}8y-\frac{4(4+15y)}{3x-1}&=\frac{16xy-107}{2x+5}\\ 2+6x+9y&=\frac{27y^2-12x^2+38}{3y-2x+1}\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $8y-\dfrac{16xy-107}{2x+5}=\dfrac{4(4+15y)}{3x-1}.$ (3)

Simplify the left member of (3), $\dfrac{40y+107}{2x+5}=\dfrac{4(4+15y)}{3x-1}.$ (4)

Multiply (4) by $(2x+5)(3x-1)$,
$$120xy+321x-40y-107=32x+120xy+80+300y. \qquad (5)$$

Multiply (2) by $3y-2x+1$,
$$2+2x+15y-12x^2+27y^2=27y^2-12x^2+38. \qquad (6)$$

Simplify (5), $17x-20y=11.$ (7)

Simplify (6), $2x+15y=36.$ (8)

Multiply (7) by 3, $51x-60y=33.$ (9)

Multiply (8) by 4, $8x+60y=144.$ (10)

Add (9) and (10), $59x = 177.$

$\therefore\ x = 3.$

Substitute the value of x in (8), $6 + 15y = 36.$

Transpose and combine, $15y = 30.$

$\therefore\ y = 2.$

Therefore, $x = 3$, and $y = 2$.

17. Solve
$$\left.\begin{aligned} x - \frac{2xy}{2y+5} &= \frac{15x+4y}{6y-2x} + \frac{5x^2+4y^2+105}{(x-3y)(2y+5)} \qquad (1)\\ 3 - \frac{7x+2y}{5x} &= 5 - \frac{5y+9}{3x} \qquad (2)\end{aligned}\right\}$$

Simplify each member of (1),
$$\frac{2xy+5x-2xy}{2y+5} = \frac{10x^2+8y^2+210-30xy-8y^2-75x-20y}{2(x-3y)(2y+5)}. \tag{3}$$

Simplify (3), $$\frac{5x}{2y+5} = \frac{10x^2-30xy-75x-20y+210}{2(x-3y)(2y+5)}. \tag{4}$$

Multiply (4) by $2(x-3y)(2y+5)$,
$$10x^2 - 30xy = 10x^2 - 30xy - 75x - 20y + 210. \tag{5}$$

Simplify (5), $15x + 4y = 42.$ (6)

Multiply (2) by $15x$, $45x - 21x - 6y = 75x - 25y - 45.$ (7)

Simplify (7), $51x - 19y = 45.$ (8)

Multiply (6) by 19, $285x + 76y = 798.$ (9)

Multiply (8) by 4, $204x - 76y = 180.$ (10)

Add (9) and (10), $489x = 978.$

$\therefore\ x = 2.$

Substitute the value of x in (6), $30 + 4y = 42.$

Transpose and combine, $4y = 12.$

$\therefore\ y = 3.$

Therefore, $x = 2$, and $y = 3$.

Exercise 74. Page 178

1. Solve
$$\left.\begin{aligned} x + y &= m \qquad (1)\\ x - y &= n \qquad (2)\end{aligned}\right\}$$

Add (1) and (2), $2x = m + n.$

$$\therefore\ x = \frac{m+n}{2}.$$

Subtract (2) from (1), $2y = m - n.$

$$\therefore\ y = \frac{m-n}{2}.$$

Therefore, $x = \dfrac{m+n}{2}$, and $y = \dfrac{m-n}{2}$.

2. Solve $$\left.\begin{aligned} x + y &= 5a - 4b \quad (1)\\ x - y &= 4a - 5b \quad (2)\end{aligned}\right\}$$

Add (1) and (2), $2x = 9a - 9b.$

$\therefore\ x = \dfrac{9a - 9b}{2}.$

Subtract (2) from (1), $2y = a + b.$

$\therefore\ y = \dfrac{a + b}{2}.$

Therefore, $x = \dfrac{9a - 9b}{2}$, and $y = \dfrac{a + b}{2}.$

3. Solve $$\left.\begin{aligned} ax - by &= c \quad (1)\\ a'x + b'y &= c' \quad (2)\end{aligned}\right\}$$

Multiply (1) by a', $aa'x - a'by = a'c.$ (3)

Multiply (2) by a, $aa'x + ab'y = ac'.$ (4)

Subtract (3) from (4), $ab'y + a'by = ac' - a'c.$

Divide by $ab' + a'b$, $y = \dfrac{ac' - a'c}{ab' + a'b}.$

Multiply (1) by b', $ab'x - bb'y = b'c.$ (5)

Multiply (2) by b, $a'bx + bb'y = bc'.$ (6)

Add (5) and (6), $ab'x + a'bx = bc' + b'c.$

Divide by $ab' + a'b$, $x = \dfrac{bc' + b'c}{ab' + a'b}.$

Therefore, $x = \dfrac{bc' + b'c}{ab' + a'b}$, and $y = \dfrac{ac' - a'c}{ab' + a'b}.$

4. Solve $$\left.\begin{aligned} ax + by &= 2ab \quad (1)\\ ay - bx &= a^2 - b^2 \quad (2)\end{aligned}\right\}$$

Multiply (1) by b, $abx + b^2y = 2ab^2.$ (3)

Multiply (2) by a, $-abx + a^2y = a^3 - ab^2.$ (4)

Add (3) and (4), $a^2y + b^2y = a^3 + ab^2.$

Divide by $a^2 + b^2$, $y = a.$

Substitute the value of y in (1), $ax + ab = 2ab.$

Transpose and combine, $ax = ab.$

$\therefore\ x = b.$

Therefore, $x = b$, and $y = a.$

5. Solve $$\left.\begin{aligned} bx + ay &= 2ab \quad (1)\\ a^2x + b^2y &= a^3 + b^3 \quad (2)\end{aligned}\right\}$$

Multiply (1) by a^2, $a^2bx + a^3y = 2a^3b.$ (3)

Multiply (2) by b, $a^2bx + b^3y = a^3b + b^4.$ (4)

Subtract (4) from (3), $a^3y - b^3y = a^3b - b^4.$ (5)
Divide (5) by $a^3 - b^3$, $y = b.$
Substitute the value of y in (1), $bx + ab = 2ab.$
Transpose and combine, $bx = ab.$
$\therefore x = a.$

Therefore, $x = a$, and $y = b$.

6. Solve $\left.\begin{aligned} b(x-c) &= a(y-c) \\ x - y &= a - b \end{aligned}\right\}$ (1) (2)
Simplify (1), $bx - ay = bc - ac.$ (3)
Multiply (2) by a, $ax - ay = a^2 - ab.$ (4)
Subtract (3) from (4), $ax - bx = a^2 - ab - bc + ac.$ (5)
Divide (5) by $a - b$, $x = a + c.$
Substitute the value of x in (2), $a + c - y = a - b.$
Transpose and combine, $y = b + c.$
Therefore, $x = a + c$, and $y = b + c$.

7. Solve $\left.\begin{aligned} ax + by &= 4a + b \\ bx + ay &= 4a - b \end{aligned}\right\}$ (1) (2)
Multiply (1) by b, $abx + b^2y = 4ab + b^2.$ (3)
Multiply (2) by a, $abx + a^2y = 4a^2 - ab.$ (4)
Subtract (3) from (4), $a^2y - b^2y = 4a^2 - 5ab - b^2.$ (5)
Divide (5) by $a^2 - b^2$, $y = \frac{4a^2 - 5ab - b^2}{a^2 - b^2}.$
Multiply (1) by a, $a^2x + aby = 4a^2 + ab.$ (6)
Multiply (2) by b, $b^2x + aby = 4ab - b^2.$ (7)
Subtract (7) from (6), $a^2x - b^2x = 4a^2 - 3ab + b^2.$ (8)
Divide (8) by $a^2 - b^2$, $x = \frac{4a^2 - 3ab + b^2}{a^2 - b^2}.$

Therefore, $x = \frac{4a^2 - 3ab + b^2}{a^2 - b^2}$, and $y = \frac{4a^2 - 5ab - b^2}{a^2 - b^2}.$

8. Solve $\left.\begin{aligned} ax + by &= abc \\ a'x + b'y &= a'b'c' \end{aligned}\right\}$ (1) (2)
Multiply (1) by a', $aa'x + a'by = aa'bc.$ (3)
Multiply (2) by a, $aa'x + ab'y = aa'b'c'.$ (4)
Subtract (3) from (4), $ab'y - a'by = aa'b'c' - aa'bc.$ (5)
Divide (5) by $ab' - a'b$, $y = \frac{aa'(b'c' - bc)}{ab' - a'b}.$

Multiply (1) by b', $ab'x + bb'y = abb'c.$ (6)
Multiply (2) by b, $a'bx + bb'y = a'bb'c'.$ (7)
Subtract (7) from (6), $ab'x - a'bx = abb'c - a'bb'c'.$ (8)
Divide (8) by $ab' - a'b$, $x = \dfrac{bb'(ac - a'c')}{ab' - a'b}.$

Therefore, $x = \dfrac{bb'(ac - a'c')}{ab' - a'b}$, and $y = \dfrac{aa'(b'c' - bc)}{ab' - a'b}.$

9. Solve
$$\left.\begin{aligned}(a+b)x + (a-b)y &= 2(a^2+b^2)\\ (a-b)x + (a+b)y &= 2(a^2-b^2)\end{aligned}\right\}\begin{aligned}&(1)\\&(2)\end{aligned}$$
Add (1) and (2), $2ax + 2ay = 4a^2.$ (3)
Divide (3) by $2a$, $x + y = 2a.$ (4)
Subtract (2) from (1), $2bx - 2by = 4b^2.$ (5)
Divide (5) by $2b$, $x - y = 2b.$ (6)
Add (4) and (6), $2x = 2a + 2b.$
$\therefore x = a + b.$
Subtract (6) from (4), $2y = 2a - 2b.$
$\therefore y = a - b.$

Therefore, $x = a + b$, and $y = a - b$.

10. Solve
$$\left.\begin{aligned}x + y &= \frac{2(a^2+b^2)}{a^2-b^2}\\ x - y &= \frac{4ab}{a^2-b^2}\end{aligned}\right\}\begin{aligned}&(1)\\&(2)\end{aligned}$$
Add (1) and (2), $2x = \dfrac{2(a^2+2ab+b^2)}{a^2-b^2} = \dfrac{2(a+b)}{a-b}.$
$\therefore x = \dfrac{a+b}{a-b}.$
Subtract (2) from (1), $2y = \dfrac{2(a^2-2ab+b^2)}{a^2-b^2} = \dfrac{2(a-b)}{a+b}.$
$\therefore y = \dfrac{a-b}{a+b}.$

Therefore, $x = \dfrac{a+b}{a-b}$, and $y = \dfrac{a-b}{a+b}.$

11. Solve
$$\left.\begin{aligned}\frac{x-y+1}{x-y-1} &= a\\ \frac{x+y+1}{x+y-1} &= b\end{aligned}\right\}\begin{aligned}&(1)\\&(2)\end{aligned}$$
Multiply (1) by $x - y - 1$, $x - y + 1 = ax - ay - a.$ (3)
Multiply (2) by $x + y - 1$, $x + y + 1 = bx + by - b.$ (4)

Simplify (3), $(a-1)x-(a-1)y=a+1.$ (5)
Simplify (4), $(b-1)x+(b-1)y=b+1.$ (6)
Multiply (5) by $b-1$,
$(a-1)(b-1)x-(a-1)(b-1)y=ab+b-a-1.$ (7)
Multiply (6) by $a-1$,
$(a-1)(b-1)x+(a-1)(b-1)y=ab-b+a-1.$ (8)
Subtract (7) from (8), $2(a-1)(b-1)y=2(a-b).$

$$\therefore\ y=\frac{a-b}{(a-1)(b-1)}.$$

Add (7) and (8), $2(a-1)(b-1)x=2ab-2.$

$$\therefore\ x=\frac{ab-1}{(a-1)(b-1)}.$$

Therefore, $x=\dfrac{ab-1}{(a-1)(b-1)}$, and $y=\dfrac{a-b}{(a-1)(b-1)}$.

12. Solve

$$\left.\begin{aligned}\frac{x-a+c}{y-a+b}&=\frac{b}{c} && (1)\\ \frac{x+c}{y+b}&=\frac{a+b}{a+c} && (2)\end{aligned}\right\}$$

Multiply (1) by $y-a+b$, $cx-ac+c^2=by-ab+b^2.$ (3)
Multiply (2) by $(y+b)(a+c)$,
$ax+cx+ac+c^2=ay+by+ab+b^2.$ (4)
Simplify (3), $cx-by=-ab+b^2+ac-c^2.$ (5)
Simplify (4), $ax+cx-ay-by=ab+b^2-ac-c^2.$ (6)
Subtract (5) from (6), $ax-ay=2ab-2ac.$ (7)
Divide (7) by a, $x-y=2b-2c.$ (8)
Multiply (8) by c, $cx-cy=2bc-2c^2.$ (9)
Subtract (5) from (9), $(b-c)y=ab-b^2+2bc-ac-c^2.$ (10)
Divide (10) by $b-c$, $y=a-b+c.$
Substitute the value of y in (8),
$x-a+b-c=2b-2c.$
Transpose and combine, $x=a+b-c.$
Therefore, $x=a+b-c$, and $y=a-b+c$.

13. Solve

$$\left.\begin{aligned}\frac{x+1}{y+1}&=\frac{a+b+c}{a-b+c} && (1)\\ \frac{x-1}{y-1}&=\frac{a+b-c}{a-b-c} && (2)\end{aligned}\right\}$$

Multiply (1) by $(y+1)(a-b+c)$,
$(a-b+c)x+a-b+c=(a+b+c)y+a+b+c.$ (3)
Multiply (2) by $(y-1)(a-b-c)$,
$(a-b-c)x-a+b+c=(a+b-c)y-a-b+c.$ (4)

Simplify (3), $(a-b+c)x-(a+b+c)y=2b.\qquad(5)$

Simplify (4), $(a-b-c)x-(a+b-c)y=-2b.\qquad(6)$

Subtract (6) from (5), $2cx-2cy=4b.\qquad(7)$

Divide (7) by $2c$, $x-y=\dfrac{2b}{c}.\qquad(8)$

Transpose, $x=y+\dfrac{2b}{c}.\qquad(9)$

Substitute the value of x in (5),

$$(a-b+c)y+\frac{2b(a-b+c)}{c}-(a+b+c)y=2b.$$

Combine, $\dfrac{2b(a-b+c)}{c}-2by=2b.$

Divide by $2b$, $\dfrac{a-b+c}{c}-y=1.$

Transpose, $y=\dfrac{a-b+c}{c}-1.$

$$\therefore\ y=\frac{a-b}{c}.$$

Substitute the value of y in (9), $x=\dfrac{a-b}{c}+\dfrac{2b}{c}.$

$$\therefore\ x=\frac{a+b}{c}.$$

Therefore, $x=\dfrac{a+b}{c}$, and $y=\dfrac{a-b}{c}.$

14. Solve

$$\left.\begin{aligned}\frac{x}{a+b}+\frac{y}{a-b}&=a+b\qquad(1)\\ \frac{x}{a}+\frac{y}{b}&=2a\qquad(2)\end{aligned}\right\}$$

Multiply (1) by $(a+b)(a-b)$,

$$ax-bx+ay+by=a^3+a^2b-ab^2-b^3.\qquad(3)$$

Multiply (2) by ab, $bx+ay=2a^2b.\qquad(4)$

Transpose (3), $ax-bx=a^3+a^2b-ab^2-b^3-ay-by.\qquad(5)$

Transpose (4), $bx=2a^2b-ay.\qquad(6)$

Divide (5) by $a-b$, $x=\dfrac{a^3+a^2b-ab^2-b^3-ay-by}{a-b}.\qquad(7)$

Divide (6) by b, $x=\dfrac{2a^2b-ay}{b}.\qquad(8)$

Equate the values of x,

$$\frac{a^3+a^2b-ab^2-b^3-ay-by}{a-b}=\frac{2a^2b-ay}{b}.$$

Multiply by $b(a-b)$,

$$a^3b+a^2b^2-ab^3-b^4-aby-b^2y=2a^3b-2a^2b^2-a^2y+aby.$$

Transpose and combine,

$$a^2y-2aby-b^2y=a^3b-3a^2b^2+ab^3+b^4.$$

$$\therefore y=\frac{a^3b-3a^2b^2+ab^3+b^4}{a^2-2ab-b^2}=ab-b^2=b(a-b).$$

Substitute the value of y in (8),

$$x=\frac{2a^2b-a^2b+ab^2}{b}=a^2+ab=a(a+b).$$

Therefore, $x=a(a+b)$, and $y=b(a-b)$.

15. Solve

$$\left.\begin{aligned}\frac{x+c}{a+b}+\frac{y+b}{a+c}&=2\\ \frac{x-b}{a-c}+\frac{y-c}{a-b}&=2\end{aligned}\right\}\quad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by $(a+b)(a+c)$,

$$ax+cx+ay+by=2a^2+ab+ac+2bc-b^2-c^2. \quad (3)$$

Multiply (2) by $(a-b)(a-c)$,

$$ax-bx+ay-cy=2a^2-ab-ac+2bc-b^2-c^2. \quad (4)$$

Subtract (4) from (3),

$$bx+cx+by+cy=2ab+2ac. \quad (5)$$

Divide (5) by $b+c$, $x+y=2a.$

Transpose, $y=2a-x.$ (6)

Substitute the value of y in (3),

$$ax+cx+2a^2-ax+2ab-bx=2a^2+ab+ac+2bc-b^2-c^2.$$

Transpose and combine, $cx-bx=ac-ab+2bc-b^2-c^2.$

Divide by $c-b$, $x=a+b-c.$

Substitute the value of x in (6), $y=2a-a-b+c.$

Combine, $y=a-b+c.$

Therefore, $x=a+b-c$, and $y=a-b+c$.

16. Solve

$$\left.\begin{aligned}\frac{x}{b^2-1}-\frac{y}{a^2-1}&=a^2-b^2\\ \frac{x}{a^2+1}+\frac{y}{b^2+1}&=a^2+b^2-2\end{aligned}\right\}\quad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by $\frac{1}{a^2+1}$, $\frac{x}{(a^2+1)(b^2-1)}-\frac{y}{a^4-1}=\frac{a^2-b^2}{a^2+1}.$ (3)

Multiply (2) by $\frac{1}{b^2-1}$, $\frac{x}{(a^2+1)(b^2-1)}+\frac{y}{b^4-1}=\frac{a^2+b^2-2}{b^2-1}.$ (4)

Subtract (3) from (4), $\dfrac{y}{b^4-1}+\dfrac{y}{a^4-1}=\dfrac{a^2+b^2-2}{b^2-1}-\dfrac{a^2-b^2}{a^2+1}.$

Simplify, $\dfrac{(a^4+b^4-2)y}{(a^4-1)(b^4-1)}=\dfrac{a^4+b^4-2}{(a^2+1)(b^2-1)}.$

Divide by a^4+b^4-2, $\dfrac{y}{(a^4-1)(b^4-1)}=\dfrac{1}{(a^2+1)(b^2-1)}.$

Multiply by $(a^4-1)(b^4-1)$, $y=(a^2-1)(b^2+1).$

Substitute the value of y in (2),

$$\frac{x}{a^2+1}+a^2-1=a^2+b^2-2.$$

Transpose and combine, $\dfrac{x}{a^2+1}=b^2-1.$

Multiply by a^2+1, $x=(a^2+1)(b^2-1).$

Therefore, $x=(a^2+1)(b^2-1)$, and $y=(a^2-1)(b^2+1)$.

17. Solve

$$\left.\begin{aligned} ax+by&=a^3+2a^2b+b^3\\ bx+ay&=a^3+2ab^2+b^3\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Add (1) and (2), $(a+b)x+(a+b)y=2a^3+2a^2b+2ab^2+2b^3.$ (3)

Divide (3) by $a+b$, $x+y=2a^2+2b^2.$ (4)

Subtract (2) from (1),

$$(a-b)x-(a-b)y=2a^2b-2ab^2. \qquad (5)$$

Divide (5) by $a-b$, $x-y=2ab.$ (6)

Add (4) and (6), $2x=2a^2+2ab+2b^2.$

$\therefore\ x=a^2+ab+b^2.$

Subtract (6) from (4), $2y=2a^2-2ab+2b^2.$

$\therefore\ y=a^2-ab+b^2.$

Therefore, $x=a^2+ab+b^2$, and $y=a^2-ab+b^2$.

18. Solve

$$\left.\begin{aligned}(a+c)x-(a-c)y&=2ab\\ (a+b)y-(a-b)x&=2ac\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Add (1) and (2), $(b+c)x+(b+c)y=2ab+2ac.$ (3)

Divide (3) by $b+c$, $x+y=2a.$ (4)

$\therefore\ y=2a-x.$ (5)

Substitute the value of y in (1),

$$ax+cx-2a^2+2ac+ax-cx=2ab.$$

Transpose and combine, $2ax=2a^2+2ab-2ac.$

$\therefore\ x=a+b-c.$

Substitute the value of x in (5), $y=2a-a-b+c.$

$\therefore\ y=a-b+c.$

Therefore, $x=a+b-c$, and $y=a-b+c$.

19. Solve
$$\left.\begin{aligned}(a+b-c)x-(a-b+c)y&=4a(b-c)\\ \frac{x}{y}&=\frac{a+b-c}{a-b+c}\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (2) by $y(a-b+c)$, $(a-b+c)x=(a+b-c)y.$ (3)

Transpose (3), $(a-b+c)x-(a+b-c)y=0.$ (4)

Add (1) and (4), $2ax-2ay=4a(b-c).$ (5)

Subtract (4) from (1),
$$(2b-2c)x+(2b-2c)y=4a(b-c).\qquad(6)$$

Divide (5) by $2a$, $x-y=2b-2c.$ (7)

Divide (6) by $2b-2c$, $x+y=2a.$ (8)

Add (7) and (8), $2x=2a+2b-2c.$
$$\therefore\ x=a+b-c.$$

Subtract (7) from (8), $2y=2a-2b+2c.$
$$\therefore\ y=a-b+c.$$

Therefore, $x=a+b-c$, and $y=a-b+c$.

20. Solve
$$\left.\begin{aligned}\frac{x+y}{x-y}&=\frac{a}{b-c}\\ \frac{x+c}{y+b}&=\frac{a+b}{a+c}\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (2) by $(y+b)(a+c)$,
$$(a+c)x+ac+c^2=(a+b)y+ab+b^2.\qquad(3)$$

Transpose (3), $(a+c)x-(a+b)y=ab+b^2-ac-c^2.$ (4)

Multiply (1) by $(x-y)(b-c)$,
$$(b-c)x+(b-c)y=ax-ay.\qquad(5)$$

Transpose (5), $(a-b+c)x=(a+b-c)y.$
$$\therefore\ x=\frac{(a+b-c)y}{a-b+c}.\qquad(6)$$

Substitute the value of x in (4),
$$\frac{(a+c)(a+b-c)y}{a-b+c}-(a+b)y=ab+b^2-ac-c^2.\qquad(7)$$

Simplify the left member of (7),
$$\frac{(a+c)(a+b-c)-(a+b)(a-b+c)y}{a-b+c}=ab+b^2-ac-c^2.$$
$$\frac{(ab-c^2+b^2-ac)y}{a-b+c}=ab+b^2-ac-c^2.$$

Divide by $ab-c^2+b^2-ac$, $\dfrac{y}{a-b+c}=1.$
$$\therefore\ y=a-b+c.$$

Substitute the value of y in (5), $$x = \frac{(a + b - c)(a - b + c)}{a - b + c}.$$

$$\therefore\ x = a + b - c.$$

Therefore, $x = a + b - c$, and $y = a - b + c$.

21. Solve
$$\left.\begin{aligned}\frac{x-a}{y-a} &= \frac{a-b}{a+b} \qquad (1)\\ \frac{x}{y} &= \frac{a^3-b^3}{a^3+b^3} \qquad (2)\end{aligned}\right\}$$

Multiply (1) by $(y - a)(a + b)$,

$$(a + b)x - a^2 - ab = (a - b)y - a^2 + ab. \qquad (3)$$

Multiply (2) by $y(a^3 + b^3)$, $\quad (a^3 + b^3)x = (a^3 - b^3)y. \qquad (4)$

Transpose and combine (3),

$$(a + b)x - (a - b)y = 2ab. \qquad (5)$$

Transpose (4), $\quad (a^3 + b^3)x - (a^3 - b^3)y = 0. \qquad (6)$

Multiply (5) by $a^2 - ab + b^2$,

$$(a^3 + b^3)x - (a^3 - 2a^2b + 2ab^2 - b^3)y = 2ab(a^2 - ab + b^2). \qquad (7)$$

Subtract (6) from (7), $\quad (2a^2b - 2ab^2)y = 2ab(a^2 - ab + b^2).$

Divide by $2ab$, $\quad (a - b)y = a^2 - ab + b^2.$

$$\therefore\ y = \frac{a^2 - ab + b^2}{a - b}.$$

Multiply (5) by $a^2 + ab + b^2$,

$$(a^3 + 2a^2b + 2ab^2 + b^3)x - (a^3 - b^3)y = 2ab(a^2 + ab + b^2). \qquad (8)$$

Subtract (6) from (8), $\quad (2a^2b + 2ab^2)x = 2ab(a^2 + ab + b^2).$

Divide by $2ab$, $\quad (a + b)x = a^2 + ab + b^2.$

$$\therefore\ x = \frac{a^2 + ab + b^2}{a + b}.$$

Therefore, $x = \dfrac{a^2 + ab + b^2}{a + b}$, and $y = \dfrac{a^2 - ab + b^2}{a - b}$.

22. Solve
$$\left.\begin{aligned}\frac{1+x}{2} &= \frac{y-1}{2} + \frac{(a-b)^2 - 2b^2}{a^2 - b^2} \qquad (1)\\ by - ax &= \frac{(3a+b)ab}{a^2-b^2} + \frac{ab}{a+b} - (a+b) \qquad (2)\end{aligned}\right\}$$

Multiply (1) by $2(a^2 - b^2)$,

$$a^2 - b^2 + (a^2 - b^2)x = (a^2 - b^2)y - a^2 + b^2 + 2a^2 - 4ab - 2b^2. \qquad (3)$$

Multiply (2) by $a^2 - b^2$,

$$b(a^2 - b^2)y - a(a^2 - b^2)x = 3a^2b + ab^2 + a^2b - ab^2 - a^3 + ab^2 - a^2b + b^3. \qquad (4)$$

Simplify (3), $(a^2 - b^2)x - (a^2 - b^2)y = -4ab. \quad (5)$

Simplify (4), $-a(a^2-b^2)x+b(a^2-b^2)y = -a^3+3a^2b+ab^2+b^3. \quad (6)$

Multiply (5) by a, $a(a^2-b^2)x-a(a^2-b^2)y = -4a^2b. \quad (7)$

Add (6) and (7), $(b-a)(a^2-b^2)y = -a^3 - a^2b + ab^2 + b^3.$

Divide by $b-a$, $(a^2-b^2)y = a^2 + 2ab + b^2.$

Divide by $a+b$, $(a-b)y = a+b.$

$$\therefore y = \frac{a+b}{a-b}.$$

Substitute the value of y in (5),

$$(a^2-b^2)x - \frac{(a^2-b^2)(a+b)}{a-b} = -4ab.$$

Simplify, $(a^2-b^2)x - (a^2+2ab+b^2) = -4ab.$

Transpose and combine, $(a^2-b^2)x = (a-b)^2.$

$$\therefore x = \frac{(a-b)^2}{a^2-b^2} = \frac{a-b}{a+b}.$$

Therefore, $x = \frac{a-b}{a+b}$, and $y = \frac{a+b}{a-b}$.

Exercise 75. Page 180

1. Solve

$$\left.\begin{aligned} \frac{5}{3x} + \frac{2}{5y} &= 7 \quad (1)\\ \frac{7}{6x} - \frac{1}{10y} &= 3 \quad (2)\end{aligned}\right\}$$

Multiply (2) by 4, $\frac{14}{3x} - \frac{2}{5y} = 12. \quad (3)$

Add (1) and (3), $\frac{19}{3x} = 19.$

$$\therefore 3x = 1.$$

$$\therefore x = \tfrac{1}{3}.$$

Substitute the value of x in (1), $5 + \frac{2}{5y} = 7.$

Transpose and combine, $\frac{2}{5y} = 2.$

$$\therefore 5y = 1.$$

$$\therefore y = \tfrac{1}{5}.$$

Therefore, $x = \frac{1}{3}$, and $y = \frac{1}{5}$.

2. Solve
$$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=a\\ \frac{1}{x}-\frac{1}{y}&=b\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Add (1) and (2),
$$\frac{2}{x}=a+b. \qquad (3)$$
$$\therefore (a+b)x=2.$$
$$\therefore x=\frac{2}{a+b}.$$

Subtract (2) from (1),
$$\frac{2}{y}=a-b.$$
$$\therefore (a-b)y=2.$$
$$\therefore y=\frac{2}{a-b}.$$

Therefore, $x=\dfrac{2}{a+b}$, and $y=\dfrac{2}{a-b}$.

3. Solve
$$\left.\begin{aligned}\frac{24}{x}+\frac{12}{y}&=10\\ \frac{16}{x}+\frac{9}{y}&=\ \ 7\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Divide (1) by 2,
$$\frac{12}{x}+\frac{6}{y}=5. \qquad (3)$$

Multiply (2) by 2,
$$\frac{32}{x}+\frac{18}{y}=14. \qquad (4)$$

Multiply (3) by 3,
$$\frac{36}{x}+\frac{18}{y}=15. \qquad (5)$$

Subtract (4) from (5),
$$\frac{4}{x}=1.$$
$$\therefore x=4.$$

Substitute the value of x in (2),
$$4+\frac{9}{y}=7.$$
$$3y=9.$$
$$\therefore y=3.$$

Therefore, $x=4$, and $y=3$.

4. Solve
$$\left.\begin{aligned}\frac{25}{x}+\frac{14}{y}&=7\\ \frac{20}{x}-\frac{21}{y}&=1\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Multiply (1) by 3,
$$\frac{75}{x}+\frac{42}{y}=21. \qquad (3)$$

Multiply (2) by 2,
$$\frac{40}{x}-\frac{42}{y}=2. \tag{4}$$

Add (3) and (4),
$$\frac{115}{x}=23.$$

Divide by 23,
$$\frac{5}{x}=1.$$
$$\therefore x=5.$$

Substitute the value of x in (1),
$$5+\frac{14}{y}=7.$$

Transpose and combine,
$$\frac{14}{y}=2.$$
$$2y=14.$$
$$\therefore y=7.$$

Therefore, $x=5$, and $y=7$.

5. Solve
$$\left.\begin{aligned}\frac{33}{x}-\frac{12}{y}&=1 \qquad (1)\\ \frac{44}{x}+\frac{30}{y}&=9 \qquad (2)\end{aligned}\right\}$$

Multiply (1) by 4,
$$\frac{132}{x}-\frac{48}{y}=4. \tag{3}$$

Multiply (2) by 3,
$$\frac{132}{x}+\frac{90}{y}=27. \tag{4}$$

Subtract (3) from (4),
$$\frac{138}{y}=23.$$

Divide by 23,
$$\frac{6}{y}=1.$$
$$\therefore y=6.$$

Substitute the value of y in (1),
$$\frac{33}{x}-2=1.$$

Transpose and combine,
$$\frac{33}{x}=3.$$

Divide by 3,
$$\frac{11}{x}=1.$$
$$\therefore x=11.$$

Therefore, $x=11$, and $y=6$.

6. Solve
$$\left.\begin{aligned}\frac{1}{5x}+\frac{1}{4y}&=\frac{1}{12} \qquad (1)\\ \frac{1}{4x}+\frac{1}{3y}&=\frac{11}{120} \qquad (2)\end{aligned}\right\}$$

Multiply (1) by $\frac{1}{3}$, $\frac{1}{15x}+\frac{1}{12y}=\frac{1}{36}$. (3)

Multiply (2) by $\frac{1}{4}$, $\frac{1}{16x}+\frac{1}{12y}=\frac{11}{480}$. (4)

Subtract (4) from (3), $\frac{1}{240x}=\frac{7}{1440}$.

Multiply by $1440x$, $6=7x$.

$\therefore x=\frac{6}{7}$.

Substitute the value of x in (1), $\frac{7}{30}+\frac{1}{4y}=\frac{1}{12}$.

Transpose and combine, $\frac{1}{4y}=-\frac{3}{20}$.

Multiply by $20y$, $5=-3y$.

$\therefore y=-\frac{5}{3}$.

Therefore, $x=\frac{6}{7}$, and $y=-\frac{5}{3}$.

7. Solve

$$\left.\begin{aligned}\frac{2}{3x}+\frac{3}{4y}&=1\tfrac{7}{12} \qquad (1)\\ \frac{3}{4x}+\frac{2}{5y}&=1\tfrac{3}{20} \qquad (2)\end{aligned}\right\}$$

Multiply (2) by $\frac{8}{9}$, $\frac{2}{3x}+\frac{16}{45y}=\frac{46}{45}$. (3)

Subtract (3) from (1), $\frac{71}{180y}=\frac{101}{180}$.

Multiply by $180y$, $71=101y$.

$\therefore y=\frac{71}{101}$.

Substitute the value of y in (2), $\frac{3}{4x}+\frac{202}{355}=\frac{23}{20}$.

Multiply by $1420x$, $1065+808x=1633x$.

Transpose and combine, $825x=1065$.

Divide by 15, $55x=71$.

$\therefore x=\frac{71}{55}$.

Therefore, $x=\frac{71}{55}$, and $y=\frac{71}{101}$.

8. Solve

$$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=c \qquad (1)\\ \frac{b}{x}+\frac{a}{y}&=d \qquad (2)\end{aligned}\right\}$$

Multiply (1) by a, $\frac{a^2}{x}+\frac{ab}{y}=ac$. (3)

Multiply (2) by b, $\frac{b^2}{x}+\frac{ab}{y}=bd$. (4)

Subtract (4) from (3), $$\frac{a^2-b^2}{x}=ac-bd.$$

$$\therefore x=\frac{a^2-b^2}{ac-bd}.$$

Multiply (1) by b, $$\frac{ab}{x}+\frac{b^2}{y}=bc. \qquad (5)$$

Multiply (2) by a, $$\frac{ab}{x}+\frac{a^2}{y}=ad. \qquad (6)$$

Subtract (5) from (6), $$\frac{a^2-b^2}{y}=ad-bc.$$

$$\therefore y=\frac{a^2-b^2}{ad-bc}.$$

Therefore, $x=\frac{a^2-b^2}{ac-bd}$, and $y=\frac{a^2-b^2}{ad-bc}$.

9. Solve
$$\left.\begin{aligned}\frac{a}{bx}+\frac{b}{ay}&=c \qquad (1)\\ \frac{b}{ax}+\frac{a}{by}&=d \qquad (2)\end{aligned}\right\}$$

Multiply (1) by $\frac{a}{b}$, $$\frac{a^2}{b^2x}+\frac{1}{y}=\frac{ac}{b}. \qquad (3)$$

Multiply (2) by $\frac{b}{a}$, $$\frac{b^2}{a^2x}+\frac{1}{y}=\frac{bd}{a}. \qquad (4)$$

Subtract (4) from (3), $$\frac{a^4-b^4}{a^2b^2x}=\frac{ac}{b}-\frac{bd}{a}.$$

Multiply by a^2b^2x, $$a^4-b^4=ab(a^2c-b^2d)x.$$

$$\therefore x=\frac{a^4-b^4}{ab(a^2c-b^2d)}.$$

Multiply (1) by $\frac{b}{a}$, $$\frac{1}{x}+\frac{b^2}{a^2y}=\frac{bc}{a}. \qquad (5)$$

Multiply (2) by $\frac{a}{b}$, $$\frac{1}{x}+\frac{a^2}{b^2y}=\frac{ad}{b}. \qquad (6)$$

Subtract (5) from (6), $$\frac{a^4-b^4}{a^2b^2y}=\frac{ad}{b}-\frac{bc}{a}.$$

Multiply by a^2b^2y, $$a^4-b^4=ab(a^2d-b^2c)y.$$

$$\therefore y=\frac{a^4-b^4}{ab(a^2d-b^2c)}.$$

Therefore, $x=\frac{a^4-b^4}{ab(a^2c-b^2d)}$, and $y=\frac{a^4-b^4}{ab(a^2d-b^2c)}$.

10. Solve

$$\left.\begin{array}{l}\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{a} \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{b}\end{array}\right\}\qquad\begin{array}{r}(1)\\(2)\end{array}$$

Add (1) and (2), $\dfrac{2}{x}=\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{b+a}{ab}.$

Multiply by abx, $2ab=(b+a)x.$

$$\therefore\ x=\frac{2ab}{b+a}.$$

Subtract (2) from (1), $\dfrac{2}{y}=\dfrac{1}{a}-\dfrac{1}{b}=\dfrac{b-a}{ab}.$

Multiply by aby, $2ab=(b-a)y.$

$$\therefore\ y=\frac{2ab}{b-a}.$$

Therefore, $x=\dfrac{2ab}{b+a}$, and $y=\dfrac{2ab}{b-a}.$

11. Solve

$$\left.\begin{array}{l}\dfrac{1}{3x}+\dfrac{1}{4y}=a \\ \dfrac{1}{4x}+\dfrac{1}{3y}=b\end{array}\right\}\qquad\begin{array}{r}(1)\\(2)\end{array}$$

Multiply (1) by $\frac{1}{3}$, $\dfrac{1}{9x}+\dfrac{1}{12y}=\dfrac{a}{3}.\qquad(3)$

Multiply (2) by $\frac{1}{4}$, $\dfrac{1}{16x}+\dfrac{1}{12y}=\dfrac{b}{4}.\qquad(4)$

Subtract (4) from (3), $\dfrac{7}{144x}=\dfrac{a}{3}-\dfrac{b}{4}=\dfrac{4a-3b}{12}.$

Multiply by $144x$, $7=12(4a-3b)x.$

$$\therefore\ x=\frac{7}{12(4a-3b)}.$$

Multiply (1) by $\frac{1}{4}$, $\dfrac{1}{12x}+\dfrac{1}{16y}=\dfrac{a}{4}.\qquad(5)$

Multiply (2) by $\frac{1}{3}$, $\dfrac{1}{12x}+\dfrac{1}{9y}=\dfrac{b}{3}.\qquad(6)$

Subtract (5) from (6), $\dfrac{7}{144y}=\dfrac{b}{3}-\dfrac{a}{4}=\dfrac{4b-3a}{12}.$

Multiply by $144y$, $7=12(4b-3a)y.$

$$\therefore\ y=\frac{7}{12(4b-3a)}.$$

Therefore, $x=\dfrac{7}{12(4a-3b)}$, and $y=\dfrac{7}{12(4b-3a)}.$

12. Solve
$$\left.\begin{aligned}\frac{9}{x}-\frac{7}{y}&=6\tfrac{3}{7}\\ \frac{7}{x}-\frac{9}{y}&=3\tfrac{8}{9}\end{aligned}\right\}\qquad\begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (1) by 7, $$\frac{63}{x}-\frac{49}{y}=45. \qquad (3)$$

Multiply (2) by 9, $$\frac{63}{x}-\frac{81}{y}=35. \qquad (4)$$

Subtract (4) from (3), $$\frac{32}{y}=10.$$

$$\therefore y=\tfrac{16}{5}.$$

Multiply (1) by 9, $$\frac{81}{x}-\frac{63}{y}=\frac{405}{7}. \qquad (5)$$

Multiply (2) by 7, $$\frac{49}{x}-\frac{63}{y}=\frac{245}{9}. \qquad (6)$$

Subtract (6) from (5), $$\frac{32}{x}=\frac{1930}{63}.$$

$$\therefore x=\frac{32\times 63}{1930}=\frac{1008}{965}.$$

Therefore, $x=\tfrac{1008}{965}$, and $y=\tfrac{16}{5}$.

13. Solve
$$\left.\begin{aligned}\frac{a}{bx}+\frac{b}{ay}&=a+b\\ \frac{b}{x}+\frac{a}{y}&=a^2+b^2\end{aligned}\right\}\qquad\begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (2) by $\frac{a}{b^2}$, $$\frac{a}{bx}+\frac{a^2}{b^2y}=\frac{a^3}{b^2}+a. \qquad (3)$$

Subtract (3) from (1), $$\frac{b}{ay}-\frac{a^2}{b^2y}=b-\frac{a^3}{b^2}. \qquad (4)$$

Simplify (4), $$\frac{b^3-a^3}{ab^2y}=\frac{b^3-a^3}{b^2}.$$

Divide by $\frac{b^3-a^3}{b^2}$, $$\frac{1}{ay}=1.$$

$$\therefore y=\frac{1}{a}.$$

Substitute the value of y in (2), $$\frac{b}{x}+a^2=a^2+b^2.$$

Transpose and combine, $$\frac{b}{x}=b^2.$$

Divide by b. $\qquad \frac{1}{x} = b.$

$$\therefore\ x = \frac{1}{b}.$$

Therefore, $x = \frac{1}{b}$, and $y = \frac{1}{a}$.

14. Solve

$$\left.\begin{aligned} \frac{20}{3x} + \frac{18}{5y} &= 2\tfrac{8}{15} \quad (1) \\ \frac{28}{5x} - \frac{10}{3y} &= \frac{2}{225} \quad (2) \end{aligned}\right\}$$

Multiply (1) by $\frac{5}{18}$, $\qquad \frac{50}{27x} + \frac{1}{y} = \frac{19}{27}. \qquad (3)$

Multiply (2) by $\frac{3}{10}$, $\qquad \frac{42}{25x} - \frac{1}{y} = \frac{1}{375}. \qquad (4)$

Add (3) and (4), $\qquad \frac{2384}{675x} = \frac{2384}{3375}.$

Divide by 2384, $\qquad \frac{1}{675x} = \frac{1}{3375}.$

Multiply by 675, $\qquad \frac{1}{x} = \frac{1}{5}.$

$$\therefore\ x = 5.$$

Substitute the value of x in (1), $\qquad \frac{20}{15} + \frac{18}{5y} = \frac{38}{15}.$

Transpose and combine, $\qquad \frac{18}{5y} = \frac{18}{15}.$

Multiply by $\frac{5}{18}$, $\qquad \frac{1}{y} = \frac{1}{3}.$

$$\therefore\ y = 3.$$

Therefore, $x = 5$, and $y = 3$.

15. Solve

$$\left.\begin{aligned} \frac{a+b}{x} + \frac{a-b}{y} &= a^2 + b^2 \quad (1) \\ \frac{a-b}{x} + \frac{a+b}{y} &= a^2 - b^2 \quad (2) \end{aligned}\right\}$$

Add (1) and (2), $\qquad \frac{2a}{x} + \frac{2a}{y} = 2a^2. \qquad (3)$

Subtract (2) from (1), $\qquad \frac{2b}{x} - \frac{2b}{y} = 2b^2. \qquad (4)$

Divide (3) by $2a$, $$\frac{1}{x}+\frac{1}{y}=a. \qquad (5)$$

Divide (4) by $2b$, $$\frac{1}{x}-\frac{1}{y}=b. \qquad (6)$$

Add (5) and (6), $$\frac{2}{x}=a+b.$$
$$\therefore x=\frac{2}{a+b}.$$

Subtract (6) from (5), $$\frac{2}{y}=a-b.$$
$$\therefore y=\frac{2}{a-b}.$$

Therefore, $x=\dfrac{2}{a+b}$, and $y=\dfrac{2}{a-b}$.

16. Solve
$$\left.\begin{aligned}\frac{a+b}{x}-\frac{a-b}{y}&=c-d \qquad (1)\\ \frac{a-b}{x}-\frac{a+b}{y}&=c+d \qquad (2)\end{aligned}\right\}$$

Multiply (1) by $a-b$,
$$\frac{a^2-b^2}{x}-\frac{a^2-2ab+b^2}{y}=ac-ad-bc+bd. \qquad (3)$$

Multiply (2) by $a+b$,
$$\frac{a^2-b^2}{x}-\frac{a^2+2ab+b^2}{y}=ac+ad+bc+bd. \qquad (4)$$

Subtract (4) from (3), $$\frac{4ab}{y}=-2ad-2bc.$$
$$\therefore y=-\frac{2ab}{ad+bc}.$$

Multiply (1) by $a+b$,
$$\frac{a^2+2ab+b^2}{x}-\frac{a^2-b^2}{y}=ac-ad+bc-bd. \qquad (5)$$

Multiply (2) by $a-b$,
$$\frac{a^2-2ab+b^2}{x}-\frac{a^2-b^2}{y}=ac+ad-bc-bd. \qquad (6)$$

Subtract (6) from (5), $$\frac{4ab}{x}=2bc-2ad.$$
$$\therefore x=\frac{2ab}{bc-ad}.$$

Therefore, $x=\dfrac{2ab}{bc-ad}$, and $y=-\dfrac{2ab}{bc+ad}$.

Substitute the value of y in (6), $x = \dfrac{(a+b-c)(a-b+c)}{a-b+c}$.

$$\therefore\ x = a + b - c.$$

Therefore, $x = a + b - c$, and $y = a - b + c$.

21. Solve
$$\left.\begin{aligned} \frac{x-a}{y-a} &= \frac{a-b}{a+b} \qquad (1)\\ \frac{x}{y} &= \frac{a^3-b^3}{a^3+b^3} \qquad (2)\end{aligned}\right\}$$

Multiply (1) by $(y-a)(a+b)$,
$$(a+b)x - a^2 - ab = (a-b)y - a^2 + ab. \qquad (3)$$
Multiply (2) by $y(a^3+b^3)$, $\quad (a^3+b^3)x = (a^3-b^3)y. \qquad (4)$

Transpose and combine (3),
$$(a+b)x - (a-b)y = 2ab. \qquad (5)$$
Transpose (4), $\quad (a^3+b^3)x - (a^3-b^3)y = 0. \qquad (6)$

Multiply (5) by $a^2 - ab + b^2$,
$$(a^3+b^3)x - (a^3 - 2a^2b + 2ab^2 - b^3)y = 2ab(a^2 - ab + b^2). \qquad (7)$$
Subtract (6) from (7), $\quad (2a^2b - 2ab^2)y = 2ab(a^2 - ab + b^2).$

Divide by $2ab$, $\quad (a-b)y = a^2 - ab + b^2.$
$$\therefore\ y = \frac{a^2 - ab + b^2}{a-b}.$$

Multiply (5) by $a^2 + ab + b^2$,
$$(a^3 + 2a^2b + 2ab^2 + b^3)x - (a^3 - b^3)y = 2ab(a^2 + ab + b^2). \qquad (8)$$
Subtract (6) from (8), $\quad (2a^2b + 2ab^2)x = 2ab(a^2 + ab + b^2).$

Divide by $2ab$, $\quad (a+b)x = a^2 + ab + b^2.$
$$\therefore\ x = \frac{a^2 + ab + b^2}{a+b}.$$

Therefore, $x = \dfrac{a^2 + ab + b^2}{a+b}$, and $y = \dfrac{a^2 - ab + b^2}{a-b}$.

22. Solve
$$\left.\begin{aligned} \frac{1+x}{2} &= \frac{y-1}{2} + \frac{(a-b)^2 - 2b^2}{a^2-b^2} \qquad (1)\\ by - ax &= \frac{(3a+b)ab}{a^2-b^2} + \frac{ab}{a+b} - (a+b) \qquad (2)\end{aligned}\right\}$$

Multiply (1) by $2(a^2 - b^2)$,
$$a^2 - b^2 + (a^2 - b^2)x = (a^2 - b^2)y - a^2 + b^2 + 2a^2 - 4ab - 2b^2. \qquad (3)$$
Multiply (2) by $a^2 - b^2$,
$$b(a^2-b^2)y - a(a^2-b^2)x = 3a^2b + ab^2 + a^2b - ab^2 - a^3 + ab^2 - a^2b + b^3. \qquad (4)$$

Simplify (3), $(a^2-b^2)x-(a^2-b^2)y=-4ab.$ (5)

Simplify (4), $-a(a^2-b^2)x+b(a^2-b^2)y=-a^3+3a^2b+ab^2+b^3.$ (6)

Multiply (5) by a, $a(a^2-b^2)x-a(a^2-b^2)y=-4a^2b.$ (7)

Add (6) and (7), $(b-a)(a^2-b^2)y=-a^3-a^2b+ab^2+b^3.$

Divide by $b-a$, $(a^2-b^2)y=a^2+2ab+b^2.$

Divide by $a+b$, $(a-b)y=a+b.$

$$\therefore y=\frac{a+b}{a-b}.$$

Substitute the value of y in (5),

$$(a^2-b^2)x-\frac{(a^2-b^2)(a+b)}{a-b}=-4ab.$$

Simplify, $(a^2-b^2)x-(a^2+2ab+b^2)=-4ab.$

Transpose and combine, $(a^2-b^2)x=(a-b)^2.$

$$\therefore x=\frac{(a-b)^2}{a^2-b^2}=\frac{a-b}{a+b}.$$

Therefore, $x=\dfrac{a-b}{a+b}$, and $y=\dfrac{a+b}{a-b}.$

Exercise 75. Page 180

1. Solve

$$\left.\begin{aligned}\frac{5}{3x}+\frac{2}{5y}&=7\\ \frac{7}{6x}-\frac{1}{10y}&=3\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (2) by 4, $\dfrac{14}{3x}-\dfrac{2}{5y}=12.$ (3)

Add (1) and (3), $\dfrac{19}{3x}=19.$

$$\therefore 3x=1.$$

$$\therefore x=\tfrac{1}{3}.$$

Substitute the value of x in (1), $5+\dfrac{2}{5y}=7.$

Transpose and combine, $\dfrac{2}{5y}=2.$

$$\therefore 5y=1.$$

$$\therefore y=\tfrac{1}{5}.$$

Therefore, $x=\frac{1}{3}$, and $y=\frac{1}{5}$.

Exercise 76. Page 183

1. Solve

$$\left.\begin{aligned} 4x+3y+2z&=16 \quad (1)\\ 3x+5y+3z&=22 \quad (2)\\ 2x+4y+5z&=25 \quad (3)\end{aligned}\right\}$$

Multiply (1) by 3, $12x+9y+6z=48.$ (4)

Multiply (2) by 2, $6x+10y+6z=44.$ (5)

Subtract (5) from (4), $6x-y=4.$ (6)

Multiply (1) by 5, $20x+15y+10z=80.$ (7)

Multiply (3) by 2, $4x+8y+10z=50.$ (8)

Subtract (8) from (7), $16x+7y=30.$ (9)

Multiply (6) by 7, $42x-7y=28.$ (10)

Add (9) and (10), $58x=58.$

$\therefore x=1.$

Substitute the value of x in (6), $6-y=4.$

Transpose and combine, $y=2.$

Substitute the values of x and y in (1),

$4+6+2z=16.$

Transpose and combine, $2z=6.$

$\therefore z=3.$

Therefore, $x=1$, $y=2$, and $z=3$.

2. Solve

$$\left.\begin{aligned} x+y+z&=5 \quad (1)\\ 2x+5y+6z&=39 \quad (2)\\ 3x+4y+5z&=28 \quad (3)\end{aligned}\right\}$$

Multiply (1) by 2, $2x+2y+2z=10.$ (4)

Subtract (4) from (2), $3y+4z=29.$ (5)

Multiply (1) by 3, $3x+3y+3z=15.$ (6)

Subtract (6) from (3), $y+2z=13.$ (7)

Multiply (7) by 2, $2y+4z=26.$ (8)

Subtract (8) from (5), $y=3.$

Substitute the value of y in (7), $3+2z=13.$

Transpose and combine, $2z=10.$

$\therefore z=5.$

Substitute the values of y and z in (1), $x+3+5=5.$

Transpose and combine, $x=-3.$

Therefore, $x=-3$, $y=3$, and $z=5$.

3. Solve

$$\left.\begin{aligned} 7x-11y+5z&=5 \quad (1)\\ 5x-7y+11z&=21 \quad (2)\\ 11x-5y+7z&=43 \quad (3)\end{aligned}\right\}$$

Multiply (1) by 5, $35x - 55y + 25z = 25.$ (4)
Multiply (2) by 7, $35x - 49y + 77z = 147.$ (5)
Subtract (4) from (5), $6y + 52z = 122.$ (6)
Divide (6) by 2, $3y + 26z = 61.$ (7)
Multiply (2) by 11, $55x - 77y + 121z = 231.$ (8)
Multiply (3) by 5, $55x - 25y + 35z = 215.$ (9)
Subtract (8) from (9), $52y - 86z = -16.$ (10)
Divide (10) by 2, $26y - 43z = -8.$ (11)
Multiply (7) by 26, $78y + 676z = 1586.$ (12)
Multiply (11) by 3, $78y - 129z = -24.$ (13)
Subtract (13) from (12), $805z = 1610.$
$\therefore z = 2.$

Substitute the value of z in (7), $3y + 52 = 61.$
Transpose and combine, $3y = 9.$
$\therefore y = 3.$

Substitute the values of y and z in (1),
$7x - 33 + 10 = 5.$
Transpose and combine, $7x = 28.$
$\therefore x = 4.$

Therefore, $x = 4$, $y = 3$, and $z = 2$.

4. Solve

$$\left.\begin{aligned} 6x - 5y + 9z &= 27 \quad (1)\\ 5x + 8y - 3z &= 7 \quad (2)\\ 7x - 3y + 6z &= 32 \quad (3). \end{aligned}\right\}$$

Multiply (2) by 3, $15x + 24y - 9z = 21.$ (4)
Add (1) and (4), $21x + 19y = 48.$ (5)
Multiply (2) by 2, $10x + 16y - 6z = 14.$ (6)
Add (3) and (6), $17x + 13y = 46.$ (7)
Multiply (5) by 17, $357x + 323y = 816.$ (8)
Multiply (7) by 21, $357x + 273y = 966.$ (9)
Subtract (9) from (8), $50y = -150.$
$\therefore y = -3.$

Substitute the value of y in (7), $17x - 39 = 46.$
Transpose and combine, $17x = 85.$
$\therefore x = 5.$

Substitute the values of x and y in (2),
$25 - 24 - 3z = 7.$
Transpose and combine, $-3z = 6.$
$\therefore z = -2.$

Therefore, $x = 5$, $y = -3$, and $z = -2$.

5. Solve

$$\left.\begin{aligned} 12x - 5y + 8z &= 2 \quad (1)\\ 6x + 10y - 5z &= 13 \quad (2)\\ 9x - 15y + 4z &= -80 \quad (3)\end{aligned}\right\}$$

Multiply (1) by 2, $24x - 10y + 16z = 4.$ (4)

Add (2) and (4), $30x + 11z = 17.$ (5)

Multiply (1) by 3, $36x - 15y + 24z = 6.$ (6)

Subtract (3) from (6), $27x + 20z = 86.$ (7)

Multiply (5) by 9, $270x + 99z = 153.$ (8)

Multiply (7) by 10, $270x + 200z = 860.$ (9)

Subtract (8) from (9), $101z = 707.$

$\therefore z = 7.$

Substitute the value of z in (5), $30x + 77 = 17.$

Transpose and combine, $30x = -60.$

$\therefore x = -2.$

Substitute the values of x and z in (1),

$-24 - 5y + 56 = 2.$

Transpose and combine, $-5y = -30.$

$\therefore y = 6.$

Therefore, $x = -2$, $y = 6$, and $z = 7$.

6. Solve

$$\left.\begin{aligned} 3x - 5y - 2z &= -37 \quad (1)\\ 2x + 3y + z &= 63 \quad (2)\\ 3x + 6y + 2z &= 115 \quad (3)\end{aligned}\right\}$$

Multiply (2) by 2, $4x + 6y + 2z = 126.$ (4)

Add (1) and (4), $7x + y = 89.$ (5)

Add (1) and (3), $6x + y = 78.$ (6)

Subtract (6) from (5), $x = 11.$

Substitute the value of x in (6), $66 + y = 78.$

Transpose and combine, $y = 12.$

Substitute the values of x and y in (2),

$22 + 36 + z = 63.$

Transpose and combine, $z = 5.$

Therefore, $x = 11$, $y = 12$, and $z = 5$.

7. Solve

$$\left.\begin{aligned} 10x - y + 3z &= 103 \quad (1)\\ 3x + 3y - 2z &= -6 \quad (2)\\ 4x + 5y + 4z &= 94 \quad (3)\end{aligned}\right\}$$

Multiply (1) by 3, $30x - 3y + 9z = 309.$ (4)

Add (2) and (4), $33x + 7z = 303.$ (5)

Multiply (1) by 5, $50x - 5y + 15z = 515.$ (6)

Add (3) and (6), $54x + 19z = 609.$ (7)

Multiply (5) by 19, $627x + 133z = 5757.$ (8)

Multiply (7) by 7, $378x + 133z = 4263.$ (9)

Subtract (9) from (8), $249x = 1494.$

$\therefore x = 6.$

Substitute the value of x in (5), $198 + 7z = 303.$

Transpose and combine, $7z = 105.$

$\therefore z = 15.$

Substitute the values of x and z in (1),

$60 - y + 45 = 103.$

Transpose and combine, $y = 2.$

Therefore, $x = 6$, $y = 2$, and $z = 15$.

8. Solve

$$\left.\begin{aligned} 7x + 8y - 4z &= 15 \quad (1)\\ 5x - 12y + 6z &= 55 \quad (2)\\ 4x - 4y - 5z &= 51 \quad (3) \end{aligned}\right\}$$

Multiply (3) by 2, $8x - 8y - 10z = 102.$ (4)

Add (1) and (4), $15x - 14z = 117.$ (5)

Multiply (3) by 3, $12x - 12y - 15z = 153.$ (6)

Subtract (2) from (6), $7x - 21z = 98.$ (7)

Divide (7) by 7, $x - 3z = 14.$ (8)

Multiply (8) by 15, $15x - 45z = 210.$ (9)

Subtract (9) from (5), $31z = -93.$

$\therefore z = -3.$

Substitute the value of z in (8), $x + 9 = 14.$

Transpose and combine, $x = 5.$

Substitute the values of x and z in (1),

$35 + 8y + 12 = 15.$

Transpose and combine, $8y = -32.$

$\therefore y = -4.$

Therefore, $x = 5$, $y = -4$, and $z = -3$.

9. Solve

$$\left.\begin{aligned} 3x + 2y + 3z &= 99 \quad (1)\\ 5x + 4y + 5z &= 171 \quad (2)\\ 7x + 8y + 6z &= 247 \quad (3) \end{aligned}\right\}$$

Multiply (1) by 2, $6x + 4y + 6z = 198.$ (4)

Subtract (2) from (4), $x + z = 27.$ (5)

Multiply (2) by 2, $10x + 8y + 10z = 342.$ (6)

Subtract (3) from (6), $3x + 4z = 95.$ (7)

Multiply (5) by 4, $4x + 4z = 108.$ (8)

Subtract (7) from (8), $x = 13.$

Substitute the value of x in (5), $13 + z = 27.$

Transpose and combine, $z = 14.$

Substitute the values of x and z in (1),

$$39 + 2y + 42 = 99.$$

Transpose and combine, $2y = 18.$

$\therefore y = 9.$

Therefore, $x = 13$, $y = 9$, and $z = 14$.

10. Solve

$$\left.\begin{aligned}5x - 4y + 3z &= 51\\ 7x + 6y - 6z &= 64\\ 9x - 8y - 9z &= 47\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\\&(3)\end{aligned}$$

Multiply (1) by 2, $10x - 8y + 6z = 102.$ (4)

Add (2) and (4), $17x - 2y = 166.$ (5)

Multiply (1) by 3, $15x - 12y + 9z = 153.$ (6)

Add (3) and (6), $24x - 20y = 200.$ (7)

Divide (7) by 4, $6x - 5y = 50.$ (8)

Multiply (5) by 5, $85x - 10y = 830.$ (9)

Multiply (8) by 2, $12x - 10y = 100.$ (10)

Subtract (10) from (9), $73x = 730.$

$\therefore x = 10.$

Substitute the value of x in (8), $60 - 5y = 50.$

Transpose and combine, $-5y = -10.$

$\therefore y = 2.$

Substitute the values of x and y in (1),

$$50 - 8 + 3z = 51.$$

Transpose and combine, $3z = 9.$

$\therefore z = 3.$

Therefore, $x = 10$, $y = 2$, and $z = 3$.

11. Solve

$$\left.\begin{aligned}x + 12y + 2z &= 86\\ 5y - 4x - 4z &= -43\\ x + y + 10z &= 103\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\\&(3)\end{aligned}$$

Subtract (3) from (1), $11y - 8z = -17.$ (4)

Multiply (3) by 4, $4x + 4y + 40z = 412.$ (5)

Add (2) and (5), $9y + 36z = 369.$ (6)

Divide (6) by 9, $y + 4z = 41.$ (7)

Multiply (7) by 2, $2y + 8z = 82.$ (8)

Add (4) and (8), $13y = 65.$

$\therefore y = 5.$

Substitute the value of y in (7), $5 + 4z = 41.$

Transpose and combine, $4z = 36.$

$\therefore\ z = 9.$

Substitute the values of y and z in (1),

$x + 60 + 18 = 86.$

Transpose and combine, $x = 8.$

Therefore, $x = 8$, $y = 5$, and $z = 9$.

12. Solve

$$\left.\begin{aligned} 6x + 2y - 7z &= 15 \\ 7x - 7y - 11z &= -4 \\ 3x + 9y + 8z &= 35 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \\ &(3) \end{aligned}$$

Multiply (1) by 7, $42x + 14y - 49z = 105.$ (4)

Multiply (2) by 2, $14x - 14y - 22z = -8.$ (5)

Add (4) and (5), $56x - 71z = 97.$ (6)

Multiply (1) by 9, $54x + 18y - 63z = 135.$ (7)

Multiply (3) by 2, $6x + 18y + 16z = 70.$ (8)

Subtract (8) from (7), $48x - 79z = 65.$ (9)

Multiply (6) by 6, $336x - 426z = 582.$ (10)

Multiply (9) by 7, $336x - 553z = 455.$ (11)

Subtract (11) from (10), $127z = 127.$

$\therefore\ z = 1.$

Substitute the value of z in (6), $56x - 71 = 97.$

Transpose and combine, $56x = 168.$

$\therefore\ x = 3.$

Substitute the values of x and z in (1),

$18 + 2y - 7 = 15.$

Transpose and combine, $2y = 4.$

$\therefore\ y = 2.$

Therefore, $x = 3$, $y = 2$, and $z = 1$.

13. Solve

$$\left.\begin{aligned} 15x - 18y + 8z &= 15 \\ 25x + 15y + 16z &= 91 \\ 20x - 21y - 12z &= -27 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \\ &(3) \end{aligned}$$

Multiply (1) by 2, $30x - 36y + 16z = 30.$ (4)

Subtract (2) from (4), $5x - 51y = -61.$ (5)

Multiply (1) by 3, $45x - 54y + 24z = 45.$ (6)

Multiply (3) by 2, $40x - 42y - 24z = -54.$ (7)

Add (6) and (7), $85x - 96y = -9.$ (8)

Multiply (5) by 17, $85x - 867y = -1037.$ (9)

Subtract (9) from (8), $771y = 1028.$

$\therefore\ y = 1\frac{1}{3}.$

Substitute the value of y in (5), $5x - 68 = -61$.

Transpose and combine, $5x = 7$.

$$\therefore\ x = 1\tfrac{2}{5}.$$

Substitute the values of x and y in (1),

$$21 - 24 + 8z = 15.$$

Transpose and combine, $8z = 18$.

$$\therefore\ z = 2\tfrac{1}{4}.$$

Therefore, $x = 1\frac{2}{5}$, $y = 1\frac{1}{3}$, and $z = 2\frac{1}{4}$.

14. Solve

$$\left.\begin{aligned} ax + by - cz &= 2ab \\ by + cz - ax &= 2bc \\ cz + ax - by &= 2ac \end{aligned}\right\} \qquad \begin{aligned} &(1)\\ &(2)\\ &(3) \end{aligned}$$

Add (1) and (2), $2by = 2ab + 2bc$.

$$\therefore\ y = a + c.$$

Add (2) and (3), $2cz = 2ac + 2bc$.

$$\therefore\ z = a + b.$$

Add (1) and (3), $2ax = 2ab + 2ac$.

$$\therefore\ x = b + c.$$

Therefore, $x = b + c$, $y = a + c$, and $z = a + b$.

15. Solve

$$\left.\begin{aligned} x + ay + a^2z + a^3 &= 0 \\ x + by + b^2z + b^3 &= 0 \\ x + cy + c^2z + c^3 &= 0 \end{aligned}\right\} \qquad \begin{aligned} &(1)\\ &(2)\\ &(3) \end{aligned}$$

Subtract (2) from (1), $(a - b)y + (a^2 - b^2)z = -(a^3 - b^3)$. (4)

Divide (4) by $a - b$, $y + (a + b)z = -a^2 - ab - b^2$. (5)

Subtract (3) from (2), $(b - c)y + (b^2 - c^2)z = -(b^3 - c^3)$. (6)

Divide (6) by $b - c$, $y + (b + c)z = -b^2 - bc - c^2$. (7)

Subtract (7) from (5), $(a - c)z = bc + c^2 - a^2 - ab$.

$$\therefore\ z = \frac{bc + c^2 - a^2 - ab}{a - c} = -(a + b + c).$$

Substitute the value of z in (5),

$$y - (a + b)(a + b + c) = -a^2 - ab - b^2.$$

Transpose, $y = (a + b)(a + b + c) - a^2 - ab - b^2$.

Expand and combine, $y = ab + ac + bc$.

Substitute the values of y and z in (1),

$$x + a(ab + ac + bc) - a^2(a + b + c) + a^3 = 0.$$

Expand, $x + a^2b + a^2c + abc - a^3 - a^2b - a^2c + a^3 = 0$.

Combine, $x + abc = 0$.

Transpose, $x = -abc$.

Therefore, $x = -abc$, $y = ab + ac + bc$, $z = -(a + b + c)$.

16. Solve

$$\left.\begin{aligned} x + y + z &= a + b + c \\ bx + cy + az &= a^2 + b^2 + c^2 \\ cx + ay + bz &= a^2 + b^2 + c^2 \end{aligned}\right\} \quad \begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

Multiply (1) by b, $bx + by + bz = ab + b^2 + bc.$ (4)

Subtract (2) from (4), $(b-c)y + (b-a)z = ab - a^2 + bc - c^2.$ (5)

Multiply (1) by c, $cx + cy + cz = ac + bc + c^2.$ (6)

Subtract (6) from (3), $(a-c)y + (b-c)z = a^2 + b^2 - ac - bc.$ (7)

Multiply (5) by $a - c$,

$(a-c)(b-c)y + (b-a)(a-c)z = (a-c)(ab - a^2 + bc - c^2).$ (8)

Multiply (7) by $b - c$,

$(a-c)(b-c)y + (b-c)^2 z = (b-c)(a^2 + b^2 - ac - bc).$ (9)

Subtract (8) from (9),

$$[(b-c)^2 + (a-b)(a-c)]z$$
$$= (b-c)(a^2 + b^2 - ac - bc) - (a-c)(ab - a^2 + bc - c^2).$$

Simplify,

$$(b^2 + c^2 + a^2 - ab - ac - bc)z$$
$$= b^3 - c^3 + a^3 - abc - 2b^2c - 2a^2c + 2ac^2 + 2bc^2.$$

$$\therefore z = \frac{b^3 - c^3 + a^3 - abc - 2b^2c - 2a^2c + 2ac^2 + 2bc^2}{b^2 + c^2 + a^2 - ab - ac - bc} = a + b - c.$$

Substitute the value of z in (5),

$$(b-c)y + (b-a)(a+b-c) = ab - a^2 + bc - c^2.$$

Transpose, $(b-c)y = (a-b)(a+b-c) + ab - a^2 + bc - c^2.$

Expand and combine, $(b-c)y = ab - ac - b^2 + 2bc - c^2.$

Factor, $(b-c)y = a(b-c) - (b-c)^2 = (b-c)(a-b+c).$

$$\therefore y = a - b + c.$$

Substitute the values of y and z in (1),

$$x + a - b + c + a + b - c = a + b + c.$$

Transpose and combine, $x = b + c - a.$

Therefore, $x = b + c - a$, $y = a + c - b$, and $z = a + b - c$.

17. Solve

$$\left.\begin{aligned} (a+b)x + (a-b)z &= 2bc \\ (b+c)y + (b-c)x &= 2ac \\ (c+a)z + (c-a)y &= 2ab \end{aligned}\right\} \quad \begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

Multiply (1) by $(b-c)$,

$(a+b)(b-c)x + (a-b)(b-c)z = 2bc(b-c).$ (4)

Multiply (2) by $(a+b)$,

$(a+b)(b-c)x + (a+b)(b+c)y = 2ac(a+b).$ (5)

Subtract (4) from (5),

$(a+b)(b+c)y - (a-b)(b-c)z = 2ac(a+b) - 2bc(b-c).$ (6)

Multiply (6) by $c - a$,

$$\begin{aligned}&(a+b)(b+c)(c-a)y-(a-b)(b-c)(c-a)z\\&\qquad=2ac(a+b)(c-a)-2bc(b-c)(c-a).\end{aligned} \tag{7}$$

Multiply (3) by $(a+b)(b+c)$,

$$\begin{aligned}&(a+b)(b+c)(c-a)y+(a+b)(b+c)(c+a)z\\&\qquad=2ab(a+b)(b+c).\end{aligned} \tag{8}$$

Subtract (7) from (8),

$$\begin{aligned}&[(a+b)(b+c)(c+a)+(a-b)(b-c)(c-a)]z\\&\qquad=2[ab(a+b)(b+c)-ac(a+b)(c-a)+bc(b-c)(c-a)].\end{aligned}$$

Expand,

$$\begin{aligned}&(a^2b+ab^2+a^2c+2abc+b^2c+ac^2+bc^2-a^2b+ab^2+a^2c-b^2c\\&\qquad-ac^2+bc^2)z\\&\qquad=2(a^2b^2+ab^3+a^2bc+ab^2c+a^3c+a^2bc-a^2c^2-abc^2+b^2c^2\\&\qquad-bc^3-ab^2c+abc^2).\end{aligned}$$

Combine,

$$\begin{aligned}&2(ab^2+a^2c+abc+bc^2)z\\&\qquad=2(a^2b^2+ab^3+2a^2bc+a^3c-a^2c^2+b^2c^2-bc^3).\end{aligned}$$

$$\therefore z=\frac{a^2b^2+ab^3+2a^2bc+a^3c-a^2c^2+b^2c^2-bc^3}{ab^2+a^2c+abc+bc^2}=a+b-c.$$

Substitute the value of z in (3),

$$(c+a)(a+b-c)+(c-a)y=2ab.$$

Transpose, $(c-a)y=2ab-(c+a)(a+b-c).$

Expand and combine, $(c-a)y=ab-a^2-bc+c^2.$

$$\therefore y=\frac{ab-a^2-bc+c^2}{c-a}=a-b+c.$$

Substitute the value of z in (1),

$$(a+b)x+(a-b)(a+b-c)=2bc.$$

Transpose, $(a+b)x=2bc-(a-b)(a+b-c).$

Expand and combine, $(a+b)x=bc-a^2+ac+b^2.$

$$\therefore x=\frac{bc-a^2+ac+b^2}{a+b}=b+c-a.$$

Therefore, $x=b+c-a$, $y=a+c-b$, and $z=a+b-c$.

18. Solve

$$\left.\begin{aligned}x+y+z&=a+b+c\\ax+by+cz&=ab+bc+ac\\(b-c)x+(c-a)y&=(b-a)z\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\\&(3)\end{aligned}$$

Multiply (1) by a, $ax+ay+az=a^2+ab+ac.$ (4)

Subtract (2) from (4), $(a-b)y+(a-c)z=a^2-bc. \qquad (5)$

Multiply (1) by $b-c$,

$$(b-c)x+(b-c)y+(b-c)z=ab+b^2-ac-c^2. \tag{6}$$

Transpose (3),

$$(b-c)x+(c-a)y+(a-b)z=0. \tag{7}$$

Subtract (7) from (6),

$$(a+b-2c)y-(a+c-2b)z=ab+b^2-ac-c^2. \tag{8}$$

Multiply (5) by $a+b-2c$,

$$\begin{aligned}(a-b)(a+b-2c)y&+(a-c)(a+b-2c)z\\ &=(a^2-bc)(a+b-2c).\end{aligned} \tag{9}$$

Multiply (8) by $a-b$,

$$\begin{aligned}(a-b)(a+b-2c)y&-(a-b)(a+c-2b)z\\ &=(a-b)(ab+b^2-ac-c^2).\end{aligned} \tag{10}$$

Subtract (10) from (9),

$$\begin{aligned}[(a-c)(a+b-2c)&+(a-b)(a+c-2b)]z\\ &=(a^2-bc)(a+b-2c)-(a-b)(ab+b^2-ac-c^2).\end{aligned}$$

Simplify, $2(a^2-ab-ac-bc+b^2+c^2)z$

$$=a^3-2abc-b^2c-a^2c+bc^2+ac^2+b^3.$$

$$\therefore\ z=\frac{a^3-2abc-b^2c-a^2c+bc^2+ac^2+b^3}{2(a^2-ab-ac-bc+b^2+c^2)}=\frac{a+b}{2}.$$

Substitute the value of z in (5),

$$(a-b)y+\frac{(a-c)(a+b)}{2}=a^2-bc.$$

Multiply by 2, $2(a-b)y+(a-c)(a+b)=2a^2-2bc.$

Transpose, $2(a-b)y=2a^2-2bc-(a-c)(a+b).$

Expand and combine, $2(a-b)y=a^2-bc-ab+ac.$

Factor, $2(a-b)y=(a+c)(a-b).$

$$\therefore\ y=\frac{a+c}{2}.$$

Substitute the values of y and z in (1),

$$x+\frac{a+c}{2}+\frac{a+b}{2}=a+b+c.$$

Multiply by 2, $2x+a+c+a+b=2a+2b+2c.$

Transpose and combine, $2x=b+c.$

$$\therefore\ x=\frac{b+c}{2}.$$

Therefore, $x=\dfrac{b+c}{2}$, $y=\dfrac{a+c}{2}$, and $z=\dfrac{a+b}{2}$.

19. Solve

$$\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}+\frac{z}{c}&=1\\ \frac{x}{a}+\frac{y}{c}+\frac{z}{b}&=1\\ \frac{x}{b}+\frac{y}{a}+\frac{z}{c}&=1\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\\&(3)\end{aligned}$$

Subtract (1) from (2), $\dfrac{(b-c)y}{bc}+\dfrac{(c-b)z}{bc}=0.$ (4)

Multiply (1) by $\dfrac{1}{b}$, $\dfrac{x}{ab}+\dfrac{y}{b^2}+\dfrac{z}{bc}=\dfrac{1}{b}.$ (5)

Multiply (3) by $\dfrac{1}{a}$, $\dfrac{x}{ab}+\dfrac{y}{a^2}+\dfrac{z}{ac}=\dfrac{1}{a}.$ (6)

Subtract (6) from (5), $\dfrac{(a^2-b^2)y}{a^2b^2}+\dfrac{(a-b)z}{abc}=\dfrac{a-b}{ab}.$ (7)

Multiply (4) by $\dfrac{bc}{b-c}$, $y-z=0.$ (8)

Multiply (7) by $\dfrac{ab}{a-b}$, $\dfrac{(a+b)y}{ab}+\dfrac{z}{c}=1.$ (9)

Transpose (8), $y=z.$ (10)

Substitute the value of y in (9),

$$\frac{(a+b)z}{ab}+\frac{z}{c}=1.$$

Multiply by abc, $(ac+bc)z+abz=abc.$

Factor, $(ab+ac+bc)z=abc.$

$$\therefore z=\frac{abc}{ab+ac+bc}.$$

Substitute the value of z in (10), $y=\dfrac{abc}{ab+ac+bc}.$

Substitute the values of y and z in (1),

$$\frac{x}{a}+\frac{ac}{ab+ac+bc}+\frac{ab}{ab+ac+bc}=1.$$

Multiply by $a(ab+ac+bc)$,

$$(ab+ac+bc)x+a^2c+a^2b=a^2b+a^2c+abc.$$

Transpose and combine, $(ab+ac+bc)x=abc.$

$$\therefore x=\frac{abc}{ab+ac+bc}.$$

Therefore, $x=\dfrac{abc}{ab+ac+bc}$, $y=\dfrac{abc}{ab+ac+bc}$, and $z=\dfrac{abc}{ab+ac+bc}$.

20. Solve

$$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}-\frac{c}{z}&=m\\ \frac{a}{x}-\frac{b}{y}+\frac{c}{z}&=n\\ -\frac{a}{x}+\frac{b}{y}+\frac{c}{z}&=p\end{aligned}\right\} \quad \begin{aligned}&(1)\\ &\\ &(3)\end{aligned}$$

Add (1) and (2), $\frac{2a}{x}=m+n.$

$$\therefore x=\frac{2a}{m+n}.$$

Add (1) and (3), $\frac{2b}{y}=m+p.$

$$\therefore y=\frac{2b}{m+p}.$$

Add (2) and (3), $\frac{2c}{z}=n+p.$

$$\therefore z=\frac{2c}{n+p}.$$

Therefore, $x=\frac{2a}{m+n}$, $y=\frac{2b}{m+p}$, and $z=\frac{2c}{n+p}$.

21. Solve

$$\left.\begin{aligned}21x+15y+16z&=-13\\ 14x+25y-13z&=15\tfrac{1}{12}\\ 17x-10y-19z&=4\tfrac{7}{12}\end{aligned}\right\} \quad \begin{aligned}&(1)\\ &(2)\\ &(3)\end{aligned}$$

Multiply (1) by 5, $105x+75y+80z=-65.$ (4)

Multiply (2) by 3, $42x+75y-39z=45\tfrac{1}{4}.$ (5)

Subtract (5) from (4), $63x+119z=-110\tfrac{1}{4}.$ (6)

Divide (6) by 7, $9x+17z=-15\tfrac{3}{4}.$ (7)

Multiply (1) by 2, $42x+30y+32z=-26.$ (8)

Multiply (3) by 3, $51x-30y-57z=13\tfrac{3}{4}.$ (9)

Add (8) and (9), $93x-25z=-12\tfrac{1}{4}.$ (10)

Multiply (7) by 31, $279x+527z=-488\tfrac{1}{4}.$ (11)

Multiply (10) by 3, $279x-75z=-36\tfrac{3}{4}.$ (12)

Subtract (12) from (11), $602z=-451\tfrac{1}{2}.$

Multiply by 2, $1204z=-903.$

Divide by 301, $4z=-3.$

$$\therefore z=-\tfrac{3}{4}.$$

Substitute the value of z in (7),

$$9x - 12\tfrac{3}{4} = -15\tfrac{3}{4}. \qquad (9)$$

Transpose and combine, $$9x = -3.$$

$$\therefore\ x = -\tfrac{1}{3}.$$

Substitute the values of x and z in (1),

$$-7 + 15y - 12 = -13.$$

Transpose and combine, $$15y = 6.$$

$$\therefore\ y = \tfrac{2}{5}.$$

Therefore, $x = -\frac{1}{3}$, $y = \frac{2}{5}$, and $z = -\frac{3}{4}$.

22. Solve

$$\left.\begin{aligned} \frac{x}{m+n} + \frac{y}{n-p} + \frac{z}{m+p} &= 2p \\ \frac{x}{m-n} - \frac{y}{n-p} + \frac{z}{p-m} &= 2m \\ \frac{x}{m-n} - \frac{y}{n-p} - \frac{z}{m+p} &= 2m - 2p \end{aligned}\right\} \qquad \begin{aligned} (1) \\ (2) \\ (3) \end{aligned}$$

Subtract (3) from (2), $$\frac{2pz}{p^2 - m^2} = 2p.$$

Divide by $2p$, $$\frac{z}{p^2 - m^2} = 1.$$

$$\therefore\ z = p^2 - m^2.$$

Substitute the value of z in (1),

$$\frac{x}{m+n} + \frac{y}{n-p} + p - m = 2p. \qquad (4)$$

Substitute the value of z in (2),

$$\frac{x}{m-n} - \frac{y}{n-p} + p + m = 2m. \qquad (5)$$

Transpose and combine (4), $$\frac{x}{m+n} + \frac{y}{n-p} = m + p. \qquad (6)$$

Transpose and combine (5), $$\frac{x}{m-n} - \frac{y}{n-p} = m - p. \qquad (7)$$

Add (6) and (7), $$\frac{2mx}{m^2 - n^2} = 2m.$$

Divide by $2m$, $$\frac{x}{m^2 - n^2} = 1.$$

$$\therefore\ x = m^2 - n^2.$$

Substitute the values of x and z in (1),

$$m - n + \frac{y}{n-p} + p - m = 2p.$$

Transpose and combine, $\frac{y}{n-p} = n + p.$

Multiply by $n - p$, $y = n^2 - p^2.$

Therefore, $x = m^2 - n^2$, $y = n^2 - p^2$, and $z = p^2 - m^2$.

23. Solve
$$\left.\begin{aligned}\frac{x}{n+p} + \frac{y}{p+m} &= n - m \quad (1)\\ \frac{x}{n+p} + \frac{z}{m+n} &= m - p \quad (2)\\ \frac{y}{p+m} + \frac{z}{m+n} &= p - n \quad (3)\end{aligned}\right\}$$

Subtract (2) from (1), $\frac{y}{p+m} - \frac{z}{m+n} = n - 2m + p.$ (4)

Add (3) and (4), $\frac{2y}{p+m} = 2p - 2m.$ (5)

Subtract (4) from (3), $\frac{2z}{m+n} = 2m - 2n.$ (6)

Multiply (5) by $\frac{p+m}{2}$, $y = p^2 - m^2.$

Multiply (6) by $\frac{m+n}{2}$, $z = m^2 - n^2.$

Substitute the value of y in (1),

$$\frac{x}{n+p} + p - m = n - m.$$

Transpose and combine, $\frac{x}{n+p} = n - p.$

$$\therefore x = n^2 - p^2.$$

Therefore, $x = n^2 - p^2$, $y = p^2 - m^2$, and $z = m^2 - n^2$.

24. Solve
$$\left.\begin{aligned}\frac{6}{x} + \frac{4}{y} + \frac{5}{z} &= 4 \quad (1)\\ \frac{3}{x} + \frac{8}{y} + \frac{5}{z} &= 4 \quad (2)\\ \frac{9}{x} + \frac{12}{y} - \frac{10}{z} &= 4 \quad (3)\end{aligned}\right\}$$

Subtract (2) from (1), $\frac{3}{x} - \frac{4}{y} = 0.$

$$\therefore 4x = 3y.$$

$$\therefore x = \frac{3y}{4}. \quad (4)$$

Substitute the value of x in (2), $\frac{4}{y}+\frac{8}{y}+\frac{5}{z}=4.$ (5)

Substitute the value of x in (3), $\frac{12}{y}+\frac{12}{y}-\frac{10}{z}=4.$ (6)

Combine (5), $\frac{12}{y}+\frac{5}{z}=4.$ (7)

Combine (6), $\frac{24}{y}-\frac{10}{z}=4.$ (8)

Divide (8) by 2, $\frac{12}{y}-\frac{5}{z}=2.$ (9)

Add (7) and (9), $\frac{24}{y}=6.$

$\therefore y=4.$

Subtract (9) from (7), $\frac{10}{z}=2.$

$\therefore z=5.$

Substitute the value of y in (4), $x=3.$

Therefore, $x=3$, $y=4$, and $z=5$.

25. Solve

$$\left.\begin{array}{r}x+2y+3z+4u=2\\2x+3y+4z+5u=3\\6x+7y-8z+9u=3\\64x+16y+4z+u=25\end{array}\right\}\begin{array}{r}(1)\\(2)\\(3)\\(4)\end{array}$$

Multiply (1) by 4, $4x+8y+12z+16u=8.$ (5)
Multiply (2) by 3, $6x+9y+12z+15u=9.$ (6)
Subtract (5) from (6), $2x+y-u=1.$ (7)
Multiply (2) by 2, $4x+6y+8z+10u=6.$ (8)
Add (3) and (8), $10x+13y+19u=9.$ (9)
Multiply (4) by 2, $128x+32y+8z+2u=50.$ (10)
Add (3) and (10), $134x+39y+11u=53.$ (11)
Multiply (7) by 19, $38x+19y-19u=19.$ (12)
Add (9) and (12), $48x+32y=28.$ (13)
Divide (13) by 4, $12x+8y=7.$ (14)
Multiply (7) by 11, $22x+11y-11u=11.$ (15)
Add (11) and (15), $156x+50y=64.$ (16)
Divide (16) by 2, $78x+25y=32.$ (17)
Multiply (14) by 25, $300x+200y=175.$ (18)
Multiply (17) by 8, $624x+200y=256.$ (19)

Subtract (18) from (19), $324x = 81.$

$\therefore x = \frac{1}{4}.$

Substitute the value of x in (14), $3 + 8y = 7.$

Transpose and combine, $8y = 4.$

$\therefore y = \frac{1}{2}.$

Substitute the values of x and y in (7),

$\frac{1}{2} + \frac{1}{2} - u = 1.$

Transpose and combine, $u = 0.$

Substitute the values of x, y, and u in (1),

$\frac{1}{4} + 1 + 3z + 0 = 2.$

Transpose and combine, $3z = \frac{3}{4}.$

$\therefore z = \frac{1}{4}.$

Therefore, $x = \frac{1}{4}$, $y = \frac{1}{2}$, $z = \frac{1}{4}$, and $u = 0.$

26. Solve

$$\left.\begin{aligned} 2x - 9y + 3z - 10u &= 121 \\ x + 7y - z - u &= 583 \\ 3x + 2y + 5z + 2u &= 255 \\ 4x - 6y - 2z - 9u &= 516 \end{aligned}\right\} \quad \begin{aligned} &(1)\\ &(2)\\ &(3)\\ &(4)\end{aligned}$$

Multiply (2) by 2, $2x + 14y - 2z - 2u = 1166.$ (5)

Subtract (1) from (5), $23y - 5z + 8u = 1045.$ (6)

Multiply (2) by 3, $3x + 21y - 3z - 3u = 1749.$ (7)

Subtract (3) from (7), $19y - 8z - 5u = 1494.$ (8)

Multiply (2) by 4, $4x + 28y - 4z - 4u = 2332.$ (9)

Subtract (4) from (9), $34y - 2z + 5u = 1816.$ (10)

Multiply (6) by 5, $115y - 25z + 40u = 5225.$ (11)

Multiply (8) by 8, $152y - 64z - 40u = 11952.$ (12)

Add (11) and (12), $267y - 89z = 17177.$ (13)

Divide (13) by 89, $3y - z = 193.$ (14)

Add (8) and (10), $53y - 10z = 3310.$ (15)

Multiply (14) by 10, $30y - 10z = 1930.$ (16)

Subtract (16) from (15), $23y = 1380.$

$\therefore y = 60.$

Substitute the value of y in (14), $180 - z = 193.$

Transpose and combine, $z = -13.$

Substitute the values of y and z in (10),

$2040 + 26 + 5u = 1816.$

Transpose and combine, $5u = -250.$

$\therefore u = -50.$

Substitute the values of y, z, and u in (2),

$$x + 420 + 13 + 50 = 583.$$

Transpose and combine, $x = 100.$

Therefore, $x = 100$, $y = 60$, $z = -13$, and $u = -50$.

27. Solve

$$\left.\begin{aligned} 3x + 4y + 5z + 6u &= 67 \\ 2x - 5y + 9z - 8u &= -127 \\ 4x - 3y + 3z + 4u &= 73 \\ 6x - 10y - 6z + 3u &= 75 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \\ &(3) \\ &(4) \end{aligned}$$

Multiply (3) by 2, $8x - 6y + 6z + 8u = 146.$ (5)

Add (2) and (5), $10x - 11y + 15z = 19.$ (6)

Multiply (4) by 2, $12x - 20y - 12z + 6u = 150.$ (7)

Subtract (1) from (7), $9x - 24y - 17z = 83.$ (8)

Multiply (3) by 3, $12x - 9y + 9z + 12u = 219.$ (9)

Multiply (4) by 4, $24x - 40y - 24z + 12u = 300.$ (10)

Subtract (9) from (10), $12x - 31y - 33z = 81.$ (11)

Multiply (6) by 9, $90x - 99y + 135z = 171.$ (12)

Multiply (8) by 10, $90x - 240y - 170z = 830.$ (13)

Subtract (13) from (12), $141y + 305z = -659.$ (14)

Multiply (8) by 4, $36x - 96y - 68z = 332.$ (15)

Multiply (11) by 3, $36x - 93y - 99z = 243.$ (16)

Subtract (15) from (16), $3y - 31z = -89.$ (17)

Multiply (17) by 47, $141y - 1457z = -4183.$ (18)

Subtract (18) from (14), $1762z = 3524.$

$$\therefore z = 2.$$

Substitute the value of z in (17), $3y - 62 = -89.$

Transpose and combine, $3y = -27.$

$$\therefore y = -9.$$

Substitute the values of y and z in (6),

$$10x + 99 + 30 = 19.$$

Transpose and combine, $10x = -110.$

$$\therefore x = -11.$$

Substitute the values of x, y, and z in (4),

$$-66 + 90 - 12 + 3u = 75.$$

Transpose and combine, $3u = 63.$

$$\therefore u = 21.$$

Therefore, $x = -11$, $y = -9$, $z = 2$, and $u = 21$.

28. Solve

$$\left.\begin{array}{l}\frac{x}{3}+\frac{y}{5}+\frac{2z}{7}=58\\[4pt] \frac{5x}{4}+\frac{y}{6}+\frac{z}{3}=76\\[4pt] \frac{x}{2}+\frac{3z}{8}+\frac{u}{5}=79\\[4pt] \frac{y}{5}+\frac{z}{2}+\frac{u}{25}=92\end{array}\right\}\qquad\begin{array}{r}(1)\\[4pt](2)\\[4pt](3)\\[4pt](4)\end{array}$$

Multiply (3) by $\frac{1}{5}$,
$$\frac{x}{10}+\frac{3z}{40}+\frac{u}{25}=\frac{79}{5}. \tag{5}$$

Subtract (4) from (5),
$$\frac{x}{10}-\frac{y}{5}-\frac{17z}{40}=-\frac{381}{5}. \tag{6}$$

Multiply (1) by $\frac{1}{6}$,
$$\frac{x}{18}+\frac{y}{30}+\frac{z}{21}=\frac{29}{3}. \tag{7}$$

Multiply (2) by $\frac{1}{5}$,
$$\frac{x}{4}+\frac{y}{30}+\frac{z}{15}=\frac{76}{5}. \tag{8}$$

Subtract (7) from (8),
$$\frac{7x}{36}+\frac{2z}{105}=\frac{83}{15}. \tag{9}$$

Multiply (9) by 3,
$$\frac{7x}{12}+\frac{2z}{35}=\frac{83}{5}. \tag{10}$$

Add (1) and (6),
$$\frac{13x}{30}-\frac{39z}{280}=-\frac{91}{5}. \tag{11}$$

Multiply (10) by 39,
$$\frac{91x}{4}+\frac{78z}{35}=\frac{3237}{5}. \tag{12}$$

Multiply (11) by 16,
$$\frac{104x}{15}-\frac{78z}{35}=-\frac{1456}{5}. \tag{13}$$

Add (12) and (13),
$$\frac{1781x}{60}=\frac{1781}{5}.$$

Divide by 1781,
$$\frac{x}{60}=\frac{1}{5}.$$

Multiply by 60,
$$x=12.$$

Substitute the value of x in (10), $7+\frac{2z}{35}=\frac{83}{5}.$

Multiply by 35, $245+2z=581.$

Transpose and combine, $2z=336.$

$$\therefore\ z=168.$$

Substitute the values of x and z in (1),

$$4+\frac{y}{5}+48=58.$$

Transpose and combine, $\frac{y}{5} = 6.$

$\therefore\ y = 30.$

Substitute the values of x and z in (3),

$$6 + 63 + \frac{u}{5} = 79.$$

Transpose and combine, $\frac{u}{5} = 10.$

$\therefore\ u = 50.$

Therefore, $x = 12$, $y = 30$, $z = 168$, and $u = 50$.

Exercise 77. Page 185

1. If A gives B $17, B will have three times as much money as A. If B gives A $17, A will have $5 more than four times as much money as B. How much has each?

Let $x =$ the number of dollars A has,
and $y =$ the number of dollars B has.

Then, if A gives B $17,

$x - 17 =$ the number of dollars A will have,
and $y + 17 =$ the number of dollars B will have.

$$\therefore\ y + 17 = 3(x - 17). \quad (1)$$

If B gives A $17,

$x + 17 =$ the number of dollars A will have,
and $y - 17 =$ the number of dollars B will have.

$$\therefore\ x + 17 = 4(y - 17) + 5. \quad (2)$$

Expand (1), $y + 17 = 3x - 51.$ (3)
Expand (2), $x + 17 = 4y - 68 + 5.$ (4)
Transpose and combine (3), $3x - y = 68.$ (5)
Transpose and combine (4), $x - 4y = -80.$ (6)
Multiply (6) by 3, $3x - 12y = -240.$ (7)
Subtract (7) from (5), $11y = 308.$
$\therefore\ y = 28.$
Substitute the value of y in (6), $x - 112 = -80.$
Transpose and combine, $x = 32.$

Therefore, A has $32, and B has $28.

2. If the smaller of two numbers is divided by the larger, the quotient is 0.32 and the remainder 0.012; but if the larger number is divided by the smaller, the quotient is 3.11 and the remainder 0.0042. Find the numbers.

Let $x =$ the smaller number,
and $y =$ the larger number.

$$\therefore \frac{x - 0.012}{y} = 0.32, \quad (1)$$

and
$$\frac{y - 0.0042}{x} = 3.11. \quad (2)$$

Multiply (1) by y, $x - 0.012 = 0.32\,y.$ (3)
Multiply (2) by x, $y - 0.0042 = 3.11\,x.$ (4)
Transpose (3), $x - 0.32\,y = 0.012.$ (5)
Transpose (4), $-3.11\,x + y = 0.0042.$ (6)
Multiply (6) by 0.32, $-0.9952\,x + 0.32\,y = 0.001344.$ (7)
Add (5) and (7), $0.0048\,x = 0.013344.$
$\therefore x = 2.78.$

Substitute the value of x in (5),
$2.78 - 0.32\,y = 0.012.$
Transpose and combine, $-0.32\,y = -2.768.$
$\therefore y = 8.65.$

Therefore, the required numbers are 2.78 and 8.65.

3. If B gave A $27, they would have equal sums of money; but if A gave B $43, A would then have half as much money as B. How much has each?

Let $x =$ the number of dollars A has,
and $y =$ the number of dollars B has.
$\therefore y - 27 = x + 27,$ (1)
and $y + 43 = 2(x - 43).$ (2)
Expand (2), $y + 43 = 2\,x - 86.$ (3)
Transpose and combine (1), $x - y = -54.$ (4)
Transpose and combine (3), $2\,x - y = 129.$ (5)
Subtract (4) from (5), $x = 183.$
Substitute the value of x in (4), $183 - y = -54.$
Transpose and combine, $-y = -237.$
$\therefore y = 237.$

Therefore, A has $183, and B has $237.

4. If the larger of two numbers is divided by the smaller, the quotient is 4 and the remainder 9; but if twenty times the smaller number is divided by twice the larger, the quotient is 2 and the remainder 152. Find the numbers.

Let $x =$ the larger number,
and $y =$ the smaller number.

$$\therefore \frac{x-9}{y} = 4, \tag{1}$$

and
$$\frac{20y-152}{2x} = 2. \tag{2}$$

Multiply (1) by y, $x - 9 = 4y$. (3)
Multiply (2) by $2x$, $20y - 152 = 4x$. (4)
Divide (4) by 4, $5y - 38 = x$. (5)
Transpose (3), $x - 4y = 9$. (6)
Transpose (5), $-x + 5y = 38$. (7)
Add (6) and (7), $y = 47$.
Substitute the value of y in (3), $x - 9 = 188$.
Transpose and combine, $x = 197$.

Therefore, the required numbers are 197 and 47.

5. If the smaller of two numbers is divided by the larger, the quotient is 0.37 and the remainder 0.01413; but if the larger number is divided by the smaller, the quotient is 2.69 and the remainder 0.00077. Find the numbers.

Let $x =$ the smaller number,
and $y =$ the larger number.

$$\therefore \frac{x-0.01413}{y} = 0.37, \tag{1}$$

and
$$\frac{y-0.00077}{x} = 2.69. \tag{2}$$

Multiply (1) by y, $x - 0.01413 = 0.37y$. (3)
Multiply (2) by x, $y - 0.00077 = 2.69x$. (4)
Transpose (3), $x - 0.37y = 0.01413$. (5)
Transpose (4), $-2.69x + y = 0.00077$. (6)
Multiply (6) by 0.37, $-0.9953x + 0.37y = 0.0002849$. (7)
Add (5) and (7), $0.0047x = 0.0144149$.
$\therefore x = 3.067$.
Substitute the value of x in (5), $3.067 - 0.37y = 0.01413$.
Transpose and combine, $-0.37y = -3.05287$.
$\therefore y = 8.251$.

Therefore, the required numbers are 8.251 and 3.067.

6. If A gives B $17, A will have twice as much money as B; but if A gives B $33, B will have one dollar less than A. How much has each?

Let $x =$ the number of dollars A has,
and $y =$ the number of dollars B has.

$\therefore x - 17 = 2(y + 17),$ (1)
and $x - 33 = y + 33 + 1.$ (2)
Expand (1), $x - 17 = 2y + 34.$ (3)
Transpose and combine (3), $x - 2y = 51.$ (4)
Transpose and combine (2), $x - y = 67.$ (5)
Subtract (4) from (5), $y = 16.$
Substitute the value of y in (5), $x - 16 = 67.$
Transpose and combine, $x = 83.$

Therefore, A has $83, and B has $16.

7. The value of a certain fraction becomes $\frac{3}{4}$ when 7 is added to the numerator, and becomes $\frac{1}{2}$ when 2 is subtracted from the denominator. Find the fraction.

Let $x =$ the numerator of the fraction,
and $y =$ the denominator of the fraction.
Then $\frac{x}{y} =$ the fraction.

$$\therefore \frac{x+7}{y} = \frac{3}{4}, \tag{1}$$

and

$$\frac{x}{y-2} = \frac{1}{2}. \tag{2}$$

Multiply (1) by $4y$, $4x + 28 = 3y.$ (3)
Multiply (2) by $2(y - 2)$, $2x = y - 2.$ (4)
Transpose (3), $4x - 3y = -28.$ (5)
Transpose (4), $2x - y = -2.$ (6)
Multiply (6) by 2, $4x - 2y = -4.$ (7)
Subtract (5) from (7), $y = 24.$
Substitute the value of y in (4), $2x = 24 - 2.$
Combine, $2x = 22.$

$$\therefore x = 11.$$

$$\frac{x}{y} = \frac{11}{24}.$$

Therefore, the required fraction is $\frac{11}{24}$.

8. A certain fraction becomes equal to $\frac{8}{9}$ when 2 is added to the numerator and 1 is subtracted from the denominator, and equal to $\frac{1}{2}$ when 9 is added to the denominator. Find the fraction.

Let $x =$ the numerator of the fraction,
and $y =$ the denominator of the fraction.

Then $\frac{x}{y} =$ the fraction.

$$\therefore \frac{x+2}{y-1} = \frac{8}{9}, \qquad (1)$$

and

$$\frac{x}{y+9} = \frac{1}{2}. \qquad (2)$$

Multiply (1) by $9(y-1)$, $9x + 18 = 8y - 8.$ (3)
Multiply (2) by $2(y+9)$, $2x = y + 9.$ (4)
Transpose and combine (3), $9x - 8y = -26.$ (5)
Transpose (4), $2x - y = 9.$ (6)
Multiply (6) by 8, $16x - 8y = 72.$ (7)
Subtract (5) from (7), $7x = 98.$
$\therefore x = 14.$
Substitute the value of x in (6), $28 - y = 9.$
Transpose and combine, $-y = -19.$
$\therefore y = 19.$

$$\frac{x}{y} = \frac{14}{19}.$$

Therefore, the required fraction is $\frac{14}{19}$.

9. A certain fraction becomes equal to $\frac{1}{4}$ when 4 is added to the denominator, and equal to $\frac{1}{2}$ when 5 is added to the numerator. Find the fraction.

Let $x =$ the numerator of the fraction,
and $y =$ the denominator of the fraction.

Then $\frac{x}{y} =$ the fraction.

$$\therefore \frac{x}{y+4} = \frac{1}{4}, \qquad (1)$$

and

$$\frac{x+5}{y} = \frac{1}{2}. \qquad (2)$$

Multiply (1) by $4(y+4)$, $4x = y + 4.$ (3)
Multiply (2) by $2y$, $2x + 10 = y.$ (4)
Transpose (3), $4x - y = 4.$ (5)
Transpose (4), $2x - y = -10.$ (6)

Subtract (6) from (5), $2x = 14.$

$\therefore x = 7.$

Substitute the value of x in (4), $14 + 10 = y.$

Transpose and combine, $y = 24.$

$$\frac{x}{y} = \frac{7}{24}.$$

Therefore, the required fraction is $\frac{7}{24}$.

10. A certain fraction becomes equal to $\frac{3}{7}$ when 6 is added to the numerator, and equal to $\frac{1}{3}$ when 8 is subtracted from the denominator. Find the fraction.

Let $x =$ the numerator of the fraction,
and $y =$ the denominator of the fraction.

Then $\frac{x}{y} =$ the fraction.

$$\therefore \frac{x+6}{y} = \frac{3}{7}, \qquad (1)$$

and
$$\frac{x}{y-8} = \frac{1}{3}. \qquad (2)$$

Multiply (1) by $7y$, $7x + 42 = 3y.$ (3)

Multiply (2) by $3(y-8)$, $3x = y - 8.$ (4)

Transpose (3), $7x - 3y = -42.$ (5)

Transpose (4) $3x - y = -8.$ (6)

Multiply (6) by 3, $9x - 3y = -24.$ (7)

Subtract (5) from (7), $2x = 18.$

$\therefore x = 9.$

Substitute the value of x in (4), $27 = y - 8.$

Transpose and combine, $-y = -35.$

$\therefore y = 35.$

$$\frac{x}{y} = \frac{9}{35}.$$

Therefore, the required fraction is $\frac{9}{35}$.

11. A certain fraction becomes equal to $\frac{4}{5}$ when 1 is subtracted from the numerator, and equal to $\frac{7}{8}$ when 1 is subtracted from the denominator. Find the fraction.

Let $x =$ the numerator of the fraction,
and $y =$ the denominator of the fraction.

Then $\frac{x}{y} =$ the fraction.

$$\therefore \frac{x-1}{y} = \frac{4}{5}, \qquad (1)$$

and $$\frac{x}{y-1} = \frac{7}{8}. \qquad (2)$$

Multiply (1) by $5y$, $5x - 5 = 4y$. (3)

Multiply (2) by $8(y-1)$, $8x = 7y - 7$. (4)

Transpose (3), $5x - 4y = 5$. (5)

Transpose (4), $8x - 7y = -7$. (6)

Multiply (5) by 7, $35x - 28y = 35$. (7)

Multiply (6) by 4, $32x - 28y = -28$. (8)

Subtract (8) from (7), $3x = 63$.

$$\therefore x = 21.$$

Substitute the value of x in (5), $105 - 4y = 5$.

Transpose and combine, $-4y = -100$.

$$\therefore y = 25.$$

$$\frac{x}{y} = \frac{21}{25}.$$

Therefore, the required fraction is $\frac{21}{25}$.

12. There are two fractions with numerators 3 and 11 respectively whose sum is $1\frac{2}{3}$. If the denominators are interchanged, the sum is 3. Find the fractions.

Let x = the denominator of the first fraction,

and y = the denominator of the second fraction.

Then $\frac{3}{x}$ and $\frac{11}{y}$ = the fractions.

$$\therefore \frac{3}{x} + \frac{11}{y} = 1\frac{2}{3}, \qquad (1)$$

and $$\frac{3}{y} + \frac{11}{x} = 3. \qquad (2)$$

Multiply (1) by 11, $\frac{33}{x} + \frac{121}{y} = \frac{55}{3}$. (3)

Multiply (2) by 3, $\frac{33}{x} + \frac{9}{y} = 9$. (4)

Subtract (4) from (3), $\frac{112}{y} = \frac{28}{3}$. (5)

Divide (5) by 28, $\frac{4}{y} = \frac{1}{3}$.

$$\therefore y = 12.$$

Substitute the value of y in (2), $\frac{1}{4}+\frac{11}{x}=3.$

Transpose and combine, $\frac{11}{x}=\frac{11}{4}.$

$\therefore x=4.$

Therefore, the required fractions are $\frac{3}{4}$ and $\frac{11}{12}$.

13. A certain fraction becomes equal to $\frac{1}{3}$ when 4 is subtracted from each term, and equal to $\frac{1}{2}$ when 4 is added to the numerator and 3 subtracted from the denominator. Find the fraction.

Let $x =$ the numerator of the fraction,
and $y =$ the denominator of the fraction.

Then $\frac{x}{y} =$ the fraction.

$$\therefore \frac{x-4}{y-4}=\frac{1}{3}, \qquad (1)$$

and
$$\frac{x+4}{y-3}=\frac{1}{2}. \qquad (2)$$

Multiply (1) by $3(y-4)$, $3x-12=y-4.$ (3)
Multiply (2) by $2(y-3)$, $2x+8=y-3.$ (4)
Transpose and combine (3), $3x-y=8.$ (5)
Transpose and combine (4), $2x-y=-11.$ (6)
Subtract (6) from (5), $x=19.$
Substitute the value of x in (6), $38-y=-11.$
Transpose and combine, $-y=-49.$

$\therefore y=49.$

$$\frac{x}{y}=\frac{19}{49}.$$

Therefore, the required fraction is $\frac{19}{49}$.

14. The sum of the three digits of a certain number is 16; the sum of the first and second digits is equal to the third digit; and if 594 is added to the number, the order of the digits is reversed. Find the number.

Let $x =$ the digit in the hundreds' place,
$y =$ the digit in the tens' place,
and $z =$ the digit in the units' place.

Then $100x+10y+z =$ the number.

$$\therefore x + y + z = 16, \quad (1)$$

$$x + y = z, \quad (2)$$

and $$100x + 10y + z + 594 = 100z + 10y + x. \quad (3)$$

Transpose and combine (3), $99x - 99z = -594.$ (4)

Divide (4) by 99, $x - z = -6.$ (5)

Substitute in (1) the value of $x + y$ from (2),

$$z + z = 16.$$

Combine, $2z = 16.$

$$\therefore z = 8.$$

Substitute the value of z in (5), $x - 8 = -6.$

Transpose and combine, $x = 2.$

Substitute the values of x and z in (1),

$$2 + y + 8 = 16.$$

Transpose and combine, $y = 6.$

$$100x + 10y + z = 268.$$

Therefore, the required number is 268.

15. The sum of the two digits of a number is 8, and if 36 is added to the number, the order of the digits is reversed. Find the number.

Let $x =$ the digit in the tens' place,
and $y =$ the digit in the units' place.
Then $10x + y =$ the number.

$$\therefore x + y = 8, \quad (1)$$

and $$10x + y + 36 = 10y + x. \quad (2)$$

Transpose and combine (2), $9x - 9y = -36.$ (3)

Divide (3) by 9, $x - y = -4.$ (4)

Add (1) and (4), $2x = 4.$

$$\therefore x = 2.$$

Subtract (4) from (1), $2y = 12.$

$$\therefore y = 6.$$

$$10x + y = 26.$$

Therefore, the required number is 26.

16. The sum of the two digits of a number is 14, and if 36 is subtracted from the number, the order of the digits is reversed. Find the number.

Let $x =$ the digit in the tens' place,
and $y =$ the digit in the units' place.
Then $10x + y =$ the number.

$$\therefore x + y = 14, \quad (1)$$

and $$10x + y - 36 = 10y + x. \quad (2)$$

Transpose and combine (2), $$9x - 9y = 36. \quad (3)$$

Divide (3) by 9, $$x - y = 4. \quad (4)$$

Add (1) and (4), $$2x = 18.$$

$$\therefore x = 9.$$

Subtract (4) from (1), $$2y = 10.$$

$$\therefore y = 5.$$

$$10x + y = 95.$$

Therefore, the required number is 95.

17. The sum of the three digits of a number is 18, and the sum of the first and third digits is equal to the second digit. If the hundreds' and units' digits are interchanged, the number is diminished by 99. Find the number.

Let $x =$ the digit in the hundreds' place,

$y =$ the digit in the tens' place,

and $z =$ the digit in the units' place.

Then $100x + 10y + z =$ the number.

$$\therefore x + y + z = 18, \quad (1)$$

$$x + z = y, \quad (2)$$

$$100z + 10y + x = 100x + 10y + z - 99. \quad (3)$$

Transpose and combine (3), $$-99x + 99z = -99. \quad (4)$$

Divide (4) by -99, $$x - z = 1. \quad (5)$$

Substitute in (1) the value of $x + z$ from (2),

$$y + y = 18.$$

Combine, $$2y = 18.$$

$$\therefore y = 9.$$

Substitute the value of y in (2), $$x + z = 9. \quad (6)$$

Add (5) and (6), $$2x = 10.$$

$$\therefore x = 5.$$

Subtract (5) from (6), $$2z = 8.$$

$$\therefore z = 4.$$

$$100x + 10y + z = 594.$$

Therefore, the required number is 594.

18. If a certain number is divided by the sum of its two digits, the quotient is 4 and the remainder 6; if the digits are interchanged and the resulting number divided by the sum of the digits, the quotient is 6 and the remainder 7. Find the number.

Let $x =$ the digit in the tens' place,
and $y =$ the digit in the units' place.
Then $10x + y =$ the number.

$$\therefore \frac{10x + y - 6}{x + y} = 4, \tag{1}$$

$$\frac{10y + x - 7}{x + y} = 6. \tag{2}$$

Multiply (1) by $x + y$, $10x + y - 6 = 4x + 4y$. (3)
Multiply (2) by $x + y$, $10y + x - 7 = 6x + 6y$. (4)
Transpose and combine (3), $6x - 3y = 6$. (5)
Divide (5) by 3, $2x - y = 2$. (6)
Transpose and combine (4), $-5x + 4y = 7$. (7)
Multiply (6) by 4, $8x - 4y = 8$. (8)
Add (7) and (8), $3x = 15$.
$\therefore x = 5$.
Substitute the value of x in (6), $10 - y = 2$.
Transpose and combine, $y = 8$.
$10x + y = 58$.

Therefore, the required number is 58.

19. A certain number, expressed by three digits, is equal to thirty-eight times the sum of the digits. If 198 is subtracted from the number, the order of the digits is reversed. The sum of the first and third digits is 2 greater than the middle digit. Find the number.

Let $x =$ the digit in the hundreds' place,
$y =$ the digit in the tens' place,
and $z =$ the digit in the units' place.
Then $100x + 10y + z =$ the number.

$$\therefore 100x + 10y + z = 38(x + y + z), \tag{1}$$

$$100x + 10y + z - 198 = 100z + 10y + x, \tag{2}$$

and

$$x + z = y + 2. \tag{3}$$

Expand (1), $100x + 10y + z = 38x + 38y + 38z$. (4)
Transpose and combine (4),

$$62x - 28y - 37z = 0. \tag{5}$$

Transpose and combine (2), $99x - 99z = 198$. (6)
Divide (6) by 99, $x - z = 2$. (7)
Transpose (3), $x - y + z = 2$. (8)
Multiply (8) by 28, $28x - 28y + 28z = 56$. (9)
Subtract (9) from (5), $34x - 65z = -56$. (10)
Multiply (7) by 34, $34x - 34z = 68$. (11)

Subtract (11) from (10), $-31z=-124.$

$\therefore\ z=4.$

Substitute the value of z in (7), $x-4=2.$

Transpose and combine, $x=6.$

Substitute the values of x and z in (8),

$$6-y+4=2.$$

Transpose and combine, $y=8.$

$$100x+10y+z=684.$$

Therefore, the required number is 684.

20. A boat's crew rows 14 miles down a river and back in 6 hours 40 minutes. The crew finds that it takes two and a third times as long to row a mile against the current as to row a mile with the current. Find the time the crew was rowing down and rowing up respectively.

Let x = the number of miles per hour the crew can row in still water,

and y = the number of miles per hour the current flows.

Then $\frac{14}{x+y}$ = the number of hours the crew was rowing down,

and $\frac{14}{x-y}$ = the number of hours the crew was rowing up.

$$\therefore\ \frac{14}{x+y}+\frac{14}{x-y}=6\tfrac{2}{3}, \quad (1)$$

and
$$\frac{2\frac{1}{3}}{x+y}=\frac{1}{x-y}. \quad (2)$$

Simplify (2), $$\frac{7}{3(x+y)}-\frac{1}{x-y}=0. \quad (3)$$

Multiply (3) by 14, $$\frac{98}{3(x+y)}-\frac{14}{x-y}=0. \quad (4)$$

Add (1) and (4), $$\frac{140}{3(x+y)}=\frac{20}{3}. \quad (5)$$

Divide (5) by $\frac{20}{3}$, $$\frac{7}{x+y}=1.$$

$$\therefore\ x+y=7. \quad (6)$$

Multiply (2) by $3(x+y)(x-y)$, $7x-7y=3x+3y.$ (7)

Transpose and combine (7), $4x-10y=0.$ (8)

Divide (8) by 2, $2x-5y=0.$ (9)

Multiply (6) by 5, $5x+5y=35.$ (10)

Add (9) and (10), $7x=35.$

$\therefore\ x=5.$

Substitute the value of x in (6), $5 + y = 7.$

Transpose and combine, $y = 2.$

$$\frac{14}{x+y} = 2.$$

$$\frac{14}{x-y} = 4\tfrac{2}{3}.$$

Therefore, it takes the crew 2 hours to row down and $4\frac{2}{3}$ hours or 4 hours 40 minutes to row up.

21. A boatman rows down a river, which flows at the rate of $1\frac{1}{2}$ miles an hour, for a certain distance in 2 hours 15 minutes; it takes him 4 hours 30 minutes to return. Find the distance he rowed down the stream and his rate of rowing in still water.

Let x = the number of miles per hour the boatman can row in still water,

and y = the number of miles he rowed down the stream.

2 hours 15 minutes = $2\frac{1}{4}$ hours; 4 hours 30 minutes = $4\frac{1}{2}$ hours.

Then $\frac{y}{x + 1\frac{1}{2}}$ = the number of hours he was rowing down,

and $\frac{y}{x - 1\frac{1}{2}}$ = the number of hours he was rowing up.

$$\therefore \frac{y}{x+1\frac{1}{2}} = 2\tfrac{1}{4}, \qquad (1)$$

and

$$\frac{y}{x-1\frac{1}{2}} = 4\tfrac{1}{2}. \qquad (2)$$

Simplify (1), $$\frac{2y}{2x+3} = \frac{9}{4}. \qquad (3)$$

Simplify (2), $$\frac{2y}{2x-3} = \frac{9}{2}. \qquad (4)$$

Multiply (3) by $4(2x + 3)$, $8y = 18x + 27.$ (5)

Multiply (4) by $2(2x - 3)$, $4y = 18x - 27.$ (6)

Subtract (6) from (5), $4y = 54.$

$\therefore y = 13\frac{1}{2}.$

Substitute the value of y in (6), $54 = 18x - 27.$

Transpose and combine, $-18x = -81.$

$\therefore x = 4\frac{1}{2}.$

Therefore, the boatman rowed $13\frac{1}{2}$ miles down the stream, and his rate of rowing in still water is $4\frac{1}{2}$ miles per hour.

22. A man rows 12 miles down a river and back again in 6 hours 15 minutes. He finds he can row 4 miles with the current in the same time he can row 3 miles against it. Find the time of his rowing down and of his rowing up the river, and the rate of the current.

Let $x =$ the number of miles per hour the man can row in still water,

and $y =$ the number of miles per hour the current flows.

Then $\frac{12}{x+y} =$ the number of hours he was rowing down,

and $\frac{12}{x-y} =$ the number of hours he was rowing up.

$$\therefore \frac{12}{x+y} + \frac{12}{x-y} = 6\tfrac{1}{4}, \tag{1}$$

and
$$\frac{4}{x+y} = \frac{3}{x-y}. \tag{2}$$

Multiply (2) by $(x+y)(x-y)$, $4x - 4y = 3x + 3y$. (3)

Transpose and combine (3), $x = 7y$. (4)

Substitute the value of x in (1), $\frac{12}{8y} + \frac{12}{6y} = \frac{25}{4}$. (5)

Simplify the terms of (5), $\frac{3}{2y} + \frac{2}{y} = \frac{25}{4}$. (6)

Multiply (6) by $4y$, $6 + 8 = 25y$.

Transpose and combine, $25y = 14$.

$$\therefore y = \tfrac{14}{25}.$$

Substitute the value of y in 4, $x = \frac{98}{25} = 3\frac{23}{25}$.

$$\frac{12}{x+y} = \frac{75}{28} = 2\tfrac{19}{28}.$$

$$\frac{12}{x-y} = \frac{25}{7} = 3\tfrac{4}{7}.$$

Therefore, the man was $2\frac{19}{28}$ hours rowing down, and $3\frac{4}{7}$ hours rowing up, and the current flows $\frac{14}{25}$ miles per hour.

23. A crew finds that it can row down a river at the rate of 9 miles an hour, and that it takes half as long again to row a mile up the river as to row a mile down. Find the rate of rowing in still water and the rate of the current.

Let $x =$ the number of miles per hour the crew can row in still water,

and $y =$ the number of miles per hour the current flows.

$$\therefore\ x + y = 9, \tag{1}$$

and
$$\frac{1\frac{1}{2}}{x+y} = \frac{1}{x-y}. \tag{2}$$

Multiply (2) by $2(x+y)(x-y)$, $3x - 3y = 2x + 2y$. (3)

Transpose and combine (3), $x = 5y$. (4)

Substitute the value of x in (1), $5y + y = 9$.

Combine, $6y = 9$.

$$\therefore\ y = 1\tfrac{1}{2}.$$

Substitute the value of y in (4), $x = 7\frac{1}{2}$.

Therefore, the crew can row $7\frac{1}{2}$ miles per hour in still water, and the rate of the current is $1\frac{1}{2}$ miles per hour.

24. A cistern has three pipes, A, B, and C. A and B together will fill the cistern in 12 hours, A and C together in 10 hours, and B and C together in 15 hours. How long will it take each pipe alone to fill the cistern?

Let x = the number of hours required for A alone,
y = the number of hours required for B alone,
and z = the number of hours required for C alone.

Then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ = the parts filled in one hour by A, B, C respectively,

and $\frac{1}{x} + \frac{1}{y}$ = the part filled in one hour by A and B together.

But $\frac{1}{12}$ = the part filled in one hour by A and B together.

$$\therefore\ \frac{1}{x} + \frac{1}{y} = \frac{1}{12}. \tag{1}$$

Likewise,
$$\frac{1}{x} + \frac{1}{z} = \frac{1}{10}, \tag{2}$$

and
$$\frac{1}{y} + \frac{1}{z} = \frac{1}{15}. \tag{3}$$

Subtract (2) from (1),
$$\frac{1}{y} - \frac{1}{z} = -\frac{1}{60}. \tag{4}$$

Add (3) and (4),
$$\frac{2}{y} = \frac{1}{20}.$$
$$\therefore\ y = 40.$$

Subtract (4) from (3),
$$\frac{2}{z} = \frac{1}{12}.$$
$$\therefore\ z = 24.$$

Substitute the value of z in (2), $\frac{1}{x}+\frac{1}{24}=\frac{1}{10}.$

Transpose and combine, $\frac{1}{x}=\frac{7}{120}.$

Multiply by $120x$, $120=7x.$

$\therefore x=17\frac{1}{7}.$

Therefore, A will fill the cistern in $17\frac{1}{7}$ hours, B in 40 hours, and C in 24 hours.

25. A and B together can do a piece of work in $13\frac{1}{3}$ days, A and C together in 15 days, B and C together in 20 days. How long will it take each alone to do the work?

Let $x=$ the number of days it will take A alone,
$y=$ the number of days it will take B alone,
and $z=$ the number of days it will take C alone.

Then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}=$ the parts done in one day by A, B, C respectively,

and $\frac{1}{x}+\frac{1}{y}=$ the part done in one day by A and B together.

But $\frac{3}{40}=$ the part done in one day by A and B together.

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{3}{40}. \qquad (1)$$

Likewise,
$$\frac{1}{x}+\frac{1}{z}=\frac{1}{15}, \qquad (2)$$

and
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{20}. \qquad (3)$$

Subtract (3) from (2),
$$\frac{1}{x}-\frac{1}{y}=\frac{1}{60}. \qquad (4)$$

Add (1) and (2), $\frac{2}{x}=\frac{11}{120}.$

Multiply by $120x$, $240=11x.$

$\therefore x=21\frac{9}{11}.$

Subtract (4) from (1), $\frac{2}{y}=\frac{7}{120}.$

Multiply by $120y$, $240=7y.$

$\therefore y=34\frac{2}{7}.$

Substitute the value of x in (2), $\frac{11}{240}+\frac{1}{z}=\frac{1}{15}.$

Multiply by $240z$, $$11z + 240 = 16z.$$
Transpose and combine, $$5z = 240.$$
$$\therefore\ z = 48.$$

Therefore, it will take A alone $21\frac{9}{11}$ days to do the work, B alone $34\frac{2}{7}$ days, and C alone 48 days.

26. A and B together can mow a field in 12 days, A and C together in 15 days, B and C together in 20 days. How long will it take each alone to mow the field?

Let $x =$ the number of days it will take A alone,
$y =$ the number of days it will take B alone,
and $z =$ the number of days it will take C alone.

Then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} =$ the parts mowed in one day by A, B, C respectively,

and $\frac{1}{x} + \frac{1}{y} =$ the part mowed in one day by A and B together.

But $\frac{1}{12} =$ the part mowed in one day by A and B together.

$$\therefore\ \frac{1}{x} + \frac{1}{y} = \frac{1}{12}. \qquad (1)$$

Likewise, $$\frac{1}{x} + \frac{1}{z} = \frac{1}{15}, \qquad (2)$$

and $$\frac{1}{y} + \frac{1}{z} = \frac{1}{20}. \qquad (3)$$

Subtract (3) from (2), $$\frac{1}{x} - \frac{1}{y} = \frac{1}{60}. \qquad (4)$$

Add (1) and (4), $$\frac{2}{x} = \frac{1}{10}.$$
$$\therefore\ x = 20.$$

Subtract (4) from (1), $$\frac{2}{y} = \frac{1}{15}.$$
$$\therefore\ y = 30.$$

Substitute the value of x in (2), $$\frac{1}{20} + \frac{1}{z} = \frac{1}{15}.$$

Multiply by $60z$, $$3z + 60 = 4z.$$
Transpose and combine, $$z = 60.$$

Therefore, it will take A alone 20 days to mow the field, B alone 30 days, and C alone 60 days.

27. **A cistern has three pipes A, B, and C. A and B together will fill it in 35 minutes, A and C together in 42 minutes, B and C together in 70 minutes. How long will it take each alone to fill the cistern?**

Let x = the number of minutes it will take A alone,

y = the number of minutes it will take B alone,

and z = the number of minutes it will take C alone.

Then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ = the parts filled in one minute by A, B, C respectively,

and $\frac{1}{x}+\frac{1}{y}$ = the part filled in one minute by A and B together.

But $\frac{1}{35}$ = the part filled in one minute by A and B together.

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{1}{35}. \quad (1)$$

Likewise,
$$\frac{1}{x}+\frac{1}{z}=\frac{1}{42}, \quad (2)$$

and
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{70}. \quad (3)$$

Subtract (3) from (2),
$$\frac{1}{x}-\frac{1}{y}=\frac{1}{105}. \quad (4)$$

Add (1) and (4),
$$\frac{2}{x}=\frac{4}{105}.$$

Multiply by $105x$,
$$210=4x.$$
$$\therefore x=52\tfrac{1}{2}.$$

Subtract (4) from (1),
$$\frac{2}{y}=\frac{2}{105}.$$

Multiply by $105y$,
$$210=2y.$$
$$\therefore y=105.$$

Substitute the value of y in (3),
$$\frac{1}{105}+\frac{1}{z}=\frac{1}{70}.$$

Multiply by $210z$,
$$2z+210=3z.$$

Transpose and combine,
$$z=210.$$

Therefore, it will take A alone $52\frac{1}{2}$ minutes to fill the cistern, B alone 105 minutes, and C alone 210 minutes.

28. **A and B together can do a piece of work in 24 days, A and C together in 32 days, B and C together in 30 days. How long will it take each alone to do the work?**

Let $x =$ the number of days it will take A alone,
$y =$ the number of days it will take B alone,
and $z =$ the number of days it will take C alone.

Then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} =$ the parts done in one day by A, B, C respectively,

and $\frac{1}{x} + \frac{1}{y} =$ the part done in one day by A and B together.

But $\frac{1}{24} =$ the part done in one day by A and B together.

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{24}. \tag{1}$$

Likewise,
$$\frac{1}{x} + \frac{1}{z} = \frac{1}{32}, \tag{2}$$

and
$$\frac{1}{y} + \frac{1}{z} = \frac{1}{30}. \tag{3}$$

Subtract (3) from (2),
$$\frac{1}{x} - \frac{1}{y} = -\frac{1}{480}. \tag{4}$$

Add (1) and (4),
$$\frac{2}{x} = \frac{19}{480}.$$

Multiply by $480x$,
$$960 = 19x.$$
$$\therefore x = 50\tfrac{10}{19}.$$

Subtract (4) from (1),
$$\frac{2}{y} = \frac{7}{160}.$$

Multiply by $160y$,
$$320 = 7y.$$
$$\therefore y = 45\tfrac{5}{7}.$$

Substitute the value of y in (3),
$$\frac{7}{320} + \frac{1}{z} = \frac{1}{30}.$$

Multiply by $960z$,
$$21z + 960 = 32z.$$

Transpose and combine,
$$-11z = -960.$$
$$\therefore z = 87\tfrac{3}{11}.$$

Therefore, it will take A alone $50\frac{10}{19}$ days to do the work, B alone $45\frac{5}{7}$ days, and C alone $87\frac{3}{11}$ days.

29. A sum of money at simple interest amounted to $2996.40 in 3 years, and to $3234 in 5 years. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the principal,
and $y =$ the rate of interest.

The interest for one year is $\frac{y}{100}$ of the principal, or $\frac{y}{100}$ of x, or $\frac{xy}{100}$; the interest for 3 years is $\frac{3xy}{100}$; for 5 years $\frac{5xy}{100}$.

$$\therefore\ x + \frac{3xy}{100} = 2996.40, \qquad (1)$$

and

$$x + \frac{5xy}{100} = 3234. \qquad (2)$$

Multiply (1) by 100, $100x + 3xy = 299640.$ (3)

Multiply (2) by 100, $100x + 5xy = 323400.$ (4)

Multiply (4) by 3 and divide by 5,

$$60x + 3xy = 194040. \qquad (5)$$

Subtract (5) from (3), $40x = 105600.$

$$\therefore\ x = 2640.$$

Substitute the value of x in (4),

$$264000 + 13200y = 323400.$$

Transpose and combine, $13200y = 59400.$

$$\therefore\ y = 4\tfrac{1}{2}.$$

Therefore, the sum is \$2640, and the rate is 4½ per cent.

30. A sum of money at simple interest amounted to \$8910 in 2 years, and to \$9240 in 3 years. Find the sum and the rate of interest.

Let x = the number of dollars in the principal,
and y = the rate of interest.

Then the interest for 2 years is $\frac{2xy}{100}$, and for 3 years is $\frac{3xy}{100}$.

$$\therefore\ x + \frac{2xy}{100} = 8910, \qquad (1)$$

and

$$x + \frac{3xy}{100} = 9240. \qquad (2)$$

Subtract (1) from (2), $\frac{xy}{100} = 330.$ (3)

Multiply (3) by 2, $\frac{2xy}{100} = 660.$ (4)

Substitute in (1) the value of $\frac{2xy}{100}$,

$$x + 660 = 8910.$$

Transpose and combine, $x = 8250.$

Substitute the value of x in (3), $82.50y = 330.$

$$\therefore\ y = 4.$$

Therefore, the sum is \$8250, and the rate is 4 per cent.

31. **A sum of money at simple interest amounted to \$12,607.50 in 6 months, and to \$12,710 in 8 months. Find the sum and the rate of interest.**

Let $x =$ the number of dollars in the principal,
and $y =$ the rate of interest.

Then the interest for 1 year is $\frac{xy}{100}$; for 1 month is $\frac{xy}{1200}$; for 6 months is $\frac{6xy}{1200} = \frac{xy}{200}$; and for 8 months is $\frac{8xy}{1200} = \frac{xy}{150}$.

$$\therefore x + \frac{xy}{200} = 12607.50, \quad (1)$$

and

$$x + \frac{xy}{150} = 12710. \quad (2)$$

Subtract (1) from (2),

$$\frac{xy}{600} = 102.50. \quad (3)$$

Multiply (3) by 3,

$$\frac{xy}{200} = 307.50. \quad (4)$$

Substitute in (1) the value of $\frac{xy}{200}$,

$$x + 307.50 = 12607.50.$$

Transpose and combine, $x = 12300.$

Substitute the value of x in (3), $20.50\,y = 102.50.$

$$\therefore y = 5.$$

Therefore, the sum is \$12,300, and the rate is 5 per cent.

32. A sum of money at simple interest amounted to \$5326.80 in 2 years 7 months 18 days, and to \$5552.20 in 3 years 5 months 12 days. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the principal,
and $y =$ the rate of interest.

2 years 7 months 18 days $= 2\frac{19}{30}$ years;
3 years 5 months 12 days $= 3\frac{9}{20}$ years.

Then the interest for 1 year is $\frac{xy}{100}$; for $2\frac{19}{30}$ years is $\frac{79xy}{3000}$; for $3\frac{9}{20}$ years is $\frac{69xy}{2000}$.

$$\therefore x + \frac{79xy}{3000} = 5326.80, \quad (1)$$

and

$$x + \frac{69xy}{2000} = 5552.20. \quad (2)$$

Subtract (1) from (2), $$\frac{49xy}{6000} = 225.40. \qquad (3)$$

Multiply (3) by $\frac{158}{49}$, $$\frac{79xy}{3000} = 726.80. \qquad (4)$$

Substitute in (1) the value of $\frac{79xy}{3000}$,

$$x + 726.80 = 5326.80.$$

$$\therefore x = 4600.$$

Substitute the value of x in (3), $$\frac{1127y}{30} = 225.40.$$

$$\therefore y = \frac{30 \times 225.40}{1127} = 6.$$

Therefore, the sum is $4600, and the rate is 6 per cent.

33. A man has invested the sum of $10,000, for a part of which he receives 4 per cent interest, and for the remainder $4\frac{1}{2}$ per cent. The income from the 4 per cent investment is $9 more than that from the $4\frac{1}{2}$ per cent. How much has he in each investment?

Let x = the number of dollars invested at 4 per cent,
and y = the number of dollars invested at $4\frac{1}{2}$ per cent.

Then $\frac{4x}{100}$ = the number of dollars interest on the first,
and $\frac{4\frac{1}{2}y}{100}$ = the number of dollars interest on the second.

$$\therefore x + y = 10000, \qquad (1)$$

and $$\frac{4x}{100} = \frac{4\frac{1}{2}y}{100} + 9. \qquad (2)$$

Multiply (2) by 200, $$8x = 9y + 1800. \qquad (3)$$

Transpose (3), $$8x - 9y = 1800. \qquad (4)$$

Multiply (1) by 9, $$9x + 9y = 90000. \qquad (5)$$

Add (4) and (5), $$17x = 91800.$$

$$\therefore x = 5400.$$

Substitute the value of x in (1), $5400 + y = 10000.$

Transpose and combine, $y = 4600.$

Therefore, the man has $5400 invested at 4 per cent, and $4600 at $4\frac{1}{2}$ per cent.

34. In running a quarter mile race A gives B a start of 32 yards and beats him by 1 second. In a second trial A gives B a start of 6 seconds and is beaten by $8\frac{4}{5}$ yards. Find the number of yards each runs a second.

Let $x =$ the number of yards A runs a second,
and $y =$ the number of yards B runs a second.

Since a quarter mile is 440 yards,

$\frac{440}{x} =$ the number of seconds it takes A to run the quarter mile.

Since B has a start of 32 yards, he runs 408 yards the first trial; and since he was 1 second longer than A,

$\frac{440}{x} + 1 =$ the number of seconds B was running.

But $\frac{408}{y} =$ the number of seconds B was running.

$$\therefore \frac{440}{x} + 1 = \frac{408}{y}. \quad (1)$$

In the second trial A runs $(440 - 8\tfrac{4}{5})$ yards $= 431\tfrac{1}{5}$ yards.

$$\therefore \frac{440}{y} = \frac{431\frac{1}{5}}{x} + 6. \quad (2)$$

Simplify (2),
$$\frac{440}{y} = \frac{2156}{5x} + 6. \quad (3)$$

Transpose (3),
$$\frac{2156}{5x} = \frac{440}{y} - 6. \quad (4)$$

$$\therefore 5x = \frac{2156}{\frac{440}{y} - 6}.$$

$$\therefore x = \frac{2156}{\frac{2200}{y} - 30}. \quad (5)$$

Transpose (1),
$$\frac{440}{x} = \frac{408}{y} - 1.$$

$$\therefore x = \frac{440}{\frac{408}{y} - 1}. \quad (6)$$

Equate the values of x,
$$\frac{440}{\frac{408}{y} - 1} = \frac{2156}{\frac{2200}{y} - 30}.$$

Simplify,
$$\frac{440y}{408 - y} = \frac{1078y}{1100 - 15y}.$$

Divide by $22y$,
$$\frac{20}{408 - y} = \frac{49}{1100 - 15y}.$$

Multiply by $(408 - y)(1100 - 15y)$,
$$22000 - 300y = 19992 - 49y.$$

Transpose and combine, $$-251y = -2008.$$

$$\therefore\ y = 8.$$

Substitute the value of y in (6), $$x = \frac{440}{\frac{408}{8} - 1}.$$

$$\therefore\ x = 8\tfrac{4}{5}.$$

Therefore, A runs $8\frac{4}{5}$ yards a second, and B 8 yards a second.

35. Two athletes, A and B, run a mile race, and A wins by 6 seconds. In a handicap race A gives B a start of 40 yards and is beaten by half a second. Find the number of yards each runs in a second and the time required by each to run a mile.

Let $x =$ the number of yards A runs a second,

and $y =$ the number of yards B runs a second.

Since a mile is 1760 yards,

$\frac{1760}{x} =$ the number of seconds it takes A to run a mile,

and $\frac{1760}{y} =$ the number of seconds it takes B to run a mile.

$$\therefore\ \frac{1760}{y} - \frac{1760}{x} = 6. \qquad (1)$$

In the handicap race B runs only 1720 yards.

$$\therefore\ \frac{1760}{x} - \frac{1720}{y} = \frac{1}{2}. \qquad (2)$$

Add (1) and (2), $$\frac{40}{y} = \frac{13}{2}.$$

Multiply by $2y$, $$80 = 13y.$$

$$\therefore\ y = 6\tfrac{2}{13}.$$

Substitute the value of y in (1), $$286 - \frac{1760}{x} = 6.$$

Transpose and combine, $$-\frac{1760}{x} = -280.$$

$$\therefore\ 280x = 1760.$$

$$\therefore\ x = 6\tfrac{2}{7}.$$

Then $$\frac{1760}{x} = 280,$$

and $$\frac{1760}{y} = 286.$$

Therefore, A runs $6\frac{2}{7}$ yards a second, and B runs $6\frac{2}{13}$ yards a second; A requires to run a mile 280 seconds, or 4 minutes 40 seconds, and B requires 286 seconds, or 4 minutes 46 seconds.

36. In running a mile race A beats B by $8\frac{4}{5}$ seconds. In a handicap race A gives B a start of 50 yards and beats him by 5 yards. Find the number of yards each runs in a second and the time required by each to run a mile.

Let $x =$ the number of yards A runs a second,
and $y =$ the number of yards B runs a second.

Since a mile is 1760 yards,

$\frac{1760}{x} =$ the number of seconds it takes A to run a mile,

and $\frac{1760}{y} =$ the number of seconds it takes B to run a mile.

$$\therefore \frac{1760}{y} - \frac{1760}{x} = 8\frac{4}{5}. \qquad (1)$$

In the handicap race B runs only 1705 yards.

$$\therefore \frac{1705}{y} - \frac{1760}{x} = 0. \qquad (2)$$

Subtract (2) from (1), $$\frac{55}{y} = \frac{44}{5}.$$

Divide by 11, $$\frac{5}{y} = \frac{4}{5}.$$

Multiply by $5y$, $$25 = 4y.$$

$$\therefore y = 6\tfrac{1}{4}.$$

Substitute the value of y in (2),

$$\frac{1364}{5} - \frac{1760}{x} = 0.$$

Transpose, $$\frac{1364}{5} = \frac{1760}{x}.$$

Divide by 44, $$\frac{31}{5} = \frac{40}{x}.$$

Multiply by $5x$, $$31x = 200.$$

$$\therefore x = 6\tfrac{14}{31}.$$

Then $$\frac{1760}{x} = 272\tfrac{4}{5},$$

and $$\frac{1760}{y} = 281\tfrac{3}{5}.$$

Therefore, A runs $6\frac{14}{31}$ yards a second, and B runs $6\frac{1}{4}$ yards a second; A requires to run a mile $272\frac{4}{5}$ seconds, or 4 minutes $32\frac{4}{5}$ seconds, and B requires $281\frac{3}{5}$ seconds, or 4 minutes $41\frac{3}{5}$ seconds.

37. A train, after traveling 2 hours from A towards B, meets with an accident which detains it 40 minutes. After the accident the train proceeds at three fourths its usual rate, and arrives 2 hours late. If the accident had happened 60 miles further on, the train would have been only an hour and a half late. Find the usual rate of the train per hour and the distance from A to B.

Let $4x =$ the number of miles the train travels per hour,
and $y =$ the number of miles from A to B.
Then $3x =$ the number of miles the train travels per hour after the accident,
$y - 8x =$ the number of miles the train travels after the accident,
$\frac{y-8x}{4x} =$ the number of hours usually required,
and $\frac{y-8x}{3x} =$ the number of hours actually required.

Since the train was detained 40 minutes, or $\frac{2}{3}$ hour, and arrived 2 hours late, the *running time* was $1\frac{1}{3}$ hours more than usual;

that is, $1\frac{1}{3} =$ the number of hours of running time lost.

But $\frac{y-8x}{3x} - \frac{y-8x}{4x} =$ the number of hours of running time lost.

$$\therefore \frac{y-8x}{3x} - \frac{y-8x}{4x} = 1\tfrac{1}{3}. \qquad (1)$$

If the accident had happened 60 miles further on, the remainder of the run would have been $(y - 8x - 60)$ miles, and the loss in running time would have been $\frac{5}{6}$ of an hour.

$$\therefore \frac{y-8x-60}{3x} - \frac{y-8x-60}{4x} = \frac{5}{6}. \qquad (2)$$

Subtract (2) from (1), $\frac{60}{3x} - \frac{60}{4x} = \frac{1}{2}.$

Simplify terms, $\frac{20}{x} - \frac{15}{x} = \frac{1}{2}.$

Combine, $\frac{5}{x} = \frac{1}{2}.$

$$\therefore x = 10.$$

Substitute the value of x in (1),

$$\frac{y-80}{30} - \frac{y-80}{40} = \frac{4}{3}.$$

Multiply by 120, $4y - 320 - 3y + 240 = 160.$
Transpose and combine, $y = 240.$
Then $4x = 40.$

Therefore, the usual rate of the train is 40 miles an hour, and the distance from A to B is 240 miles.

38. A train, after traveling 1 hour 20 minutes from A towards B, meets with an accident which detains it 30 minutes. After the accident the train proceeds at two thirds its usual rate, and arrives 2 hours 20 minutes late. If the accident had happened 55 miles further on, the train would have been only an hour and a half late. Find the usual rate of the train per hour and the distance from A to B.

Let $3x =$ the number of miles the train travels per hour,
and $y =$ the number of miles from A to B.
Then $2x =$ the number of miles the train travels per hour after the accident,
$y - 4x =$ the number of miles the train travels after the accident,
$\frac{y-4x}{3x} =$ the number of hours usually required,
and $\frac{y-4x}{2x} =$ the number of hours actually required.

Since the train was detained 30 minutes, or $\frac{1}{2}$ hour, and arrived 2 hours 20 minutes, or $2\frac{1}{3}$ hours late, the running time was $1\frac{5}{6}$ hours more than usual;

that is, $1\frac{5}{6} =$ the number of hours of running time lost.
But $\frac{y-4x}{2x} - \frac{y-4x}{3x} =$ the number of hours of running time lost.

$$\therefore \frac{y-4x}{2x} - \frac{y-4x}{3x} = 1\frac{5}{6}. \qquad (1)$$

If the accident had happened 55 miles further on, the remainder of the run would have been $(y - 4x - 55)$ miles, and the loss in running time would have been 1 hour.

$$\therefore \frac{y-4x-55}{2x} - \frac{y-4x-55}{3x} = 1. \qquad (2)$$

Subtract (2) from (1), $\frac{55}{2x} - \frac{55}{3x} = \frac{5}{6}.$

Divide by 5, $\frac{11}{2x} - \frac{11}{3x} = \frac{1}{6}.$

Multiply by $6x$, $33 - 22 = x$.
Transpose and combine, $x = 11$.
Substitute the value of x in (1),

$$\frac{y-44}{22} - \frac{y-44}{33} = \frac{11}{6}.$$

Multiply by 66, $3y - 132 - 2y + 88 = 121$.
Transpose and combine, $y = 165$.
Then $3x = 33$.

Therefore, the usual rate of the train is 33 miles an hour, and the distance from A to B is 165 miles.

39. The sum of the three angles of any triangle is 180°. If one angle of a triangle exceeds half the sum of the other two angles by 15°, and their difference by 30°, find the angles.

Let $x =$ the number of degrees in the first angle,
$y =$ the number of degrees in the second angle,
and $z =$ the number of degrees in the third angle.

$$\therefore x + y + z = 180, \tag{1}$$

$$x - \frac{y+z}{2} = 15, \tag{2}$$

and $$x - (y - z) = 30. \tag{3}$$

Multiply (2) by 2, $$2x - y - z = 30. \tag{4}$$
Expand (3), $$x - y + z = 30. \tag{5}$$
Add (1) and (4), $3x = 210$.
$\therefore x = 70$.
Add (1) and (5), $$2x + 2z = 210. \tag{6}$$
Divide (6) by 2, $$x + z = 105. \tag{7}$$
Substitute the value of x in (7), $70 + z = 105$.
Transpose and combine, $z = 35$.
Substitute the values of x and z in (1),

$$70 + y + 35 = 180.$$

Transpose and combine, $y = 75$.

Therefore, the three angles of the triangle are 70°, 75°, and 35°.

40. An angle of a triangle is $\frac{5}{13}$ the sum of the other two angles and is $\frac{5}{6}$ their difference. Find the angles.

Let $x =$ the number of degrees in the first angle,
$y =$ the number of degrees in the second angle,
and $z =$ the number of degrees in the third angle.

$$\therefore\ x + y + z = 180, \qquad (1)$$

$$x = \frac{5(y+z)}{13}, \qquad (2)$$

and $$x = \frac{5(y-z)}{6}. \qquad (3)$$

Multiply (2) by 13, $13x = 5y + 5z. \qquad (4)$

Multiply (3) by 6, $6x = 5y - 5z. \qquad (5)$

Add (4) and (5), $19x = 10y.$

$$\therefore\ y = \frac{19x}{10}. \qquad (6)$$

Subtract (5) from (4), $7x = 10z.$

$$\therefore\ z = \frac{7x}{10}. \qquad (7)$$

Substitute the values of y and z in (1),

$$x + \frac{19x}{10} + \frac{7x}{10} = 180.$$

Multiply by 10, $10x + 19x + 7x = 1800.$

Combine, $36x = 1800.$

$\therefore\ x = 50.$

Substitute the value of x in (6), $y = 95.$

Substitute the value of x in (7), $z = 35.$

Therefore, the three angles of the triangle are 50°, 95°, and 35°.

41. A number of young men purchase a camp. If there had been two more men in the company, each would have paid $12 less; and if there had been three men less, each would have paid $24 more. How many men were there and how much did each pay?

Let $x =$ the number of men in the company,

and $y =$ the number of dollars each paid.

Then $xy =$ the number of dollars in the cost of the camp.

$$\therefore\ (x+2)(y-12) = xy, \qquad (1)$$

and $$(x-3)(y+24) = xy. \qquad (2)$$

Expand (1), $xy - 12x + 2y - 24 = xy. \qquad (3)$

Expand (2), $xy + 24x - 3y - 72 = xy. \qquad (4)$

Transpose and combine (3), $-12x + 2y = 24. \qquad (5)$

Transpose and combine (4), $24x - 3y = 72. \qquad (6)$

Multiply (5) by 2, $-24x + 4y = 48. \qquad (7)$

Add (6) and (7), $y = 120.$

Substitute the value of y in (6), $24x - 360 = 72$.

Transpose and combine, $24x = 432$.

$$\therefore x = 18.$$

Therefore, there were 18 men in the company and each paid $120.

42. A sum of money at simple interest amounted in a years to m dollars, and in b years to n dollars. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the principal,

and $y =$ the rate of interest.

The interest for one year is $\frac{xy}{100}$; for a years, $\frac{axy}{100}$; for b years, $\frac{bxy}{100}$.

$$\therefore x + \frac{axy}{100} = m, \tag{1}$$

and

$$x + \frac{bxy}{100} = n. \tag{2}$$

Subtract (2) from (1), $\frac{(a-b)xy}{100} = m - n$.

Divide by $a - b$, $\frac{xy}{100} = \frac{m-n}{a-b}$.

Substitute the value of $\frac{xy}{100}$ in (1),

$$x + \frac{a(m-n)}{a-b} = m.$$

Transpose, $x = m - \frac{a(m-n)}{a-b}$.

Simplify, $x = \frac{an-bm}{a-b}$.

Substitute the value of x in (1),

$$\frac{an-bm}{a-b} + \frac{ay(an-bm)}{100(a-b)} = m.$$

Transpose and combine, $\frac{ay(an-bm)}{100(a-b)} = \frac{a(m-n)}{a-b}$.

Multiply by $100(a-b)$, $ay(an-bm) = 100a(m-n)$.

$$\therefore y = \frac{100(m-n)}{an-bm}.$$

Therefore, the sum is $\frac{an-bm}{a-b}$ dollars, and the rate is $\frac{100(m-n)}{an-bm}$ per cent.

43. A sum of money at simple interest amounted in a months to m dollars, and in b months to n dollars. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the principal,
and $y =$ the rate of interest.

The interest for one year is $\frac{xy}{100}$; for one month, $\frac{xy}{1200}$; for a months, $\frac{axy}{1200}$; for b months, $\frac{bxy}{1200}$.

$$\therefore\ x + \frac{axy}{1200} = m, \qquad (1)$$

and

$$x + \frac{bxy}{1200} = n. \qquad (2)$$

Subtract (2) from (1), $\frac{(a-b)xy}{1200} = m - n.$

Divide by $a - b$, $\frac{xy}{1200} = \frac{m-n}{a-b}.$

Substitute the value of $\frac{xy}{1200}$ in (1),

$$x + \frac{a(m-n)}{a-b} = m.$$

Transpose, $x = m - \frac{a(m-n)}{a-b}.$

Simplify, $x = \frac{an - bm}{a-b}.$

Substitute the value of x in (1),

$$\frac{an-bm}{a-b} + \frac{ay(an-bm)}{1200(a-b)} = m.$$

Transpose and combine, $\frac{ay(an-bm)}{1200(a-b)} = \frac{a(m-n)}{a-b}.$

Multiply by $1200(a-b)$, $ay(an-bm) = 1200\,a(m-n).$

$$\therefore\ y = \frac{1200(m-n)}{an-bm}.$$

Therefore, the sum is $\frac{an-bm}{a-b}$ dollars, and the rate is $\frac{1200(m-n)}{an-bm}$ per cent.

44. A boat's crew can row 15 miles an hour downstream. The crew can row a certain distance in still water in 15 minutes and requires 20 minutes to row the same distance up the stream. Find the rate of rowing in still water and the rate of the stream.

Let x = the number of miles per hour the crew can row in still water,

and y = the number of miles per hour the current flows.

15 minutes = $\frac{1}{4}$ hour, and 20 minutes = $\frac{1}{3}$ hour.

$$\therefore\ x + y = 15, \tag{1}$$

and

$$\frac{x}{4} = \frac{x - y}{3}. \tag{2}$$

Multiply (2) by 12, $3x = 4x - 4y.$ (3)

Transpose and combine (3), $x = 4y.$ (4)

Substitute the value of x in (1), $4y + y = 15.$

Combine, $5y = 15.$

$\therefore\ y = 3.$

Substitute the value of y in (4), $x = 12.$

Therefore, the rate of the crew in still water is 12 miles an hour, and the rate of the stream is 3 miles an hour.

45. If the length of a rectangle was 7 feet greater and the breadth 2 feet less, the area would be 159 square feet greater. If the length was 5 feet less and the breadth 4 feet greater, the area would be 33 square feet less. Find the dimensions.

Let x = the number of feet in the length of the rectangle,

and y = the number of feet in the breadth of the rectangle.

Then xy = the number of square feet in the area of the rectangle.

$$\therefore\ (x + 7)(y - 2) = xy + 159, \tag{1}$$

and

$$(x - 5)(y + 4) = xy - 33. \tag{2}$$

Expand (1), $xy - 2x + 7y - 14 = xy + 159.$ (3)

Expand (2), $xy + 4x - 5y - 20 = xy - 33.$ (4)

Transpose and combine (3), $-2x + 7y = 173.$ (5)

Transpose and combine (4), $4x - 5y = -13.$ (6)

Multiply (5) by 2, $-4x + 14y = 346.$ (7)

Add (6) and (7), $9y = 333.$

$\therefore\ y = 37.$

Substitute the value of y in (5), $-2x + 259 = 173.$

Transpose and combine, $-2x = -86.$

$\therefore\ x = 43.$

Therefore, the length of the rectangle is 43 feet, and the breadth 37 feet.

46. In running a mile race A gives B a start of 44 yards and is beaten by one second. In a second trial A gives B a start of 7 seconds and beats him by $4\frac{8}{9}$ yards. Find the number of yards each runs per second.

Let $x =$ the number of yards A runs in one second,
and $y =$ the number of yards B runs in one second.

The number of yards in a mile is 1760.

$$\therefore \frac{1760}{x} - \frac{1760 - 44}{y} = 1, \tag{1}$$

and

$$\frac{1760 - 4\frac{8}{9}}{y} - \frac{1760}{x} = 7. \tag{2}$$

Add (1) and (2), $\frac{44 - 4\frac{8}{9}}{y} = 8.$

Simplify, $\frac{352}{9y} = 8.$

Multiply by $9y$, $72y = 352.$

$\therefore y = 4\frac{8}{9}.$

Substitute the value of y in (1),

$$\frac{1760}{x} - \frac{1716}{4\frac{8}{9}} = 1.$$

Simplify, $\frac{1760}{x} - 351 = 1.$

Transpose and combine, $\frac{1760}{x} = 352.$

$\therefore x = 5.$

Therefore, A runs 5 yards per second, and B $4\frac{8}{9}$ yards per second.

47. A courtyard, whose area is 1300 square feet, is paved with 50 stones of one size and 195 stones of another size. Ten stones of the first size and seven of the second size cover together 100 square feet. Find the area of each kind.

Let $x =$ the number of square feet in the area of the first size of stone,
and $y =$ the number of square feet in the area of the second size of stone.

$$\therefore 50x + 195y = 1300, \tag{1}$$

and

$$10x + 7y = 100. \tag{2}$$

Divide (1) by 5, $10x + 39y = 260.$ (3)

Subtract (2) from (3), $32y = 160.$

$\therefore y = 5.$

Substitute the value of y in (2), $10x + 35 = 100.$

Transpose and combine, $10x = 65.$

$\therefore x = 6\frac{1}{2}.$

Therefore, the area of the first size is $6\frac{1}{2}$ square feet, and of the second size is 5 square feet.

48. The sum of the three digits of a number is 16, and the sum of the first digit and the third digit is equal to the second. If the units' digit and the tens' digit are interchanged, the resulting number is less than the original number by 27. Find the number.

Let $x =$ the digit in the hundreds' place,
$y =$ the digit in the tens' place,
and $z =$ the digit in the units' place.

Then $100x + 10y + z =$ the number.

$$\therefore\ x + y + z = 16, \tag{1}$$

$$x + z = y, \tag{2}$$

and
$$100x + 10z + y = 100x + 10y + z - 27. \tag{3}$$

Transpose (2),
$$x - y + z = 0. \tag{4}$$

Transpose and combine (3),
$$-9y + 9z = -27. \tag{5}$$

Divide (5) by -9,
$$y - z = 3. \tag{6}$$

Subtract (4) from (1),
$$2y = 16.$$
$$\therefore\ y = 8.$$

Substitute the value of y in (6),
$$8 - z = 3.$$

Transpose and combine,
$$z = 5.$$

Substitute the values of y and z in (1),
$$x + 8 + 5 = 16.$$

Transpose and combine,
$$x = 3.$$
$$100x + 10y + z = 385.$$

Therefore, the number required is 385.

49. The report of an explosion traveled 1039 feet per second against the wind and 1061 feet per second with the wind. Find the velocity of the sound in still air and the velocity of the wind.

Let $x =$ the number of feet per second the sound traveled,
and $y =$ the number of feet per second the wind traveled.

$$\therefore\ x - y = 1039, \tag{1}$$

and
$$x + y = 1061. \tag{2}$$

Add (1) and (2),
$$2x = 2100.$$
$$\therefore\ x = 1050.$$

Subtract (1) from (2),
$$2y = 22.$$
$$\therefore\ y = 11.$$

Therefore, the velocity of sound in still air was 1050 feet per second, and the velocity of the wind was 11 feet per second, or $7\frac{1}{2}$ miles per hour.

50. The sum of the ages of A, B, and C is 171 years. Fifteen years ago A's age was equal to the sum of the ages of B and C; and 34 years ago A's age was four times the difference between the ages of B and C. Find the age of each.

Let $x =$ the number of years in A's age,
$y =$ the number of years in B's age,
and $z =$ the number of years in C's age.

$$\therefore x + y + z = 171, \tag{1}$$

$$x - 15 = y - 15 + z - 15, \tag{2}$$

and
$$x - 34 = 4[(y-34)-(z-34)]. \tag{3}$$

Transpose and combine (2), $x - y - z = -15.$ (4)
Expand (3), $x - 34 = 4y - 4z.$ (5)
Transpose (5), $x - 4y + 4z = 34.$ (6)
Add (1) and (4), $2x = 156.$
$\therefore x = 78.$
Substitute the value of x in (4), $78 - y - z = -15.$ (7)
Substitute the value of x in (6), $78 - 4y + 4z = 34.$ (8)
Transpose and combine (7), $y + z = 93.$ (9)
Transpose and combine (8), $-4y + 4z = -44.$ (10)
Divide (10) by -4, $y - z = 11.$ (11)
Add (9) and (11), $2y = 104.$
$\therefore y = 52.$
Subtract (11) from (9), $2z = 82.$
$\therefore z = 41.$

Therefore, A's age is 78 years; B's age, 52 years; and C's age, 41 years.

51. A, B, and C together subscribed \$100. If A had subscribed a tenth less and B a tenth more, C must have subscribed \$2 more to make up the sum; but if A had subscribed an eighth more, and B an eighth less, C must have subscribed \$17.50. What was the subscription of each?

Let $x =$ the number of dollars in A's subscription,
$y =$ the number of dollars in B's subscription,
and $z =$ the number of dollars in C's subscription.

$$\therefore x + y + z = 100, \tag{1}$$

$$\frac{9x}{10} + \frac{11y}{10} + z + 2 = 100, \tag{2}$$

and
$$\frac{9x}{8} + \frac{7y}{8} + 17\tfrac{1}{2} = 100. \tag{3}$$

Multiply (2) by 10, $9x + 11y + 10z + 20 = 1000.$ (4)

Multiply (3) by 8, $9x + 7y + 140 = 800.$ (5)

Transpose and combine (4), $9x + 11y + 10z = 980.$ (6)

Transpose and combine (5), $9x + 7y = 660.$ (7)

Multiply (1) by 10, $10x + 10y + 10z = 1000.$ (8)

Subtract (6) from (8), $x - y = 20.$ (9)

Multiply (9) by (7), $7x - 7y = 140.$ (10)

Add (7) and (10), $16x = 800.$

$\therefore x = 50.$

Substitute the value of x in (9), $50 - y = 20.$

$\therefore y = 30.$

Substitute the values of x and y in (1), $50 + 30 + z = 100.$

Transpose and combine, $z = 20.$

Therefore, A subscribed $50; B, $30; and C, $20.

52. A and B together can do a piece of work in 16 days, A and C together in 18 days, and B and C together in 20 days. In how many days can each alone do the work?

Let x = the number of days required for A alone,
y = the number of days required for B alone,
and z = the number of days required for C alone.

Then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ = the parts done in one day by A, B, C respectively,

and $\frac{1}{x} + \frac{1}{y}$ = the part done in one day by A and B together.

But $\frac{1}{16}$ = the part done in one day by A and B together.

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{16}. \qquad (1)$$

Likewise,

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{18}, \qquad (2)$$

and

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{20}. \qquad (3)$$

Subtract (3) from (2),

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{180}. \qquad (4)$$

Add (1) and (4),

$$\frac{2}{x} = \frac{49}{720}.$$

Multiply by $720x$,

$$1440 = 49x.$$

$$\therefore x = 29\tfrac{19}{49}.$$

Subtract (4) from (1),

$$\frac{2}{y} = \frac{41}{720}.$$

Multiply by $720\,y$, $1440 = 41\,y$.

$\therefore\ y = 35\frac{5}{41}$.

Substitute the value of x in (2), $\frac{49}{1440} + \frac{1}{z} = \frac{1}{18}$.

Multiply by $1440\,z$, $49\,z + 1440 = 80\,z$.

Transpose and combine, $-31\,z = -1440$.

$\therefore\ z = 46\frac{14}{31}$.

Therefore, A can do the work alone in $29\frac{19}{49}$ days, B in $35\frac{5}{41}$ days, and C in $46\frac{14}{31}$ days.

53. A boat travels $10\frac{1}{2}$ miles an hour down a river. It takes the boat three times as long to go a certain distance up the river as to go the same distance down the river. Find the rate at which the river flows and the rate of the boat in still water.

Let x the number of miles per hour the boat travels in still water,
and y = the number of miles per hour the river flows.

$$\therefore\ x + y = 10\tfrac{1}{2}, \qquad (1)$$

and

$$\frac{3}{x+y} = \frac{1}{x-y}. \qquad (2)$$

Multiply (1) by 2, $2x + 2y = 21$. (3)

Multiply (2) by $(x + y)(x - y)$, $3x - 3y = x + y$. (4)

Transpose and combine (4), $2x = 4y$.

$\therefore\ x = 2y$. (5)

Substitute the value of x in (3), $4y + 2y = 21$.

Combine, $6y = 21$.

$\therefore\ y = 3\frac{1}{2}$.

Substitute the value of y in (5), $x = 7$.

Therefore, the river flows at the rate of $3\frac{1}{2}$ miles an hour, and the boat travels in still water at the rate of 7 miles an hour.

54. A sum of money at simple interest amounted in 7 months to \$2118.30 and in 10 months to \$2139. Find the sum and the rate of interest.

Let x = the number of dollars in the principal,
and y = the rate of interest.

The interest for 1 year is $\frac{xy}{100}$; and for 1 month $\frac{xy}{1200}$.

$$\therefore\ x + \frac{7\,xy}{1200} = 2118.30, \qquad (1)$$

and

$$x + \frac{10\,xy}{1200} = 2139. \qquad (2)$$

Subtract (1) from (2), $\frac{3xy}{1200} = 20.70.$ (3)

Divide (3) by 3, $\frac{xy}{1200} = 6.90.$ (4)

Substitute the value of $\frac{xy}{1200}$ in (2), $x + 69 = 2139.$

Transpose and combine, $x = 2070.$

Substitute the value of x in (4), $\frac{2070y}{1200} = 6.90.$

Multiply by 1200, $2070y = 8280.$

$\therefore y = 4.$

Therefore, the sum is $2070, and the rate is 4 per cent.

55. A boy bought 570 oranges, some at 16 for 25 cents, and the remainder at 18 for 25 cents. He sold them all at the rate of 15 for 25 cents, and made a profit of 75 cents. How many oranges at each price did he buy?

Let x = the number of oranges the boy bought at 16 for 25 cents,

and y = the number of oranges he bought at 18 for 25 cents.

Then $\frac{25x}{16}$ = the number of cents the first lot cost,

$\frac{25y}{18}$ = the number of cents the second lot cost,

and $570 \times \frac{25}{15} = 950$ = the number of cents the boy received.

$\therefore x + y = 570,$ (1)

and $\frac{25x}{16} + \frac{25y}{18} = 950 - 75.$ (2)

Divide (2) by 25, $\frac{x}{16} + \frac{y}{18} = 35.$ (3)

Multiply (3) by 144, $9x + 8y = 5040.$ (4)

Multiply (1) by 8, $8x + 8y = 4560.$ (5)

Subtract (5) from (4), $x = 480.$

Substitute the value of x in (1), $480 + y = 570.$

Transpose and combine, $y = 90.$

Therefore, the boy bought 480 oranges at 16 for 25 cents, and 90 oranges at 18 for 25 cents.

56. A person has a certain capital invested at a certain rate per cent. Another person has $4000 more capital invested at one per cent better than the first, and receives $300 more income. A third person has $6000 more capital invested at two per cent better than the first, and receives $560 more income. Find the capital of each and the rate at which it is invested.

Let $x =$ the number of dollars in the capital of the first person,
and $y =$ the rate of interest on this capital.
Then $x + 4000 =$ the number of dollars in the capital of the second person,
$y + 1 =$ the rate of interest on this capital,
$x + 6000 =$ the number of dollars in the capital of the third person,
and $y + 2 =$ the rate of interest on this capital.

$$\therefore\ (x + 4000)\left(\frac{y+1}{100}\right) = \frac{xy}{100} + 300, \qquad (1)$$

and

$$(x + 6000)\left(\frac{y+2}{100}\right) = \frac{xy}{100} + 560. \qquad (2)$$

Multiply (1) by 100, $(x + 4000)(y + 1) = xy + 30000.$ (3)
Multiply (2) by 100, $(x + 6000)(y + 2) = xy + 56000.$ (4)
Expand (3), $xy + x + 4000y + 4000 = xy + 30000.$ (5)
Expand (4), $xy + 2x + 6000y + 12000 = xy + 56000.$ (6)
Transpose and combine (5), $x + 4000y = 26000.$ (7)
Transpose and combine (6), $2x + 6000y = 44000.$ (8)
Divide (8) by 2, $x + 3000y = 22000.$ (9)
Subtract (9) from (7), $1000y = 4000.$
$\therefore\ y = 4.$
Substitute the value of y in (7), $x + 16000 = 26000.$
Transpose and combine, $x = 10000.$
Then $x + 4000 = 14000,$
$x + 6000 = 16000,$
$y + 1 = 5,$
and $y + 2 = 6.$

Therefore, the first person has $10,000 invested at 4 per cent; the second, $14,000 at 5 per cent; and the third, $16,000 at 6 per cent.

57. A dealer sold a certain quantity of eggs. If he had sold 125 dozen more at 1 cent a dozen less, or 120 dozen less at 1 cent a dozen

more, he would have received the same amount of money for his eggs. Find the number of dozen he sold and the price per dozen.

Let $x =$ the number of dozen eggs the dealer sold,
and $y =$ the number of cents he received for each dozen.
$\therefore (x + 125)(y - 1) = xy,$ (1)
and $(x - 120)(y + 1) = xy.$ (2)
Expand (1), $xy - x + 125y - 125 = xy.$ (3)
Expand (2), $xy + x - 120y - 120 = xy.$ (4)
Transpose and combine (3), $-x + 125y = 125.$ (5)
Transpose and combine (4), $x - 120y = 120.$ (6)
Add (5) and (6), $5y = 245.$
$\therefore y = 49.$
Substitute the value of y in (6), $x - 5880 = 120.$
Transpose and combine, $x = 6000.$

Therefore, the dealer sold 6000 dozen eggs at 49 cents per dozen.

58. C sets out from the town A and walks toward B at the rate of 3 miles an hour. A quarter of an hour later D sets out from the town B and walks toward A at the rate of $3\frac{3}{4}$ miles an hour. If D walks 2 miles beyond the half-way point between B and A before he meets C, find the distance between A and B.

Let $x =$ the number of miles from A to B,
and $y =$ the number of hours C walks before meeting D.
Then $y - \frac{1}{4} =$ the number of hours D walks.
$\therefore 3y + (y - \frac{1}{4})3\frac{3}{4} = x,$ (1)
and $3y + 2 = (y - \frac{1}{4})3\frac{3}{4} - 2.$ (2)
Expand (1), $3y + \frac{15y}{4} - \frac{15}{16} = x.$ (3)
Expand (2), $3y + 2 = \frac{15y}{4} - \frac{15}{16} - 2.$ (4)
Multiply (3) by 16, $48y + 60y - 15 = 16x.$ (5)
Multiply (4) by 16, $48y + 32 = 60y - 15 - 32.$ (6)
Transpose and combine (5), $108y - 16x = 15.$ (7)
Transpose and combine (6), $-12y = -79.$
$\therefore y = 6\frac{7}{12}.$
Substitute the value of y in (7), $711 - 16x = 15.$
Transpose and combine, $-16x = -696.$
$\therefore x = 43\frac{1}{2}.$

Therefore, the distance from A to B is $43\frac{1}{2}$ miles.

59. A and B together can do a piece of work in 12 days. After working together 9 days, however, they call on C to aid them, and the three finish the work in 2 days. C finds that he can do as much work in 5 days as A does in 6 days. In how many days can each alone do the work?

Let $x =$ the number of days required for A alone,
$y =$ the number of days required for B alone,
and $z =$ the number of days required for C alone.

Then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} =$ the parts done in one day by A, B, C respectively.

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{12}, \tag{1}$$

$$\frac{9}{x} + \frac{9}{y} + \frac{2}{x} + \frac{2}{y} + \frac{2}{z} = 1, \tag{2}$$

and

$$\frac{5}{z} = \frac{6}{x}. \tag{3}$$

Combine (2),

$$\frac{11}{x} + \frac{11}{y} + \frac{2}{z} = 1. \tag{4}$$

Multiply (1) by 11,

$$\frac{11}{x} + \frac{11}{y} = \frac{11}{12}. \tag{5}$$

Subtract (5) from (4),

$$\frac{2}{z} = \frac{1}{12}.$$

$$\therefore z = 24.$$

Substitute the value of z in (3),

$$\frac{5}{24} = \frac{6}{x}.$$

Multiply by $24x$,

$$5x = 144.$$

$$\therefore x = 28\tfrac{4}{5}.$$

Substitute the values of x and z in (1),

$$\frac{5}{144} + \frac{1}{y} = \frac{1}{12}.$$

Multiply by $144y$,

$$5y + 144 = 12y.$$

Transpose and combine,

$$-7y = -144.$$

$$\therefore y = 20\tfrac{4}{7}.$$

Therefore, A can do the work in $28\frac{4}{5}$ days, B in $20\frac{4}{7}$ days, and C in 24 days.

60. An income of $670 a year is obtained from two investments, one in $4\frac{1}{2}$ per cent stock and the other in 5 per cent stock. If the $4\frac{1}{2}$ per cent stock should be sold at 110, and the 5 per cent stock at 125, the sum realized from the sales would be $16,600. How much of each stock is there?

Let x = the number of dollars invested at $4\frac{1}{2}$ per cent,
and y = the number of dollars invested at 5 per cent.

$$\therefore \frac{9x}{200} + \frac{5y}{100} = 670, \qquad (1)$$

and $$\frac{110x}{100} + \frac{125y}{100} = 16600. \qquad (2)$$

Multiply (1) by 200, $$9x + 10y = 134000. \qquad (3)$$

Simplify the terms of (2), $$\frac{11x}{10} + \frac{5y}{4} = 16600. \qquad (4)$$

Multiply (4) by 20, $$22x + 25y = 332000. \qquad (5)$$

Multiply (3) by 5, $$45x + 50y = 670000. \qquad (6)$$

Multiply (5) by 2, $$44x + 50y = 664000. \qquad (7)$$

Subtract (7) from (6), $$x = 6000.$$

Substitute the value of x in (3), $$54000 + 10y = 134000.$$

Transpose and combine, $$10y = 80000.$$

$$\therefore y = 8000.$$

Therefore, there is $6000 of the $4\frac{1}{2}$ per cent stock, and $8000 of the 5 per cent.

61. A box contains a mixture of 12 bushels of oats and 18 of corn, and another box contains a mixture of 12 bushels of oats and 4 of corn. How many bushels must be taken from each box in order to have a mixture of 14 bushels, half oats and half corn?

Let x = the number of bushels to be taken from the first box,
and y = the number of bushels to be taken from the second.

Then $\frac{12x}{30}$ = the number of bushels of oats from the first box,

$\frac{18x}{30}$ = the number of bushels of corn from the first box,

$\frac{12y}{16}$ = the number of bushels of oats from the second box,

and $\frac{4y}{16}$ = the number of bushels of corn from the second box.

$$\therefore \frac{12x}{30} + \frac{12y}{16} = 7, \qquad (1)$$

and $$\frac{18x}{30} + \frac{4y}{16} = 7. \qquad (2)$$

Simplify (1), $\frac{2x}{5} + \frac{3y}{4} = 7.$ (3)

Simplify (2), $\frac{3x}{5} + \frac{y}{4} = 7.$ (4)

Multiply (3) by 20, $8x + 15y = 140.$ (5)

Multiply (4) by 20, $12x + 5y = 140.$ (6)

Multiply (6) by 3, $36x + 15y = 420.$ (7)

Subtract (5) from (7), $28x = 280.$

$\therefore x = 10.$

Substitute the value of x in (6), $120 + 5y = 140.$

Transpose and combine, $5y = 20.$

$\therefore y = 4.$

Therefore, 10 bushels must be taken from the first box, and 4 bushels from the second box.

Exercise 78. Page 198

1. At the present time A is a years old and B is b years old. In how many years will A be c times as old as B?

Discuss the result (1) when b is greater than a; (2) when c is less than unity; (3) when $a = b$; (4) when $c = 1$.

Let $x =$ the required number of years.

Then $a + x =$ the number of years in A's age x years hence,

and $b + x =$ the number of years in B's age x years hence.

But x years hence A will be c times as old as B.

$$\therefore a + x = c(b + x).$$

Expand, $a + x = bc + cx.$

Transpose, $cx - x = a - bc.$

$$\therefore x = \frac{a - bc}{c - 1}.$$

Therefore, A will be c times as old as B in $\frac{a - bc}{c - 1}$ years.

(1) If b is greater than a, the value of x is negative. In this case B is older than A by $(b - a)$ years, and A will *never* be c times as old as B, provided c is greater than unity.

(2) If c is less than unity, the denominator of the value of x is negative, and therefore b is greater than a. The problem now means, in how many years will A be the fraction c as old as B.

(3) If $a = b$, A is just the age of B and always will be just as old as B. Hence, the value of c must be unity, and the problem is indeterminate.

(4) If $c = 1$, it follows that a must be equal to bc, and that $a = b$, as in (3).

2. A boatman, moving with the tide, rows n miles in t hours. In returning he rows against a tide m times as strong as the first and makes the distance in t' hours. Find the velocity of the second tide in miles per hour.

Discuss the result (1) when $m = 1$; (2) when $t' = t$; (3) when t' is less than t; (4) when m is negative; (5) when n is negative.

Let $x =$ the number of miles per hour the first tide flows.

Then $mx =$ the number of miles per hour the second tide flows,

$\frac{n}{t} =$ the number of miles per hour he makes with the first tide,

$\frac{n}{t} - x =$ the number of miles per hour he actually rows,

$\frac{n}{t'} =$ the number of miles per hour he makes against the second tide,

and $\frac{n}{t'} + mx =$ the number of miles per hour he actually rows.

$$\therefore \frac{n}{t'} + mx = \frac{n}{t} - x.$$

Multiply by tt', $nt + mtt'x = nt' - tt'x.$

Transpose, $mtt'x + tt'x = nt' - nt.$

Factor, $tt'(m+1)x = n(t' - t).$

$$\therefore x = \frac{n(t'-t)}{tt'(m+1)}.$$

$$mx = \frac{mn(t'-t)}{tt'(m+1)}.$$

Therefore, the velocity of the second tide is $\frac{mn(t'-t)}{tt'(m+1)}$ miles per hour.

(1) If $m = 1$, the boatman in returning is rowing against a tide just as strong as the first, and the velocity of the second tide reduces to $\frac{n(t'-t)}{2tt'}$ miles per hour.

(2) If $t' = t$, the value of mx reduces to $\frac{0}{t^2(m+1)}$, a form which shows the impossibility of the problem. It is evident that a boatman cannot row a certain distance against a tide however weak in the same time that it takes him to row the same distance with a tide, whether strong or weak. The only interpretation of this case is that he is rowing with a tide just as strong as the first.

(3) If t' is less than t, the value of mx is negative, which shows that the boatman is rowing *with* the second tide and not *against* it.

(4) If m is negative and greater than 1, the boatman is evidently rowing *with* the second tide and not *against* it.

(5) If n is negative, the value of mx is negative, which shows that the conditions are reversed, and that the boatman is rowing *against* the first tide and *with* the second.

3. A merchant owns two kinds of sugar that cost him, one a cents a pound and the other b cents a pound. He wishes to make a mixture of c pounds that shall cost him d cents a pound. How many pounds of each kind shall he take?

Discuss the result (1) when $a = b$; (2) when $a = d$ or $b = d$; (3) when $a = b = d$; (4) when a is greater than b and less than d; (5) when a is greater than b and b is greater than d.

Let $x =$ the number of pounds at a cents,
and $y =$ the number of pounds at b cents.

Then
$$x + y = c, \tag{1}$$
and
$$ax + by = d(x + y). \tag{2}$$

Substitute the value of $x + y$ from (1) in (2),
$$ax + by = cd. \tag{3}$$

Multiply (1) by a,
$$ax + ay = ac. \tag{4}$$

Subtract (3) from (4),
$$(a - b)y = (a - d)c.$$
$$\therefore\ y = \frac{(a - d)c}{a - b}.$$

Multiply (1) by b,
$$bx + by = bc. \tag{5}$$

Subtract (5) from (3),
$$(a - b)x = (d - b)c.$$
$$\therefore\ x = \frac{(d - b)c}{a - b}.$$

Therefore, he shall take $\frac{(d - b)c}{a - b}$ pounds at a cents, and $\frac{(a - d)c}{a - b}$ pounds at b cents.

We will suppose that, if the two prices a and b are different, a is greater than b. Then it is evident that the price of any mixture must be less than a but greater than b cents per pound; that is, a is greater than d, but d is greater than b.

(1) If $a = b$, the price of any mixture will evidently be the same as that of either kind. In this case the results become $x = \frac{0}{0}$, and $y = \frac{0}{0}$; that is, x and y are indeterminate. For the merchant may take as many pounds of each kind as he chooses, so long as he takes c pounds in all.

(2) If $d = a$, and a and b are different, then all the c pounds must be taken from the kind that costs a cents a pound; for any admixture of the second kind would change the price. In this case the results become $x = c$, and $y = 0$.

If $d = b$, the results become $x = 0$, and $y = c$.

(3) If $a = b = d$, the same reasoning applies as in (1).

(4) a cannot be greater than b and less than d, for then the price of the mixture would be higher than that of either of its components. The results show that y is negative; that is, the statement of the problem is in error.

(5) a and b cannot both be greater than d, for then the price of the mixture would be less than that of either of its components. In this case x is negative; that is, the statement of the problem is in error.

4. A party of a young men hired a camp, agreeing to share the expense equally; but, as b of the party were unable to go, each one who went paid c dollars more than he expected to pay. What was the rent of the camp?

Discuss the result (1) when $b = 0$; (2) when $b = a$; (3) when b is greater than a; (4) when c is negative.

Let $x =$ the number of dollars each expected to pay.
Then $ax =$ the number of dollars in the whole rent,
$a - b =$ the number who went to the camp,
$x + c =$ the number of dollars each paid,
and $(a - b)(x + c) =$ the number of dollars in the whole rent.

$$\therefore\ ax = (a - b)(x + c).$$

Expand, $$ax = ax + ac - bx - bc.$$

Transpose and combine, $$bx = ac - bc.$$

$$\therefore\ x = \frac{(a - b)c}{b}.$$

$$ax = \frac{(a - b)ac}{b}.$$

Therefore, the rent of the camp was $\frac{(a - b)ac}{b}$ dollars.

(1) If $b = 0$, the rent becomes $\frac{a^2c}{0}$, a form which shows the impossibility of the problem under this condition. It is evident that if $b = 0$, all the members of the party went to the camp, and therefore each

member paid 0 dollars more than he expected. Hence, if $b = 0$, c must equal 0.

(2) If $b = a$, the rent becomes $\frac{0}{a}$, a form which shows the impossibility of the problem under this condition. It is evident that if $b = a$, all the members of the party remained away from the camp, and therefore no member paid any rent. Hence, no rent would be paid.

(3) If b is greater than a, the rent becomes a negative number of dollars. It is evident that no more than all the party could be prevented from going to the camp.

(4) If c is negative, each member paid c dollars *less* than he expected, and the total rent is a negative number of dollars. This shows that b must be negative; in other words, that all the party went to the camp and b others besides the original party of a members.

Exercise 79. Page 203

1. In the figure on page 203 determine the coördinates of B; of M; of N; of R; of S; of H; of L; of A; of F; of D; of C.

B is the point $(-3, 4)$;
M is the point $(6, 4)$;
N is the point $(-3, -2)$;
R is the point $(6, 0)$;
S is the point $(0, 3)$;
H is the point $(-6, 0)$;
L is the point $(0, -3)$;
A is the point $(3, 2)$;
F is the point $(6, -2)$;
D is the point $(-4, -4)$;
C is the point $(-4, 0)$.

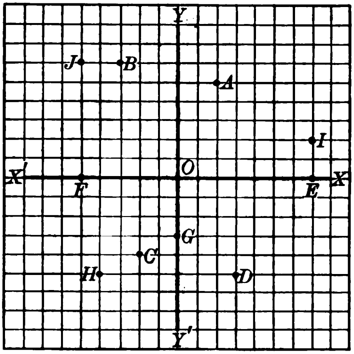
2. What is the abscissa of a point on the axis of y? What is the ordinate of a point on the axis of x?

The abscissa of a point on the axis of y is 0. The ordinate of a point on the axis of x is 0.

3. Where must a point lie if its ordinate is zero? if its abscissa is zero? if both abscissa and ordinate are zero?

If the ordinate of a point is zero, the point must lie on the axis of x. If the abscissa of a point is zero, the point must lie on the axis of y. If both the abscissa and the ordinate of a point are zero, the point must lie at the intersection of the axes of coördinates, that is, at the origin.

4. Plot the following points: $(2, 5)$, $(-3, 6)$, $(-2, -4)$, $(3, -5)$, $(7, 0)$, $(-5, 0)$, $(0, 0)$, $(0, -3)$, $(-4, -5)$, $(7, 2)$.



(i) To plot the point $(2, 5)$, count 2 units to the right from *O*, and then 5 units upwards. The point *A* is the point required.

(ii) To plot the point $(-3, 6)$, count 3 units to the left from *O*, and then 6 units upwards. The point *B* is the point required.

(iii) To plot the point $(-2, -4)$, count 2 units to the left from *O*, and then 4 units downwards. The point *C* is the point required.

(iv) To plot the point $(3, -5)$, count 3 units to the right from *O*, and then 5 units downwards. The point *D* is the point required.

(v) To plot the point $(7, 0)$, count 7 units to the right from *O*, and then 0 units in a vertical direction. The point *E* is the point required.

(vi) To plot the point $(-5, 0)$, count 5 units to the left from *O*, and then 0 units in a vertical direction. The point *F* is the point required.

(vii) To plot the point $(0, 0)$, count 0 units from *O* in both a horizontal and a vertical direction. The point *O* is the point required.

(viii) To plot the point $(0, -3)$, count 0 units from *O* in a horizontal direction, and then 3 units downwards. The point *G* is the point required.

(ix) To plot the point $(-4, -5)$, count 4 units to the left from *O*, and then 5 units downwards. The point *H* is the point required.

(x) To plot the point $(7, 2)$, count 7 units to the right from *O*, and then 2 units upwards. The point *I* is the point required.

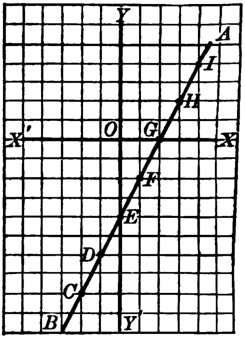
5. In what quadrant does a point lie if its coördinates are both positive? if both are negative? if the ordinate is positive and the abscissa negative? if the abscissa is positive and the ordinate negative?

If the coördinates of a point are both positive, the point lies in Quadrant I.

If the coördinates of a point are both negative, the point lies in Quadrant III.

If the ordinate of a point is positive and the abscissa negative, the point lies in Quadrant II.

If the abscissa of a point is positive and the ordinate negative, the point lies in Quadrant IV.



6. Plot the points $(-2, -8)$, $(-1, -6)$, $(0, -4)$, $(1, -2)$, $(2, 0)$, $(3, 2)$, $(4, 4)$. Do these points lie in a straight line? Is the equation $2x - y = 4$ satisfied if the abscissas are substituted in turn for x, and the corresponding ordinates for y?

Plot the points $(-2, -8)$, $(-1, -6)$, $(0, -4)$, $(1, -2)$, $(2, 0)$, $(3, 2)$, $(4, 4)$, which are the points C, D, E, F, G, H, I respectively.

These points evidently lie in the straight line AB.

Substitute in turn in the equation $2x - y = 4$ the abscissas of the given points for x and the corresponding ordinates for y.

Then $-4 + 8 = 4$, $\qquad 4 - 0 = 4$,

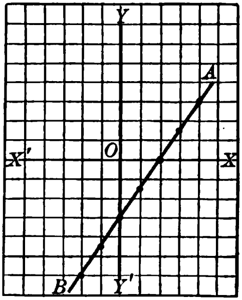
$-2 + 6 = 4$, $\qquad 6 - 2 = 4$,

$0 + 4 = 4$, $\qquad 8 - 4 = 4$.

$2 + 2 = 4$,

Therefore, the equation $2x - y = 4$ is satisfied in each case.

Exercise 80. Page 205



1. Plot the graph of $3x - 2y = 6$.

Transpose, $\qquad 2y = 3x - 6$.

Divide by 2, $\qquad y = \dfrac{3x - 6}{2}$.

If $\quad x = 0$, $\qquad y = -3$;

$x = +1$, $\qquad y = -1\frac{1}{2}$;

$x = +2$, $\qquad y = 0$;

$x = +3$, $\qquad y = +1\frac{1}{2}$;

$x = +4$, $\qquad y = +3$;

$x = -1$, $\qquad y = -4\frac{1}{2}$;

$x = -2$, $\qquad y = -6$.

The above seven points are plotted in the figure, and all lie on the straight line AB, which is the graph required.

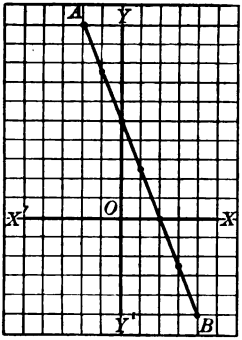
2. Plot the graph of $5x + 2y = 10$.

Transpose, $2y = 10 - 5x$.

Divide by 2, $y = \frac{10 - 5x}{2}$.

If $x = 0$, $y = +5$;
$x = +1$, $y = +2\frac{1}{2}$;
$x = +2$, $y = 0$;
$x = +3$, $y = -2\frac{1}{2}$;
$x = +4$, $y = -5$;
$x = -1$, $y = +7\frac{1}{2}$;
$x = -2$, $y = +10$.

The above seven points are plotted in the figure, and all lie on the straight line AB, which is the graph required.

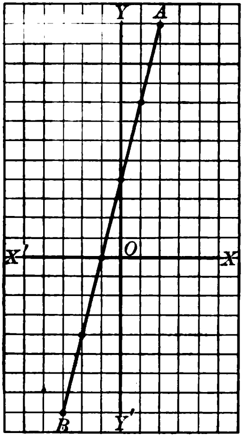


3. Plot the graph of $4x - y + 4 = 0$.

Transpose, $y = 4x + 4$.

If $x = 0$, $y = +4$;
$x = +1$, $y = +8$;
$x = +2$, $y = +12$;
$x = -1$, $y = 0$;
$x = -2$, $y = -4$;
$x = -3$, $y = -8$.

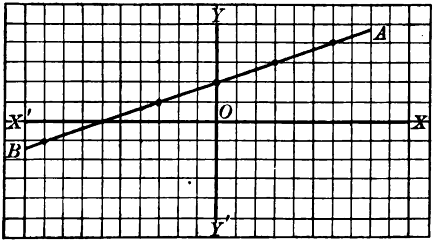
The above six points are plotted in the figure, and all lie on the straight line AB, which is the graph required.



4. Plot the graph of $-x + 3y = 6$.

Transpose, $x = 3y - 6$.

If $y = 0$, $x = -6$;
$y = +1$, $x = -3$;
$y = +2$, $x = 0$;
$y = +3$, $x = +3$;
$y = +4$, $x = +6$;
$y = -1$, $x = -9$.



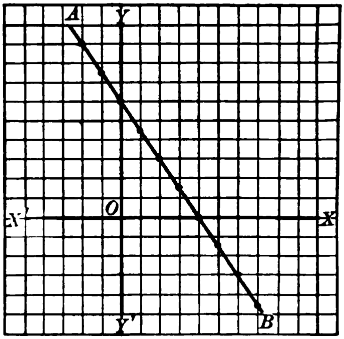
The above six points are plotted in the figure, and all lie on the straight line AB, which is the graph required.

5. Plot the graph of $3x + 2y = 12$.

Transpose, $2y = 12 - 3x$.

Divide by 2, $y = \frac{12 - 3x}{2}$.

If $x = 0$, $y = +6$;
$x = +1$, $y = +4\frac{1}{2}$;
$x = +2$, $y = +3$;
$x = +3$, $y = +1\frac{1}{2}$;
$x = +4$, $y = 0$;
$x = +5$, $y = -1\frac{1}{2}$;
$x = +6$, $y = -3$;
$x = +7$, $y = -4\frac{1}{2}$;
$x = -1$, $y = +7\frac{1}{2}$;
$x = -2$, $y = +9$.

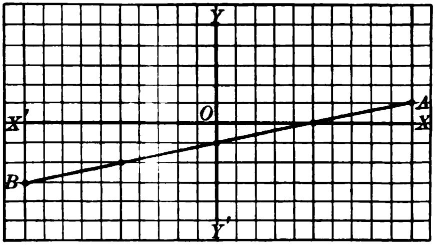


The above ten points are plotted in the figure, and all lie on the straight line AB, which is the graph required.

6. Plot the graph of $x - 5y = 5$.

Transpose,
$x = 5y + 5$.

If $y = 0$, $x = +5$;
$y = +1$, $x = +10$;
$y = -1$, $x = 0$;
$y = -2$, $x = -5$;
$y = -3$, $x = -10$.



The above five points are plotted in the figure, and all lie on the straight line AB, which is the graph required.

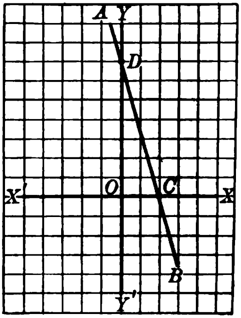
7. Plot the graph of $7x + 2y - 14 = 0$ by finding the points in which the graph cuts the axes.

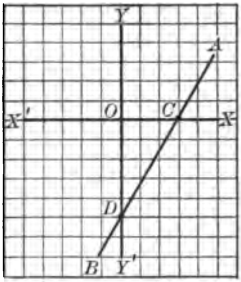
We have $7x + 2y - 14 = 0$.

If $x = 0$, $y = +7$;
and if $y = 0$, $x = +2$.

The point C, (2, 0), and the point D, (0, 7), are plotted in the figure.

The straight line AB, drawn through the points C and D, is the graph required.



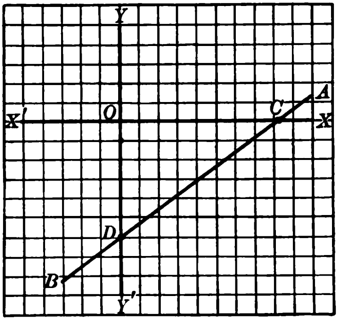


8. Plot the graph of $5x - 3y - 15 = 0$ by finding the points in which the graph cuts the axes.

We have $5x - 3y - 15 = 0$.

If $x = 0$, $y = -5$;
and if $y = 0$, $x = +3$.

The point C, $(3, 0)$, and the point D, $(0, -5)$, are plotted in the figure.
The straight line AB, drawn through the points C and D, is the graph required.



9. Plot the graph of

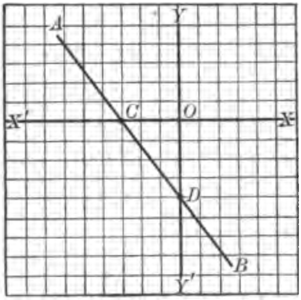
$$3x - 4y - 24 = 0$$

by finding the points in which the graph cuts the axes.

We have $3x - 4y - 24 = 0$.

If $x = 0$, $y = -6$;
and if $y = 0$, $x = +8$.

The point C, $(8, 0)$, and the point D, $(0, -6)$, are plotted in the figure.
The straight line AB, drawn through the points C and D, is the graph required.



10. Plot the graph of

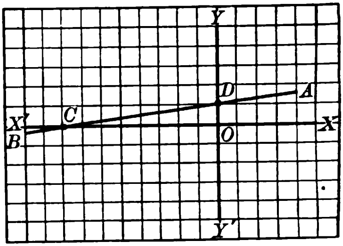
$$4x + 3y + 12 = 0$$

by finding the points in which the graph cuts the axes.

We have $4x + 3y + 12 = 0$.

If $x = 0$, $y = -4$;
and if $y = 0$, $x = -3$.

The point C, $(-3, 0)$, and the point D, $(0, -4)$, are plotted in the figure.
The straight line AB, drawn through the points C and D, is the graph required.



11. Plot the graph of

$$x - 8y + 8 = 0$$

by finding the points in which the graph cuts the axes.

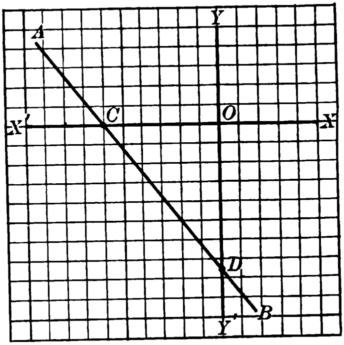
We have $x - 8y + 8 = 0$.

If $x = 0$, $y = +1$;

and if $y = 0$, $x = -8$.

The point C, $(-8, 0)$, and the point D, $(0, 1)$, are plotted in the figure.

The straight line AB, drawn through the points C and D, is the graph required.



12. Plot the graph of

$$5x + 4y + 30 = 0$$

by finding the points in which the graph cuts the axes.

We have $5x + 4y + 30 = 0$.

If $x = 0$, $y = -7\frac{1}{2}$;

and if $y = 0$, $x = -6$.

The point C, $(-6, 0)$, and the point D, $(0, -7\frac{1}{2})$, are plotted in the figure.

The straight line AB, drawn through the points C and D, is the graph required.

13. Plot the graph of the equation

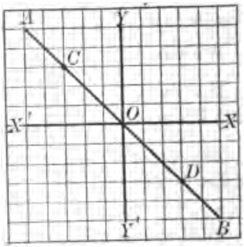
$$x + y = 0$$

by finding any two points.

We have $x + y = 0$.

If $x = -3$, $y = +3$;

$x = +3$, $y = -3$.



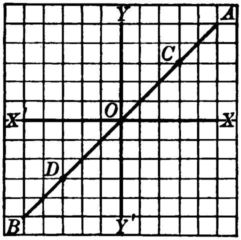
The point C, $(-3, 3)$, and the point D, $(3, -3)$, are plotted in the figure.

The straight line AB, drawn through the points C and D, is the graph required.

14. Plot the graph of the equation

$$x - y = 0$$

by finding any two points.



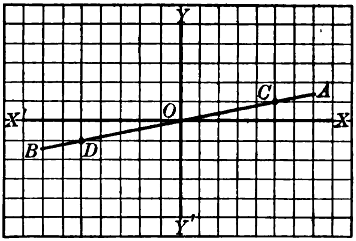
We have $x - y = 0$.

If $x = +3$, $y = +3$;
$x = -3$, $y = -3$.

The point C, $(3, 3)$, and the point D, $(-3, -3)$, are plotted in the figure.

The straight line AB, drawn through the points C and D, is the graph required.

15. Plot the graph of the equation $x - 5y = 0$ by finding any two points.



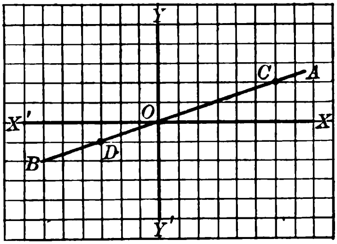
We have $x - 5y = 0$.

If $x = +5$, $y = +1$;
$x = -5$, $y = -1$.

The point C, $(5, 1)$, and the point D, $(-5, -1)$, are plotted in the figure.

The straight line AB, drawn through the points C and D, is the graph required.

16. Plot the graph of the equation $2x = 6y$ by finding any two points.



We have $2x = 6y$.

If $x = +6$, $y = +2$;
$x = -3$, $y = -1$.

The point C, $(6, 2)$, and the point D, $(-3, -1)$, are plotted in the figure.

The straight line AB, drawn through the points C and D, is the graph required.

17. Plot the graph of the equation $5x + 4y = 0$ by finding any two points.

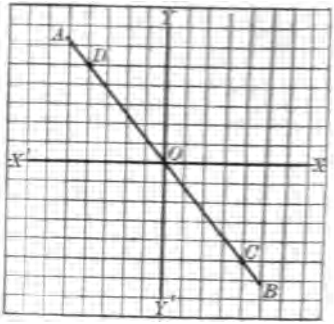
We have $5x + 4y = 0$.

If $x = +4$, $y = -5$;

$x = -4$, $y = +5$.

The point C, $(4, -5)$, and the point D, $(-4, 5)$, are plotted in the figure.

The straight line AB, drawn through the points C and D, is the graph required.



18. Plot the graph of the equation $7x - 5y = 0$ by finding any two points.

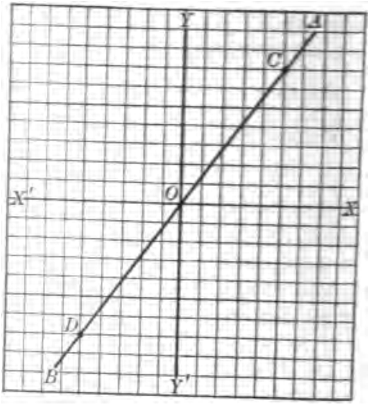
We have $7x - 5y = 0$.

If $x = +5$, $y = +7$;

$x = -5$, $y = -7$.

The point C, $(5, 7)$, and the point D, $(-5, -7)$, are plotted in the figure.

The straight line AB, drawn through the points C and D, is the graph required.



19. In what respect do the equations of Examples 1–12 differ from the equations of Examples 13–18?

Each of the equations of Examples 1–12 contains a constant term, while each of the equations of Examples 13–18 does not contain a constant term.

The graphs of the equations of Examples 1–12 do not pass through the origin, while the graphs of the equations of Examples 13–18 pass through the origin.

20. Does the graph of the equation $ax \pm by = 0$ pass through the origin? Why?

The graph of the equation $ax \pm by = 0$ passes through the origin.

The graph passes through the origin because the equation contains no constant term so that if $x = 0$, then $y = 0$. Hence, the origin is one point of the graph.

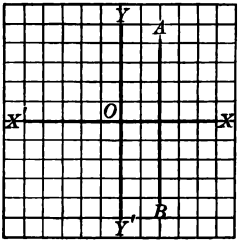
21. The equation of XX' is $y = 0$. Find the equation of YY'.

The equation of YY' is $x = 0$.

22. What is the position of a graph if its equation does not contain x? if its equation does not contain y?

If an equation does not contain x, its graph is a straight line parallel to XX'.

If an equation does not contain y, its graph is a straight line parallel to YY'.



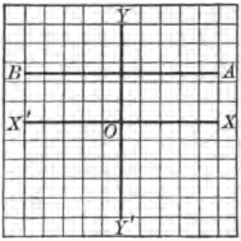
23. Plot the graph of $3x = 6$.

We have $3x = 6$.

Divide by 3, $x = 2$.

Hence, the graph of the equation $3x = 6$ is a straight line parallel to YY' and at a distance of 2 units to the right of it.

AB is the graph required.



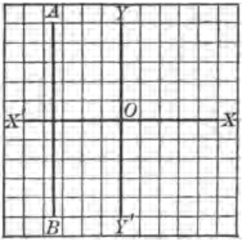
24. Plot the graph of $2y = 5$.

We have $2y = 5$.

Divide by 2, $y = 2\frac{1}{2}$.

Hence, the graph of the equation $2y = 5$ is a straight line parallel to XX' and at a distance of $2\frac{1}{2}$ units above it.

AB is the graph required.



25. Plot the graph of $x = -\frac{7}{2}$.

The graph of the equation $x = -\frac{7}{2}$ is a straight line parallel to YY' and at a distance of $\frac{7}{2}$ units to the left of it.

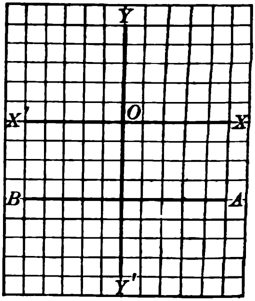
AB is the graph required.

26. Plot the graph of $y = -\frac{4}{5}$.

The graph of the equation $y = -\frac{4}{5}$ is a straight line parallel to XX' and at a distance of $\frac{4}{5}$ of a unit below it.

For convenience and accuracy in plotting the graph, choose as the unit 5 spaces of the coördinate paper.

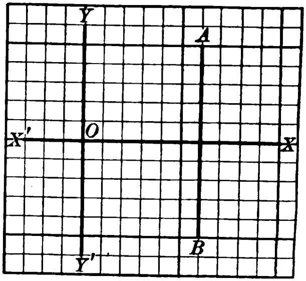
AB is the graph required.



27. Plot the graph of $5x = 30$.

The graph of the equation $5x = 30$ or $x = 6$ is a straight line parallel to YY' and at a distance of 6 units to the right of it.

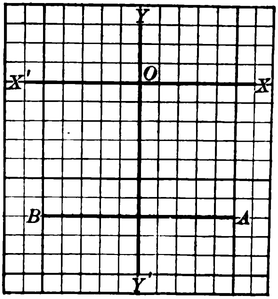
AB is the graph required.



28. Plot the graph of $6y = -42$.

The graph of the equation $6y = -42$ or $y = -7$ is a straight line parallel to XX' and at a distance of 7 units below it.

AB is the graph required.



Exercise 81. Page 207

1. Find by graphs exact values of x and y in the equations

$$\left.\begin{aligned} 2x - 5y &= 0 \\ 4x + 2y &= 24 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

and verify by solving the equations.

Transpose (1), $2x = 5y$.

$\therefore x = \frac{5}{2}y$.

If $y = +2$, $x = +5$;
$y = -2$, $x = -5$.

AB is the graph of equation (1).

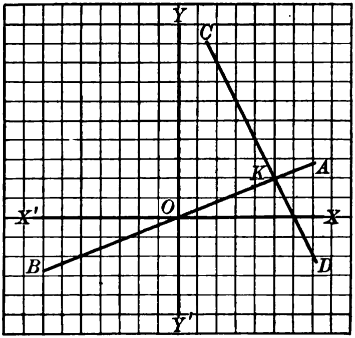
Transpose (2),

$2y = 24 - 4x$.

$\therefore y = 12 - 2x$.

If $x = +2$, $y = +8$;
$x = +6$, $y = 0$.

CD is the graph of equation (2).



The point K, the point of intersection of AB and CD, is the point (5, 2).

Therefore, $x = 5$ and $y = 2$.

Verification.

$$\left.\begin{aligned} 2x - 5y &= 0 \\ 4x + 2y &= 24 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Divide (2) by 2, $2x + y = 12$. (3)

Subtract (1) from (3), $6y = 12$.

$\therefore y = 2$.

Substitute the value of y in (3), $2x + 2 = 12$.

Transpose and combine, $2x = 10$.

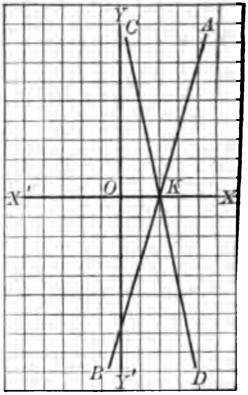
$\therefore x = 5$.

Therefore, $x = 5$, and $y = 2$.

2. Find by graphs exact values of x and y in the equations

$$\left.\begin{aligned} 7x - 2y &= 14 \\ 5x + y &= 10 \end{aligned}\right\} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

and verify by solving the equations.



Transpose (1), $2y = 7x - 14$.

$$\therefore\ y = \frac{7x - 14}{2}.$$

If $x = 0$, $y = -7$;
$x = +4$, $y = +7$.

AB is the graph of equation (1).

Transpose (2), $y = 10 - 5x$.

If $x = +1$, $y = +5$;
$x = +3$, $y = -5$.

CD is the graph of equation (2).

The point K, the point of intersection of AB and CD, is the point (2, 0).
Therefore, $x = 2$, and $y = 0$.

Verification.

$$\left.\begin{aligned} 7x - 2y &= 14 \\ 5x + y &= 10 \end{aligned}\right\} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Multiply (2) by 2, $10x + 2y = 20$. (3)

Add (1) and (3), $17x = 34$.

$$\therefore\ x = 2.$$

Substitute the value of x in (2), $10 + y = 10$.

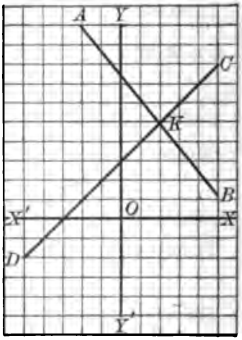
Transpose and combine, $y = 0$.

Therefore, $x = 2$, and $y = 0$.

3. Find by graphs exact values of x and y in the equations

$$\left.\begin{aligned} 5x + 4y &= 30 \\ x - y &= -3 \end{aligned}\right\} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

and verify by solving the equations.



Transpose (1), $4y = 30 - 5x$.

Divide by 4, $y = \dfrac{30 - 5x}{4}$.

If $x = -2$, $y = +10$;
$x = +2$, $y = +5$.

AB is the graph of equation (1).

Transpose (2), $y = x + 3$.

If $x = -3$, $y = 0$;
$x = +3$, $y = +6$.

CD is the graph of equation (2).

The point K, the point of intersection of AB and CD, is the point (2, 5).

Therefore, $x = 2$, and $y = 5$.

Verification.

$$\left.\begin{aligned} 5x + 4y &= 30 \\ x - y &= -3 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (2) by 4, $4x - 4y = -12.$ (3)

Add (1) and (3), $9x = 18.$

$\therefore x = 2.$

Substitute the value of x in (2), $2 - y = -3.$

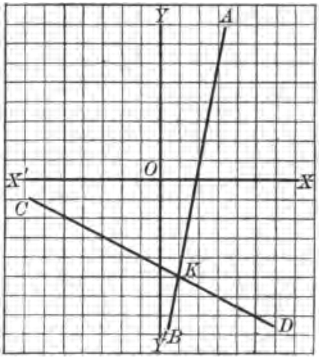
Transpose and combine, $y = 5.$

Therefore, $x = 2$, and $y = 5$.

4. Find by graphs exact values of x and y in the equations

$$\left.\begin{aligned} 11x - 2y &= 21 \\ 2x + 4y &= -18 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

and verify by solving the equations.



Transpose (1), $2y = 11x - 21.$

Divide by 2, $y = \dfrac{11x - 21}{2}.$

If $x = +3$, $y = +6$;

$x = +1$, $y = -5$.

AB is the graph of equation (1).

Transpose (2), $2x = -4y - 18.$

Divide by 2, $x = -2y - 9.$

If $y = -2$, $x = -5$;

$y = -7$, $x = +5$.

CD is the graph of equation (2).

The point K, the point of intersection of AB and CD, is the point (1, -5).

Therefore, $x = 1$, and $y = -5$.

Verification.

$$\left.\begin{aligned} 11x - 2y &= 21 \\ 2x + 4y &= -18 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Divide (2) by 2, $x + 2y = -9.$ (3)

Add (1) and (3), $12x = 12.$

$\therefore x = 1.$

Substitute the value of x in (3), $1 + 2y = -9.$

Transpose and combine, $2y = -10.$

$\therefore y = -5.$

Therefore, $x = 1$, and $y = -5$.

5. Find by graphs exact values of x and y in the equations

$$\left.\begin{aligned} 5x + 8y &= 20 \\ 2x - 3y &= -23 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

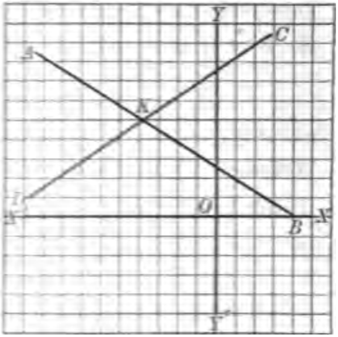
and verify by solving the equations.

Transpose (1), $8y = 20 - 5x.$

Divide by 8, $y = \dfrac{20 - 5x}{8}.$

If $x = +4, \quad y = 0;$
$x = -4, \quad y = +5.$

AB is the graph of equation (1).



Transpose (2),

$$3y = 2x + 23.$$

Divide by 3,

$$y = \frac{2x + 23}{3}.$$

If $x = -7, \quad y = +3;$
$x = +2, \quad y = +9.$

CD is the graph of equation (2).

The point K, the point of intersection of AB and CD, is the point $(-4, 5)$.

Therefore, $x = -4$, and $y = 5$.

Verification.

$$\left.\begin{aligned} 5x + 8y &= 20 \\ 2x - 3y &= -23 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (1) by 2, $10x + 16y = 40.$ (3)

Multiply (2) by 5, $10x - 15y = -115.$ (4)

Subtract (4) from (3), $31y = 155.$

$\therefore y = 5.$

Substitute the value of y in (2), $2x - 15 = -23.$

Transpose and combine, $2x = -8.$

$\therefore x = -4.$

Therefore, $x = -4$, and $y = 5$.

6. Find by graphs exact values of x and y in the equations

$$\left.\begin{aligned}3x + 4y &= 30\\5x - 6y &= 12\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

and verify by solving the equations.

Transpose (1),

$$4y = 30 - 3x.$$

Divide by 4,

$$y = \frac{30 - 3x}{4}.$$

If $x = -2$, $y = +9$;

$x = +10$, $y = 0$.

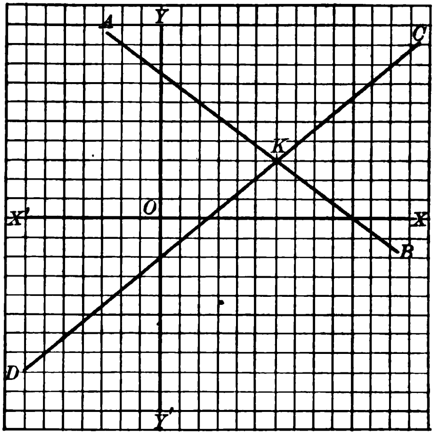
AB is the graph of equation (1).

Transpose (2),

$$5x = 12 + 6y.$$

Divide by 5,

$$x = \frac{12 + 6y}{5}.$$



If $y = -7$, $x = -6$;

$y = +8$, $x = +12$.

CD is the graph of equation (2).

The point *K*, the point of intersection of *AB* and *CD*, is the point (6, 3).

Therefore, $x = 6$, and $y = 3$.

Verification.

$$\left.\begin{aligned}3x + 4y &= 30\\5x - 6y &= 12\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 3, $9x + 12y = 90.$ (3)

Multiply (2) by 2, $10x - 12y = 24.$ (4)

Add (3) and (4), $19x = 114.$

$\therefore x = 6.$

Substitute the value of x in (1), $18 + 4y = 30.$

Transpose and combine, $4y = 12.$

$\therefore y = 3.$

Therefore, $x = 6$, and $y = 3$.

7. Find by graphs approximate values of x and y in

$$\left.\begin{aligned} 4x - 5y &= 10 \\ 2x + 3y &= 9 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Transpose (1), $4x = 10 + 5y$.

Divide by 4, $x = \dfrac{10 + 5y}{4}$.

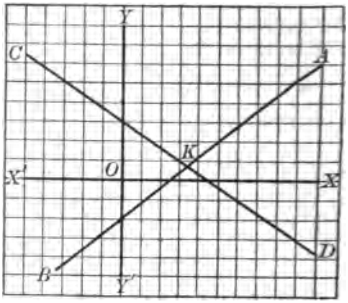
If $y = +6, \quad x = +10;$
$y = -2, \quad x = 0.$

AB is the graph of equation (1).

Transpose (2), $2x = 9 - 3y$.

Divide by 2, $x = \dfrac{9 - 3y}{2}$.

If $y = +5, \quad x = -3;$
$y = -3, \quad x = +9.$



CD is the graph of equation (2).

The point K, the point of intersection of AB and CD, is approximately the point $(3\frac{1}{2}, \frac{3}{4})$.

Therefore, $x = 3\frac{1}{2}$, and $y = \frac{3}{4}$, approximately.

8. Find by graphs approximate values of x and y in

$$\left.\begin{aligned} 8x + y &= 20 \\ 2x - 5y &= 10 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Transpose (1), $y = 20 - 8x$.

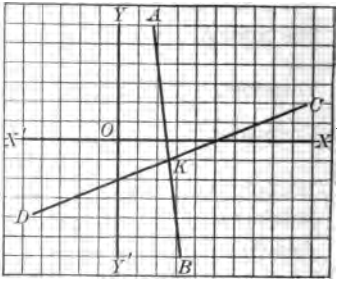
If $x = +2, \quad y = +4;$
$x = +3, \quad y = -4.$

AB is the graph of equation (1).

Transpose (2), $2x = 10 + 5y$.

Divide by 2, $x = \dfrac{10 + 5y}{2}$.

If $y = 0, \quad x = +5;$
$y = -2, \quad x = 0.$



CD is the graph of equation (2).

The point K, the point of intersection of AB and CD, is approximately the point $(2\frac{1}{2}, -1)$.

Therefore, $x = 2\frac{1}{2}$, and $y = -1$, approximately.

9. Find by graphs approximate values of x and y in

$$\left.\begin{aligned} 7x - 2y &= 14 \\ 5x + 3y &= 15 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Transpose (1), $2y = 7x - 14$.

Divide by 2, $y = \dfrac{7x - 14}{2}$.

If $x = 0, \quad y = -7;$

$x = +2, \quad y = 0.$

AB is the graph of equation (1).

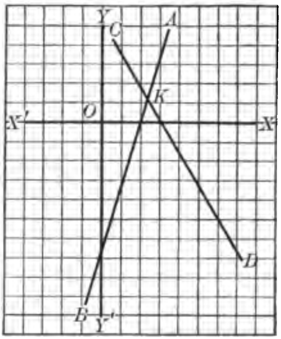
Transpose (2), $3y = 15 - 5x$.

Divide by 3, $y = \dfrac{15 - 5x}{3}$.

If $x = +3, \quad y = 0;$

$x = +6, \quad y = -5.$

CD is the graph of equation (2).



The point K, the point of intersection of AB and CD, is approximately the point $(2\frac{1}{3}, 1\frac{1}{6})$.

Therefore, $x = 2\frac{1}{3}$, and $y = 1\frac{1}{6}$, approximately.

10. Find by graphs approximate values of x and y in

$$\left.\begin{aligned} 9x - 4y &= 18 \\ 2x + 5y &= 20 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Transpose (1),

$4y = 9x - 18.$

Divide by 4, $y = \dfrac{9x - 18}{4}$.

If $x = +2, \quad y = 0;$

$x = -2, \quad y = -9.$

AB is the graph of equation (1).

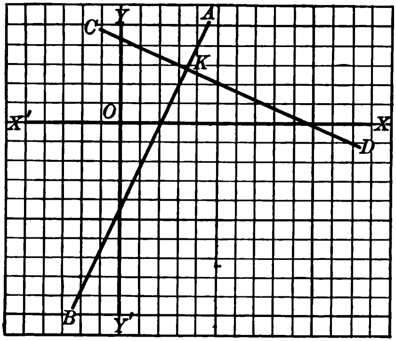
Transpose (2),

$5y = 20 - 2x.$

Divide by 5, $y = \dfrac{20 - 2x}{5}$.

If $x = +5, \quad y = +2;$

$x = +10, \quad y = 0.$



CD is the graph of equation (2).

The point K, the point of intersection of AB and CD, is approximately the point $(3\frac{1}{4}, 2\frac{3}{4})$.

Therefore, $x = 3\frac{1}{4}$, and $y = 2\frac{3}{4}$, approximately.

11. **The graphs of the equations $3x+5y=5$, $2x-y=12$, and $x+3y=-1$ meet in a point. Are the equations simultaneous? Give reason.**

We have

$$\left.\begin{aligned} 2x+3y&=4\\ 2x-y&=12\\ x+3y&=-1\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

Transpose (1),

$$3y=4-2x.$$

Divide by 3,

$$y=\frac{4-2x}{3}.$$

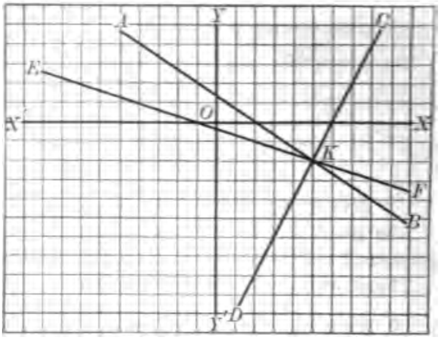
If $x=+5$, $y=-2$;
$x=-4$, $y=+4$.

AB is the graph of equation (1).

Transpose (2),

$$y=2x-12.$$

If $x=+8$, $y=+4$;
$x=+2$, $y=-8$.



CD is the graph of equation (2).

Transpose (3), $x=-3y-1$.

If $y=+2$, $x=-7$;
$y=0$, $x=-1$.

EF is the graph of equation (3).

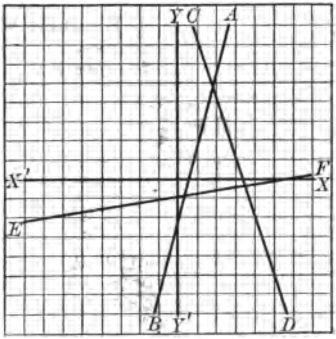
AB, *CD*, and *EF* intersect at the point *K*, $(5, -2)$.

The equations are simultaneous because the unknown numbers, x and y, stand for the same numbers in all three equations.

12. Do the graphs of the equations $4x-y=2$, $3x+y=10$, and $x-6y=5$ meet in a point? Are the equations simultaneous? Are the equations inconsistent?

We have

$$\left.\begin{aligned} 4x-y&=2\\ 3x+y&=10\\ x-6y&=5\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\\(3)\end{matrix}$$



Transpose (1), $y = 4x - 2$.

If $x = -1$, $y = -6$;
$x = +2$, $y = +6$.

AB is the graph of equation (1).

Transpose (2), $y = 10 - 3x$.

If $x = +5$, $y = -5$;
$x = +1$, $y = +7$.

CD is the graph of equation (2).

Transpose (3), $x = 5 + 6y$.

If $y = 0$, $x = +5$;
$y = -2$, $x = -7$.

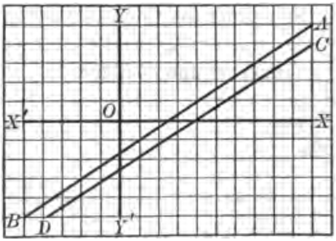
EF is the graph of equation (3).

The three graphs evidently do not intersect in a common point.

Hence, the equations are not simultaneous and are inconsistent.

13. Are the equations $2x - 3y = 5$ and $2x - 3y = 8$ simultaneous? What is shown by their graphs?

We have
$$\left.\begin{aligned}2x - 3y &= 5\\ 2x - 3y &= 8\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$



Transpose (1), $2x = 5 + 3y$.

Divide by 2, $x = \frac{5 + 3y}{2}$.

If $y = +3$, $x = +7$;
$y = -3$, $x = -2$.

AB is the graph of equation (1).

Transpose (2), $2x = 8 + 3y$.

Divide by 2, $x = \frac{8 + 3y}{2}$.

If $y = +2$, $x = +7$;
$y = -2$, $x = +1$.

CD is the graph of equation (2).

The graphs AB and CD evidently do not meet. Therefore, the equations are not simultaneous.

The graphs are parallel lines.

Exercise 82. Page 209

1. Solve in positive integers $x + y = 9.$

Transpose, $y = 9 - x.$

If $x = 1,\quad y = 8;\qquad x = 5,\quad y = 4;$

$x = 2,\quad y = 7;\qquad x = 6,\quad y = 3;$

$x = 3,\quad y = 6;\qquad x = 7,\quad y = 2;$

$x = 4,\quad y = 5;\qquad x = 8,\quad y = 1.$

2. Solve in positive integers $x + 9y = 27.$

Transpose, $x = 27 - 9y.$

If $y = 1,\ x = 18;$

$y = 2,\ x = 9.$

3. Solve in positive integers $3x + 7y = 46. \quad (1)$

Transpose, $3x = 46 - 7y.$

Divide by 3, $x = 15 - 2y + \dfrac{1-y}{3}. \quad (2)$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\dfrac{1-y}{3} = m$, an integer.

Then $1 - y = 3m.$

$\therefore\ y = 1 - 3m.$

Substitute the value of y in (1),

$$3x + 7 - 21m = 46.$$

Transpose and combine, $3x = 39 + 21m.$

$\therefore\ x = 13 + 7m.$

For $m = 0,\ x = 13,$ and $y = 1;$

for $m = -1,\ x = 6,$ and $y = 4.$

4. Solve in positive integers $4x + 5y = 33. \quad (1)$

Transpose, $4x = 33 - 5y.$

Divide by 4, $x = 8 - y + \dfrac{1-y}{4}. \quad (2)$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\dfrac{1-y}{4} = m$, an integer.

Then $1 - y = 4m.$

$\therefore\ y = 1 - 4m.$

Substitute the value of y in (1),

$$4x + 5 - 20m = 33.$$

Transpose and combine, $4x = 28 + 20m.$

$$\therefore\ x = 7 + 5m.$$

For $m = 0,\ x = 7,$ and $y = 1;$
for $m = -1,\ x = 2,$ and $y = 5.$

5. Solve in positive integers $8x + 7y = 112.$ (1)

Transpose, $7y = 112 - 8x.$

Divide by 7, $y = 16 - x - \frac{x}{7}.$ (2)

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{x}{7} = m$, an integer.

$$\therefore\ x = 7m.$$

Substitute the value of x in (1), $56m + 7y = 112.$

Transpose, $7y = 112 - 56m.$

$$\therefore\ y = 16 - 8m.$$

For $m = 1,\ x = 7,$ and $y = 8.$

6. Solve in positive integers $7x + 10y = 100.$ (1)

Transpose, $7x = 100 - 10y.$

Divide by 7, $x = 14 - y + \frac{2 - 3y}{7}.$

Multiply by 5, $5x = 70 - 5y + \frac{10 - 15y}{7}.$

$$5x = 70 - 5y + 1 - 2y + \frac{3 - y}{7}.$$

Combine, $5x = 71 - 7y + \frac{3 - y}{7}.$ (2).

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{3 - y}{7} = m$, an integer.

Then $3 - y = 7m.$

$$\therefore\ y = 3 - 7m.$$

Substitute the value of y in (1),

$$7x + 30 - 70m = 100.$$

Transpose and combine, $7x = 70 + 70m.$

$$\therefore\ x = 10 + 10m.$$

For $m = 0,\ x = 10,$ and $y = 3.$

7. Solve in positive integers $3x + 8y = 25.$ (1)

Transpose, $3x = 25 - 8y.$

Divide by 3, $x = 8 - 3y + \frac{1+y}{3}.$ (2)

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{1+y}{3} = m$, an integer.

Then $1 + y = 3m.$

$\therefore\ y = 3m - 1.$

Substitute the value of y in (1),

$$3x + 24m - 8 = 25.$$

Transpose and combine, $3x = 33 - 24m.$

$\therefore\ x = 11 - 8m.$

For $m = 1,\ x = 3,$ and $y = 2.$

8. Solve in positive integers $15x + 9y = 87.$

Divide by 3, $5x + 3y = 29.$ (1)

Transpose, $3y = 29 - 5x.$

Divide by 3, $y = 9 - x + \frac{2-2x}{3}.$

$\therefore\ y = 9 - x + 2\left(\frac{1-x}{3}\right).$ (2)

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{1-x}{3} = m$, an integer.

Then $1 - x = 3m.$

$\therefore\ x = 1 - 3m.$

Substitute the value of x in (1),

$$5 - 15m + 3y = 29.$$

Transpose and combine, $3y = 24 + 15m.$

$\therefore\ y = 8 + 5m.$

For $m = 0,\ x = 1,$ and $y = 8;$

for $m = -1,\ x = 4,$ and $y = 3.$

9. Solve in positive integers $5x + 9y = 34$. (1)

Transpose, $5x = 34 - 9y$.

Divide by 5, $x = 6 - y + \frac{4 - 4y}{5}$.

$$\therefore x = 6 - y + 4\left(\frac{1 - y}{5}\right). \quad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{1 - y}{5} = m$, an integer.

Then $1 - y = 5m$.

$\therefore y = 1 - 5m$.

Substitute the value of y in (1),

$$5x + 9 - 45m = 34.$$

Transpose and combine, $5x = 25 + 45m$.

$\therefore x = 5 + 9m$.

For $m = 0$, $x = 5$, and $y = 1$.

10. Solve in positive integers $8x + 13y = 413$. (1)

Transpose, $8x = 413 - 13y$.

Divide by 8, $x = 51 - y + \frac{5 - 5y}{8}$.

$$\therefore x = 51 - y + 5\left(\frac{1 - y}{8}\right). \quad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{1 - y}{8} = m$, an integer.

Then $1 - y = 8m$.

$\therefore y = 1 - 8m$.

Substitute the value of y in (1),

$$8x + 13 - 104m = 413.$$

Transpose and combine, $8x = 400 + 104m$.

$\therefore x = 50 + 13m$.

For $m = 0$, $x = 50$, and $y = 1$;
for $m = -1$, $x = 37$, and $y = 9$;
for $m = -2$, $x = 24$, and $y = 17$;
for $m = -3$, $x = 11$, and $y = 25$.

11. Solve in least positive integers

$$4x - 9y = 53. \qquad (1)$$

Transpose, $4x = 53 + 9y.$

Divide by 4, $x = 13 + 2y + \frac{1 + y}{4}. \qquad (2)$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{1 + y}{4} = m$, an integer.

Then $1 + y = 4m.$

$\therefore\ y = 4m - 1.$

Substitute the value of y in (1),

$$4x - 36m + 9 = 53.$$

Transpose and combine, $4x = 44 + 36m.$

$\therefore\ x = 11 + 9m.$

For $m = 1,\ x = 20,$ and $y = 3.$

12. Solve in least positive integers

$$15x - 47y = 1. \qquad (1)$$

Transpose, $15x = 1 + 47y.$

Divide by 15, $x = 3y + \frac{1 + 2y}{15}.$

Multiply by 8, $8x = 24y + \frac{8 + 16y}{15}.$

$8x = 24y + y + \frac{8 + y}{15}.$

Combine, $8x = 25y + \frac{8 + y}{15}. \qquad (2)$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{8 + y}{15} = m$, an integer.

Then $8 + y = 15m.$

$\therefore\ y = 15m - 8.$

Substitute the value of y in (1),

$$15x - 705m + 376 = 1.$$

Transpose and combine, $15x = 705m - 375.$

$\therefore\ x = 47m - 25.$

For $m = 1,\ x = 22,$ and $y = 7.$

13. Solve in least positive integers

$$13y - 10x = 19. \qquad (1)$$

Transpose, $$10x = 13y - 19.$$

Divide by 10, $$x = y - 1 + \frac{3y - 9}{10}.$$

$$\therefore\ x = y - 1 + 3\left(\frac{y-3}{10}\right). \qquad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $$\frac{y-3}{10} = m, \text{ an integer.}$$

Then $$y - 3 = 10m.$$

$$\therefore\ y = 10m + 3.$$

Substitute the value of y in (1),

$$130m + 39 - 10x = 19.$$

Transpose and combine, $$10x = 130m + 20.$$

$$\therefore\ x = 13m + 2.$$

For $$m = 0,\ x = 2, \text{ and } y = 3.$$

14. Solve in least positive integers

$$7x - 3y = 17. \qquad (1)$$

Transpose, $$3y = 7x - 17.$$

Divide by 3, $$y = 2x - 5 + \frac{x-2}{3}. \qquad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $$\frac{x-2}{3} = m, \text{ an integer.}$$

Then $$x - 2 = 3m.$$

$$\therefore\ x = 3m + 2.$$

Substitute the value of x in (1),

$$21m + 14 - 3y = 17.$$

Transpose and combine, $$3y = 21m - 3.$$

$$\therefore\ y = 7m - 1.$$

For $$m = 1,\ x = 5, \text{ and } y = 6.$$

15. Solve in least positive integers

$$11x - 9y = 31. \qquad (1)$$

Transpose, $9y = 11x - 31.$

Divide by 9, $y = x - 3 + \frac{2x - 4}{9}.$

$$\therefore y = x - 3 + 2\left(\frac{x - 2}{9}\right). \qquad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{x - 2}{9} = m$, an integer.

Then $x - 2 = 9m.$

$$\therefore x = 9m + 2.$$

Substitute the value of x in (1),

$$99m + 22 - 9y = 31.$$

Transpose and combine, $9y = 99m - 9.$

$$\therefore y = 11m - 1.$$

For $m = 1,\ x = 11,$ and $y = 10.$

16. Solve in least positive integers

$$13x - 5y = 113. \qquad (1)$$

Transpose, $5y = 13x - 113.$

Divide by 5, $y = 2x - 22 + \frac{3x - 3}{5}.$

$$\therefore y = 2x - 22 + 3\left(\frac{x - 1}{5}\right). \qquad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{x - 1}{5} = m$, an integer.

Then $x - 1 = 5m.$

$$\therefore x = 5m + 1.$$

Substitute the value of x in (1),

$$65m + 13 - 5y = 113.$$

Transpose and combine, $5y = 65m - 100.$

$$\therefore y = 13m - 20.$$

For $m = 2,\ x = 11,$ and $y = 6.$

17. Solve in least positive integers

$$14x - 9y = 229. \qquad (1)$$

Transpose, $9y = 14x - 229.$

Divide by 9, $y = x - 25 + \frac{5x - 4}{9}.$

Multiply by 2, $2y = 2x - 50 + \frac{10x - 8}{9}.$

$$2y = 2x - 50 + x + \frac{x - 8}{9}.$$

Combine, $2y = 3x - 50 + \frac{x - 8}{9}. \qquad (2)$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{x - 8}{9} = m$, an integer.

Then $x - 8 = 9m.$

$$\therefore x = 9m + 8.$$

Substitute the value of x in (1),

$$126m + 112 - 9y = 229.$$

Transpose and combine, $9y = 126m - 117.$

$$\therefore y = 14m - 13.$$

For $m = 1,\ x = 17,$ and $y = 1.$

18. Solve in least positive integers

$$13x - 23y = 42. \qquad (1)$$

Transpose, $13x = 23y + 42.$

Divide by 13, $x = y + 3 + \frac{10y + 3}{13}.$

Multiply by 4, $4x = 4y + 12 + \frac{40y + 12}{13}.$

$$4x = 4y + 12 + 3y + \frac{y + 12}{13}.$$

Combine, $4x = 7y + 12 + \frac{y + 12}{13}. \qquad (2)$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{y+12}{13} = m$, an integer.

Then $y + 12 = 13m$.

$\therefore y = 13m - 12$.

Substitute the value of y in (1),

$$13x - 299m + 276 = 42.$$

Transpose and combine, $13x = 299m - 234$.

$\therefore x = 23m - 18$.

For $m = 1$, $x = 5$, and $y = 1$.

19. **Find two numbers which, multiplied respectively by 3 and by 20, have for the sum of their products 368.**

Let $x =$ the first number,

and $y =$ the second number.

Then $3x + 20y = 368$. (1)

Transpose, $3x = 368 - 20y$.

Divide by 3, $x = 122 - 6y + \frac{2-2y}{3}$.

$\therefore x = 122 - 6y + 2\left(\frac{1-y}{3}\right)$. (2)

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{1-y}{3} = m$, an integer.

Then $1 - y = 3m$.

$\therefore y = 1 - 3m$.

Substitute the value of y in (1),

$$3x + 20 - 60m = 368.$$

Transpose and combine, $3x = 60m + 348$.

$\therefore x = 20m + 116$.

For $m = 0$, $x = 116$, and $y = 1$;

for $m = -1$, $x = 96$, and $y = 4$;

for $m = -2$, $x = 76$, and $y = 7$;

for $m = -3$, $x = 56$, and $y = 10$;

for $m = -4$, $x = 36$, and $y = 13$;

for $m = -5$, $x = 16$, and $y = 16$.

Therefore, the numbers required are 116 and 1; 96 and 4; 76 and 7; 56 and 10; 36 and 13; or 16 and 16.

20. Divide 76 into two parts such that one part shall be a positive multiple of 8 and the other a positive multiple of 9.

Let $8x =$ the first part,

and $9y =$ the second part.

Then $$8x + 9y = 76. \quad (1)$$

Transpose, $$8x = 76 - 9y.$$

Divide by 8, $$x = 9 - y + \frac{4-y}{8}. \quad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{4-y}{8} = m$, an integer.

Then $$4 - y = 8m.$$
$$\therefore y = 4 - 8m.$$

Substitute the value of y in (1),
$$8x + 36 - 72m = 76.$$

Transpose and combine, $$8x = 40 + 72m.$$
$$\therefore x = 5 + 9m.$$

For $m = 0$, $x = 5$, and $y = 4$.
$$\therefore 8x = 40,$$

and $$9y = 36.$$

Therefore, the required parts are 40 and 36.

21. How may a man pay a debt of $59, giving only bills of $5 and $2?

Let $x =$ the number of five-dollar bills,

and $y =$ the number of two-dollar bills.

Then $$5x + 2y = 59. \quad (1)$$

Transpose, $$2y = 59 - 5x.$$

Divide by 2, $$y = 29 - 2x + \frac{1-x}{2}. \quad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{1-x}{2} = m$, an integer.

Then $$1 - x = 2m.$$
$$\therefore x = 1 - 2m.$$

Substitute the value of x in (1),
$$5 - 10m + 2y = 59.$$

Transpose and combine, $$2y = 54 + 10m.$$
$$\therefore y = 27 + 5m.$$

For $m = 0$, $x = 1$, and $y = 27$;
for $m = -1$, $x = 3$, and $y = 22$;
for $m = -2$, $x = 5$, and $y = 17$;
for $m = -3$, $x = 7$, and $y = 12$;
for $m = -4$, $x = 9$, and $y = 7$;
for $m = -5$, $x = 11$, and $y = 2$.

Therefore, the man must use of five-dollar bills and two-dollar bills respectively 1 and 27; 3 and 22; 5 and 17; 7 and 12; 9 and 7; or 11 and 2.

22. A man paid $126 for calves at $7 each, and pigs at $3 each. How many calves and how many pigs did he buy?

Let $x =$ the number of calves,
and $y =$ the number of pigs.

Then $$7x + 3y = 126. \qquad (1)$$

Transpose, $$3y = 126 - 7x.$$

Divide by 3, $$y = 42 - 2x - \frac{x}{3}. \qquad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $$\frac{x}{3} = m, \text{ an integer.}$$

Then $$x = 3m.$$

Substitute the value of x in (2), $$y = 42 - 6m - m.$$

$$\therefore y = 42 - 7m.$$

For $m = 1$, $x = 3$, and $y = 35$;
for $m = 2$, $x = 6$, and $y = 28$;
for $m = 3$, $x = 9$, and $y = 21$;
for $m = 4$, $x = 12$, and $y = 14$;
for $m = 5$, $x = 15$, and $y = 7$.

Therefore, the man bought 3 calves and 35 pigs; 6 calves and 28 pigs; 9 calves and 21 pigs; 12 calves and 14 pigs; or 15 calves and 7 pigs.

23. Solve $11x - 21y = 7$, so that x and y may both be positive integers, and x a positive multiple of y.

We have $$11x - 21y = 7. \qquad (1)$$

Let $x = my$, where m is a positive integer.

Substitute the value of x in (1),

$$11my - 21y = 7.$$

$$\therefore y = \frac{7}{11m - 21}.$$

$$\therefore x = \frac{7m}{11m - 21}.$$

For $m = 2,\ x = 14,$ and $y = 7.$

Therefore, $x = 14,$ and $y = 7.$

24. A woman paid $106 for silk at $2.50 a yard and velvet at $3.50 a yard. How many yards of each did she buy?

Let $x =$ the number of yards of silk,

and $y =$ the number of yards of velvet.

Then $$\frac{5x}{2} + \frac{7y}{2} = 106.$$

Multiply by 2, $$5x + 7y = 212. \quad (1)$$

Transpose, $$5x = 212 - 7y.$$

Divide by 5, $$x = 42 - y + \frac{2 - 2y}{5}.$$

$$\therefore x = 42 - y + 2\left(\frac{1 - y}{5}\right). \quad (2)$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $$\frac{1 - y}{5} = m, \text{ an integer.}$$

Then $$1 - y = 5m.$$

$$\therefore y = 1 - 5m.$$

Substitute the value of y in (1),

$$5x + 7 - 35m = 212.$$

Transpose and combine, $$5x = 35m + 205.$$

$$\therefore x = 7m + 41,$$

For $m = 0,\ x = 41,$ and $y = 1;$

for $m = -1,\ x = 34,$ and $y = 6;$

for $m = -2,\ x = 27,$ and $y = 11;$

for $m = -3,\ x = 20,$ and $y = 16;$

for $m = -4,\ x = 13,$ and $y = 21;$

for $m = -5,\ x = 6,$ and $y = 26.$

Therefore, the woman bought 41 yards of silk and 1 yard of velvet; 34 yards of silk and 6 yards of velvet; 27 yards of silk and 11 yards of velvet; 20 yards of silk and 16 yards of velvet; 13 yards of silk and 21 yards of velvet; or 6 yards of silk and 26 yards of velvet.

25. A man paid \$72 for lambs at \$5.80 each, and geese at \$1.40 each. How many lambs and how many geese did he buy?

Let $x =$ the number of lambs,

and $y =$ the number of geese.

$$\$5.80 = \$5\tfrac{4}{5} = \$\tfrac{29}{5};\ \$1.40 = \$1\tfrac{2}{5} = \$\tfrac{7}{5}.$$

Then $$\frac{29x}{5} + \frac{7y}{5} = 72.$$

Multiply by 5, $$29x + 7y = 360. \tag{1}$$

Transpose, $$7y = 360 - 29x.$$

Divide by 7, $$y = 51 - 4x + \frac{3 - x}{7}. \tag{2}$$

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $$\frac{3 - x}{7} = m, \text{ an integer.}$$

Then $$3 - x = 7m.$$

$$\therefore x = 3 - 7m.$$

Substitute the value of x in (1),

$$87 - 203m + 7y = 360.$$

Transpose and combine, $$7y = 203m + 273.$$

$$\therefore y = 29m + 39.$$

For $m = 0$, $x = 3$, and $y = 39$;

for $m = -1$, $x = 10$, and $y = 10$.

Therefore, the man bought 3 lambs and 39 geese; or 10 lambs and 10 geese.

26. The diameter of a five-franc piece is 3.7 centimeters, and that of a one-franc piece 2.3 centimeters. How many coins of each kind must a man use to obtain a length of 1 meter?

Let $x =$ the number of five-franc pieces,

and $y =$ the number of one-franc pieces.

Then $3.7x =$ the number of centimeters in the length of the five-franc pieces,

and $2.3y =$ the number of centimeters in the length of the one-franc pieces.

One meter $= 100$ centimeters.

Then $$3.7x + 2.3y = 100.$$

Multiply by 10, $$37x + 23y = 1000. \tag{1}$$

Transpose. $$23y = 1000 - 37x.$$

Divide by 23, $y = 43 - x + \frac{11 - 14x}{23}.$

Multiply by 5, $5y = 215 - 5x + \frac{55 - 70x}{23}.$

$5y = 215 - 5x + 2 - 3x + \frac{9 - x}{23}.$

Combine, $5y = 217 - 8x + \frac{9 - x}{23}.$ (2)

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{9 - x}{23} = m$, an integer.

Then $9 - x = 23m.$

$\therefore x = 9 - 23m.$

Substitute the value of x in (1),

$333 - 851m + 23y = 1000.$

Transpose and combine, $23y = 851m + 667.$

$\therefore y = 37m + 29.$

For $m = 0,\ x = 9,$ and $y = 29.$

Therefore, to obtain a length of 1 meter a man must use 9 five-franc pieces and 29 one-franc pieces.

27. An excursion cost for each adult \$7.15, and for each child \$3.10. The total expense was \$97.25. How many adults and how many children went on the excursion?

Let $x =$ the number of adults,
and $y =$ the number of children.

Then $7.15x + 3.10y = 97.25.$

Multiply by 20, $143x + 62y = 1945.$ (1)

Transpose, $62y = 1945 - 143x.$

Divide by 62, $y = 31 - 2x + \frac{23 - 19x}{62}.$

Multiply by 13, $13y = 403 - 26x + \frac{299 - 247x}{62}.$

$13y = 403 - 26x + 5 - 4x - \frac{11 - x}{62}.$

Combine, $13y = 408 - 30x - \frac{11 - x}{62}.$ (2)

Since the left member of (2) is integral, the right member is integral and the fraction is an integer.

Let $\frac{11-x}{62}=m$, an integer.

Then $11-x=62m$.

$\therefore x=11-62m$.

Substitute the value of x in (2), $13y=408-330+1860m-m$.

Combine, $13y=78+1859m$.

$\therefore y=6+143m$.

For $m=0$, $x=11$, and $y=6$.

Therefore, there went on the excursion 11 adults and 6 children.

28. A box contains between 300 and 400 pears. If the pears are divided among some children, 13 pears being given to each child, 9 pears are left. If 15 pears are given to each child, 4 pears are left. How many pears are there in the box?

Let $x=$ the number of pears.

Then $\frac{x-9}{13}=$ an integer,

and $\frac{x-4}{15}=$ an integer.

Let $\frac{x-9}{13}=m$,

and $\frac{x-4}{15}=n$.

Then $x=13m+9$,

and $x=15n+4$.

$\therefore 13m+9=15n+4$.

Transpose and combine, $13m=15n-5$.

Divide by 13, $m=n+\frac{2n-5}{13}$.

Multiply by 7, $7m=7n+\frac{14n-35}{13}$.

$7m=7n+n-2+\frac{n-9}{13}$.

Transpose and combine, $7m-8n+2=\frac{n-9}{13}$. (1)

Since the left member of (1) is integral, the right member is integral.

Let $\frac{n-9}{13} = p$, an integer.

Then $n - 9 = 13p$.

$\therefore n = 13p + 9.$

Substitute the value of n in (1),

$$7m - 104p - 72 + 2 = p.$$

Transpose and combine, $7m = 105p + 70.$

$\therefore m = 15p + 10.$

For $p = 0,\ m = 10,$ and $n = 9;$

for $p = 1,\ m = 25,$ and $n = 22;$

for $p = 2,\ m = 40,$ and $n = 35;$

. . . .

$\therefore x = 13m + 9$, or $15n + 4$

$= 139, 334, 529, \cdots$

Since the number of pears is between 300 and 400, the required number is 334.

29. A farmer sells 12 calves, 16 lambs, and 9 pigs, and receives \$187. At the same price he sells 14 calves, 10 lambs, and 15 pigs, and receives \$197. What is the price of each?

Let x = the number of dollars received for each calf,

y = the number of dollars received for each lamb,

and z = the number of dollars received for each pig.

Then $12x + 16y + 9z = 187,$ (1)

and $14x + 10y + 15z = 197.$ (2)

Multiply (1) by 5, $60x + 80y + 45z = 935.$ (3)

Multiply (2) by 3, $42x + 30y + 45z = 591.$ (4)

Subtract (4) from (3), $18x + 50y = 344.$

Divide by 2, $9x + 25y = 172.$ (5)

Transpose, $9x = 172 - 25y.$

Divide by 9, $x = 19 - 2y + \frac{1-7y}{9}.$ (6)

Multiply by 4, $4x = 76 - 8y + \frac{4-28y}{9}.$

$4x = 76 - 8y - 3y + \frac{4-y}{9}.$

$\therefore 4x = 76 - 11y + \frac{4-y}{9}.$ (7)

Since the left member of (7) is integral, the right member is integral and the fraction is an integer.

Let $\frac{4-y}{9}=m$, an integer.

Then $4-y=9m$.

$\therefore y=4-9m$.

Substitute the value of y in (6), $x=19-8+18m-3+7m$.

$\therefore x=8+25m$.

For $m=0$, $x=8$, and $y=4$.

Substitute the values of x and y in (1),

$$96+64+9z=187.$$

Transpose and combine, $9z=27$.

$\therefore z=3$.

Therefore, the farmer receives \$8 for each calf, \$4 for each lamb, and \$3 for each pig.

30. Divide 151 into three parts such that the first part is divisible by 3, the second by 5, the third by 7, and twice the first part, three times the second, and four times the third is 483.

Let $3x=$ the first part,

$5y=$ the second part,

and $7z=$ the third part.

Then $3x+5y+7z=151, \qquad (1)$

and $6x+15y+28z=483. \qquad (2)$

Multiply (1) by 3, $9x+15y+21z=453. \qquad (3)$

Subtract (2) from (3), $3x-7z=-30$.

Transpose, $3x=7z-30$.

Divide by 3, $x=2z-10+\frac{z}{3}. \qquad (4)$

Since the left member of (4) is integral the right member is integral and the fraction is an integer.

Let $\frac{z}{3}=m$, an integer.

Then $z=3m$.

Substitute the value of z in (4), $x=6m-10+m$.

$\therefore x=7m-10$.

Substitute the values of x and z in (1),

$$21m-30+5y+21m=151.$$

Transpose and combine, $5y=181-42m$.

Then $3x=21m-30$,

and $7z=21m$.

For $m=3$, $3x=33$, $5y=55$, and $7z=63$.

Therefore, the required parts are 33, 55, and 63.

Exercise 83. Page 213

1. Find one limit of x, given

$$7x - 12 > 3x + 4.$$

Transpose and combine, $4x > 16.$

$$\therefore x > 4.$$

Therefore, the lower limit of the values of x is 4.

2. Find one limit of x, given

$$21x - 4 < 14x + 17.$$

Transpose and combine, $7x < 21.$

$$\therefore x < 3.$$

Therefore, the upper limit of the values of x is 3.

3. Find one limit of x, given

$$3x + 13 > 4x + 6.$$

Transpose and combine, $7 > x.$

$$\therefore x < 7.$$

Therefore, the upper limit of the values of x is 7.

4. Find one limit of x, given

$$\frac{3x+7}{4x+5} > \frac{3x+5}{4x+3}.$$

Multiply by $(4x + 5)(4x + 3)$,

$$12x^2 + 37x + 21 > 12x^2 + 35x + 25.$$

Transpose and combine, $2x > 4.$

$$\therefore x > 2.$$

Therefore, the lower limit of the values of x is 2.

5. Find one limit of x, given

$$\frac{1}{a} + \frac{a}{x+a} < \frac{x+a}{ax}.$$

Multiply by $ax(x+a)$, $x^2 + ax + a^2x < x^2 + 2ax + a^2.$

Transpose and combine, $a^2x - ax < a^2.$

Divide by a, $ax - x < a.$

$$\therefore x < \frac{a}{a-1}.$$

Therefore, the upper limit of the values of x is $\frac{a}{a-1}$.

6. Find the limits of x, given

$$\frac{9x+5}{14}+\frac{8x-7}{6x+2}<\frac{9x+14}{14}, \qquad (1)$$

and
$$\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}>\frac{2x-1}{2x+1}. \qquad (2)$$

Multiply (1) by 14, $9x+5+\frac{56x-49}{3x+1}<9x+14.$

Transpose and combine, $\frac{56x-49}{3x+1}<9.$

Multiply by $3x+1$, $56x-49<27x+9.$

Transpose and combine, $29x<58.$

$\therefore x<2.$

Multiply (2) by $4x^2-1$, $4x^2+4x+1-8>4x^2-4x+1.$

Transpose and combine, $8x>8.$

$\therefore x>1.$

Therefore, the upper limit of the values of x is 2 and the lower limit is 1.

7. Find the integral value of x, given

$$\tfrac{1}{3}(4x-1)-\tfrac{3}{4}>\tfrac{1}{6}(x-4)+\tfrac{1}{4}(3x+5), \qquad (1)$$

and
$$\tfrac{1}{3}(2x+1)+\tfrac{1}{4}(3x+1)<5\tfrac{1}{12}+\tfrac{1}{7}(5x-2). \qquad (2)$$

Multiply (1) by 12, $16x-4-9>2x-8+9x+15.$

Transpose and combine, $5x>20.$

$\therefore x>4.$

Multiply (2) by 84, $56x+28+63x+21<427+60x-24.$

Transpose and combine, $59x<354.$

$\therefore x<6.$

Hence the value of x lies between 4 and 6.

Therefore, the only possible integral value of x is 5.

8. Three times a certain integral number increased by 5 is not greater than 23; and five times the number diminished by 2 is not less than 28. Find the number.

Let $x=$ the number.

Then $3x+5\not>23, \qquad (1)$

and $5x-2\not<28. \qquad (2)$

Transpose and combine (1), $3x\not>18.$

$\therefore x\not>6.$

Transpose and combine (2), $5x\not<30.$

$\therefore x\not<6.$

Hence, x is not greater than 6 and not less than 6. Therefore, $x=6$.

Exercise 84. Page 214

1. Show that if the letters are unequal and positive

$$a^2 + b^2 + c^2 > ab + ac + bc.$$

Now by § 241, $a^2 + b^2 > 2\,ab,$ (1)

$a^2 + c^2 > 2\,ac,$ (2)

and $b^2 + c^2 > 2\,bc.$ (3)

Add (1), (2), and (3), $2\,a^2 + 2\,b^2 + 2\,c^2 > 2\,ab + 2\,ac + 2\,bc.$

Divide by 2, $a^2 + b^2 + c^2 > ab + ac + bc.$

2. Show that if the letters are unequal and positive

$$(ab + cd)(ac + bd) > 4\,abcd.$$

$(ab + cd)(ac + bd) >$ or $< 4\,abcd,$

according as $a^2bc + ab^2d + ac^2d + bcd^2 >$ or $< 4\,abcd.$

Now $a^2 + d^2 > 2\,ad.$ (§ 241)

$\therefore\ bc(a^2 + d^2) > 2\,ad(bc).$

$\therefore\ a^2bc + bcd^2 > 2\,abcd.$ (1)

Again, $b^2 + c^2 > 2\,bc.$ (§ 241)

$\therefore\ ad(b^2 + c^2) > 2\,bc(ad).$

$\therefore\ ab^2d + ac^2d > 2\,abcd.$ (2)

Add (1) and (2),

$$a^2bc + ab^2d + ac^2d + bcd^2 > 4\,abcd.$$

$$\therefore\ (ab + cd)(ac + bd) > 4\,abcd.$$

3. Show that if the letters are unequal and positive

$$ab + ac + bc < (a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2.$$

Now $ab + ac + bc >$ or $< (a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2,$

according as

$$\begin{aligned} ab + ac + bc > \text{or} < a^2 + b^2 + c^2 &+ 2\,ab - 2\,ac - 2\,bc \\ + a^2 + b^2 + c^2 &- 2\,ab + 2\,ac - 2\,bc \\ + a^2 + b^2 + c^2 &- 2\,ab - 2\,ac + 2\,bc, \end{aligned}$$

according as

$$ab + ac + bc > \text{or} < 3\,a^2 + 3\,b^2 + 3\,c^2 - 2\,ab - 2\,ac - 2\,bc,$$

according as

$$3\,ab + 3\,ac + 3\,bc > \text{or} < 3\,a^2 + 3\,b^2 + 3\,c^2,$$

according as

$$ab + ac + bc > \text{or} < a^2 + b^2 + c^2.$$

But $ab + ac + bc < a^2 + b^2 + c^2.$ (Example 1)

$$\therefore\ ab + ac + bc < (a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2.$$

4. **Show that if the letters are unequal and positive**

$$(a+b+c)^3 > 3(a+b)(a+c)(b+c).$$

Now $(a+b+c)^3 >$ or $< 3(a+b)(a+c)(b+c)$,

according as $a^3+b^3+c^3+3a^2b+3a^2c+3ab^2+3ac^2+3b^2c$
$+3bc^2+6abc$
$>$ or $< 3a^2b+3ab^2+6abc$
$+3b^2c+3a^2c+3ac^2+3bc^2$,

according as $a^3+b^3+c^3 >$ or < 0.

But $a^3+b^3+c^3 > 0$.

$$\therefore (a+b+c)^3 > 3(a+b)(a+c)(b+c).$$

5. **Show that if the letters are unequal and positive**

$$a^2b+a^2c+ab^2+b^2c+ac^2+bc^2 > 6abc.$$

Now $a^2+c^2 > 2ac$. (§ 241)

$\therefore (a^2+c^2)b > 2abc$.

$\therefore a^2b+bc^2 > 2abc$. (1)

Again, $a^2+b^2 > 2ab$. (§ 241)

$\therefore (a^2+b^2)c > 2abc$.

$\therefore a^2c+b^2c > 2abc$. (2)

Also, $b^2+c^2 > 2bc$. (§ 241)

$\therefore (b^2+c^2)a > 2abc$.

$\therefore ab^2+ac^2 > 2abc$. (3)

Add (1), (2), and (3),

$$a^2b+a^2c+ab^2+b^2c+ac^2+bc^2 > 6abc.$$

6. **Show that if the letters are unequal and positive**

$$(a^2+b^2)(a^4+b^4) > (a^3+b^3)^2.$$

Now $(a^2+b^2)(a^4+b^4) >$ or $< (a^3+b^3)^2$,

according as $a^6+a^4b^2+a^2b^4+b^6 >$ or $< a^6+2a^3b^3+b^6$,

according as $a^4b^2+a^2b^4 >$ or $< 2a^3b^3$,

according as $a^2+b^2 >$ or $< 2ab$.

But $a^2+b^2 > 2ab$. (§ 241)

$$\therefore (a^2+b^2)(a^4+b^4) > (a^3+b^3)^2.$$

7. Show that if $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$,

$$xy \nless ac + bd, \text{ or } ad + bc.$$

Now $xy \nless ac + bd,$

if $x^2y^2 \nless a^2c^2 + 2\,abcd + b^2d^2.$

But $x^2 = a^2 + b^2,$

and $y^2 = c^2 + d^2.$

$\therefore\ xy \nless ac + bd,$

if $(a^2 + b^2)(c^2 + d^2) \nless a^2c^2 + 2\,abcd + b^2d^2,$

if $a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 \nless a^2c^2 + 2\,abcd + b^2d^2,$

if $b^2c^2 + a^2d^2 \nless 2\,abcd.$

But $b^2c^2 + a^2d^2 \nless 2\,abcd.$ (§ 241)

$\therefore\ xy \nless ac + bd.$

Again, $xy \nless ad + bc,$

if $a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 \nless a^2d^2 + 2\,abcd + b^2c^2,$

if $a^2c^2 + b^2d^2 \nless 2\,abcd.$

But $a^2c^2 + b^2d^2 \nless 2\,abcd.$ (§ 241)

$\therefore\ xy \nless ad + bc.$

8. Which is the greater, $(a^2 + b^2)(c^2 + d^2)$ or $(ac + bd)^2$?

Now $(a^2 + b^2)(c^2 + d^2) > \text{or} < (ac + bd)^2,$

according as $a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 > \text{or} < a^2c^2 + 2\,abcd + b^2d^2,$

according as $a^2d^2 + b^2c^2 > \text{or} < 2\,abcd.$

But $a^2d^2 + b^2c^2 > 2\,abcd.$ (§ 241)

$\therefore\ (a^2 + b^2)(c^2 + d^2) > (ac + bd)^2.$

9. Which is the greater, $a^4 - b^4$ or $4\,a^3(a - b)$ when $a > b$?

Now $a^4 - b^4 > \text{or} < 4\,a^3(a - b),$

according as $(a - b)(a^3 + a^2b + ab^2 + b^3) > \text{or} < 4\,a^3(a - b),$

according as $a^3 + a^2b + ab^2 + b^3 > \text{or} < 4\,a^3,$

according as $a^2b + ab^2 + b^3 > \text{or} < 3\,a^3.$

Since $b < a,$

then $a^2b < a^3,$

$ab^2 < a^2b < a^3,$

and $b^3 < ab^2 < a^3.$

$\therefore\ a^2b + ab^2 + b^3 < 3\,a^3.$

$\therefore\ a^4 - b^4 < 4\,a^3(a - b),$ when $a > b.$

10. Which is the greater, $\frac{a+b}{2}$ or $\frac{2ab}{a+b}$?

Now $$\frac{a+b}{2} > \text{or} < \frac{2ab}{a+b},$$
according as $$a^2 + 2ab + b^2 > \text{or} < 4ab,$$
according as $$a^2 + b^2 > \text{or} < 2ab.$$
But $$a^2 + b^2 > 2ab. \qquad (\S\ 241)$$
$$\therefore \frac{a+b}{2} > \frac{2ab}{a+b}.$$

11. Which is the greater, $\frac{a+4b}{a+5b}$ or $\frac{a+6b}{a+7b}$?

Now $$\frac{a+4b}{a+5b} > \text{or} < \frac{a+6b}{a+7b},$$
according as $$(a+4b)(a+7b) > \text{or} < (a+5b)(a+6b),$$
according as $$a^2 + 11ab + 28b^2 > \text{or} < a^2 + 11ab + 30b^2,$$
according as $$0 > \text{or} < 2b^2.$$
But $$0 < 2b^2.$$
$$\therefore \frac{a+6b}{a+7b} > \frac{a+4b}{a+5b}.$$

12. Show that the sum of any positive fraction and its reciprocal is greater than 2.

Let $\frac{x}{y}$ = any positive fraction.

Then $\frac{y}{x}$ = its reciprocal,

and $\frac{x}{y} + \frac{y}{x}$ = their sum.

Now $$\frac{x}{y} + \frac{y}{x} > 2,$$
if $$x^2 + y^2 > 2xy.$$
But $$x^2 + y^2 > 2xy. \qquad (\S\ 241)$$
$$\therefore \frac{x}{y} + \frac{y}{x} > 2.$$

Therefore, the sum of any positive fraction and its reciprocal is greater than 2.

Exercise 85. Page 218

Perform the indicated operation:

1. $(x^3)^5 = x^{15}.$

2. $(x^2y^3)^6 = x^{12}y^{18}.$

3. $\left(\frac{2\,a^3b^2}{3\,x^3y^4}\right)^3 = \frac{8\,a^9b^6}{27\,x^9y^{12}}.$

4. $(-3\,x^2y^3)^4 = 81\,x^8y^{12}.$

5. $(-5\,a^3b^5)^3 = -125\,a^9b^{15}.$

6. $\left(-\frac{3\,x^2y^3}{4\,ab^2}\right)^5 = -\frac{243\,x^{10}y^{15}}{1024\,a^5b^{10}}.$

7. $(2\,a^2b^3yz^4)^5 = 32\,a^{10}b^{15}y^5z^{20}.$

8. $(-4\,c^3dx^2)^4 = 256\,c^{12}d^4x^8.$

9. $\frac{(-5\,ab^3)^5}{(10\,a^2xy^3)^3} = \frac{-5^5\,a^5b^{15}}{2^3 \times 5^3a^6x^3y^9} = \frac{-5^2b^{15}}{2^3ax^3y^9} = -\frac{25\,b^{15}}{8\,ax^3y^9}.$

10.
$$\begin{aligned}(a-3)^5 &= (a)^5 - 5(a)^4(3) + 10(a)^3(3)^2 - 10(a)^2(3)^3 \\ &\quad + 5(a)(3)^4 - (3)^5 \\ &= a^5 - 15\,a^4 + 90\,a^3 - 270\,a^2 + 405\,a - 243.\end{aligned}$$

11.
$$\begin{aligned}(a^2+2\,ab)^4 &= (a^2)^4 + 4(a^2)^3(2\,ab) + 6(a^2)^2(2\,ab)^2 + 4(a^2)(2\,ab)^3 \\ &\quad + (2\,ab)^4 \\ &= a^8 + 8\,a^7b + 24\,a^6b^2 + 32\,a^5b^3 + 16\,a^4b^4.\end{aligned}$$

12.
$$\begin{aligned}(2\,x+3\,y)^5 &= (2\,x)^5 + 5(2\,x)^4(3\,y) + 10(2\,x)^3(3\,y)^2 + 10(2\,x)^2(3\,y)^3 \\ &\quad + 5(2\,x)(3\,y)^4 + (3\,y)^5 \\ &= 32x^5 + 240\,x^4y + 720\,x^3y^2 + 1080\,x^2y^3 + 810\,xy^4 + 243\,y^5.\end{aligned}$$

13.
$$\begin{aligned}(3\,a-2\,b)^4 &= (3\,a)^4 - 4(3\,a)^3(2\,b) + 6(3\,a)^2(2\,b)^2 - 4(3\,a)(2\,b)^3 + (2\,b)^4 \\ &= 81\,a^4 - 216\,a^3b + 216\,a^2b^2 - 96\,ab^3 + 16\,b^4.\end{aligned}$$

14.
$$\begin{aligned}(a+b)^7 &= a^7 + 7\,a^6b + 21\,a^5b^2 + 35\,a^4b^3 + 35\,a^3b^4 + 21\,a^2b^5 \\ &\quad + 7\,ab^6 + b^7.\end{aligned}$$

15.
$$\begin{aligned}(xy-ab)^5 &= (xy)^5 - 5(xy)^4(ab) + 10(xy)^3(ab)^2 - 10(xy)^2(ab)^3 \\ &\quad + 5(xy)(ab)^4 - (ab)^5 \\ &= x^5y^5 - 5\,x^4y^4ab + 10\,x^3y^3a^2b^2 - 10\,x^2y^2a^3b^3 \\ &\quad + 5\,xya^4b^4 - a^5b^5.\end{aligned}$$

16.
$$\begin{aligned}(3\,n^2-2)^3 &= (3\,n^2)^3 - 3(3\,n^2)^2(2) + 3(3\,n^2)(2)^2 - (2)^3 \\ &= 27\,n^6 - 54\,n^4 + 36\,n^2 - 8.\end{aligned}$$

17.
$$\begin{aligned}(ab-3)^6 &= (ab)^6 - 6(ab)^5(3) + 15(ab)^4(3)^2 - 20(ab)^3(3)^3 \\ &\quad + 15(ab)^2(3)^4 - 6(ab)(3)^5 + (3)^6 \\ &= a^6b^6 - 18\,a^5b^5 + 135\,a^4b^4 - 540\,a^3b^3 + 1215\,a^2b^2 \\ &\quad - 1458\,ab + 729.\end{aligned}$$

18.
$$\begin{aligned}(1-2x)^5 &= (1)^5 - 5(1)^4(2x) + 10(1)^3(2x)^2 - 10(1)^2(2x)^3\\ &\quad + 5(1)(2x)^4 - (2x)^5\\ &= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5.\end{aligned}$$

19.
$$\begin{aligned}(3a+5b)^4 &= (3a)^4 + 4(3a)^3(5b) + 6(3a)^2(5b)^2 + 4(3a)(5b)^3 + (5b)^4\\ &= 81a^4 + 540a^3b + 1350a^2b^2 + 1500ab^3 + 625b^4.\end{aligned}$$

20.
$$\begin{aligned}(x+2y)^8 &= (x)^8 + 8(x)^7(2y) + 28(x)^6(2y)^2 + 56(x)^5(2y)^3\\ &\quad + 70(x)^4(2y)^4 + 56(x)^3(2y)^5 + 28(x)^2(2y)^6\\ &\quad + 8(x)(2y)^7 + (2y)^8\\ &= x^8 + 16x^7y + 112x^6y^2 + 448x^5y^3 + 1120x^4y^4\\ &\quad + 1792x^3y^5 + 1792x^2y^6 + 1024xy^7 + 256y^8.\end{aligned}$$

21.
$$\begin{aligned}(1-x+x^2)^3 &= [(1-x)+x^2]^3\\ &= (1-x)^3 + 3(1-x)^2(x^2) + 3(1-x)(x^2)^2 + (x^2)^3\\ &= 1 - 3x + 3x^2 - x^3 + 3x^2 - 6x^3 + 3x^4 + 3x^4 - 3x^5 + x^6\\ &= 1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6.\end{aligned}$$

22.
$$\begin{aligned}\left(\frac{3}{2x}-\frac{2x}{3}\right)^5 &= \left(\frac{3}{2x}\right)^5 - 5\left(\frac{3}{2x}\right)^4\left(\frac{2x}{3}\right) + 10\left(\frac{3}{2x}\right)^3\left(\frac{2x}{3}\right)^2\\ &\quad - 10\left(\frac{3}{2x}\right)^2\left(\frac{2x}{3}\right)^3 + 5\left(\frac{3}{2x}\right)\left(\frac{2x}{3}\right)^4 - \left(\frac{2x}{3}\right)^5\\ &= \frac{243}{32x^5} - \frac{135}{8x^3} + \frac{15}{x} - \frac{20x}{3} + \frac{40x^3}{27} - \frac{32x^5}{243}.\end{aligned}$$

23.
$$\begin{aligned}\left(1-\frac{2x^2}{3}\right)^6 &= (1)^6 - 6(1)^5\left(\frac{2x^2}{3}\right) + 15(1)^4\left(\frac{2x^2}{3}\right)^2 - 20(1)^3\left(\frac{2x^2}{3}\right)^3\\ &\quad + 15(1)^2\left(\frac{2x^2}{3}\right)^4 - 6(1)\left(\frac{2x^2}{3}\right)^5 + \left(\frac{2x^2}{3}\right)^6\\ &= 1 - 4x^2 + \frac{20x^4}{3} - \frac{160x^6}{27} + \frac{80x^8}{27} - \frac{64x^{10}}{81} + \frac{64x^{12}}{729}.\end{aligned}$$

24.
$$\begin{aligned}\left(\frac{2}{x}-\frac{3x^2}{2}\right)^7 &= \left(\frac{2}{x}\right)^7 - 7\left(\frac{2}{x}\right)^6\left(\frac{3x^2}{2}\right) + 21\left(\frac{2}{x}\right)^5\left(\frac{3x^2}{2}\right)^2\\ &\quad - 35\left(\frac{2}{x}\right)^4\left(\frac{3x^2}{2}\right)^3 + 35\left(\frac{2}{x}\right)^3\left(\frac{3x^2}{2}\right)^4\\ &\quad - 21\left(\frac{2}{x}\right)^2\left(\frac{3x^2}{2}\right)^5 + 7\left(\frac{2}{x}\right)\left(\frac{3x^2}{2}\right)^6 - \left(\frac{3x^2}{2}\right)^7\\ &= \frac{128}{x^7} - \frac{672}{x^4} + \frac{1512}{x} - 1890x^2 + \frac{2835x^5}{2} - \frac{5103x^8}{8}\\ &\quad + \frac{5103x^{11}}{32} - \frac{2187x^{14}}{128}.\end{aligned}$$

25. $(1 - 2x + 3x^2 + 4x^3)^3$

$$\begin{aligned}
&= [(1 - 2x) + (3x^2 + 4x^3)]^3\\
&= (1 - 2x)^3 + 3(1 - 2x)^2(3x^2 + 4x^3)\\
&\quad + 3(1 - 2x)(3x^2 + 4x^3)^2 + (3x^2 + 4x^3)^3\\
&= (1 - 6x + 12x^2 - 8x^3) + 3(1 - 4x + 4x^2)(3x^2 + 4x^3)\\
&\quad + 3(1 - 2x)(9x^4 + 24x^5 + 16x^6) + (3x^2)^3 + 3(3x^2)^2(4x^3)\\
&\quad + 3(3x^2)(4x^3)^2 + (4x^3)^3\\
&= 1 - 6x + 12x^2 - 8x^3 + 9x^2 - 24x^3 - 12x^4 + 48x^5 + 27x^4\\
&\quad + 18x^5 - 96x^6 - 96x^7 + 27x^6 + 108x^7 + 144x^8 + 64x^9\\
&= 1 - 6x + 21x^2 - 32x^3 + 15x^4 + 66x^5 - 69x^6 + 12x^7\\
&\quad + 144x^8 + 64x^9.
\end{aligned}$$

26. $(1 - 3x + x^2)^4$

$$\begin{aligned}
&= [(1 - 3x) + x^2]^4\\
&= (1 - 3x)^4 + 4(1 - 3x)^3(x^2) + 6(1 - 3x)^2(x^2)^2\\
&\quad + 4(1 - 3x)(x^2)^3 + (x^2)^4\\
&= [(1)^4 - 4(1)^3(3x) + 6(1)^2(3x)^2 - 4(1)(3x)^3 + (3x)^4]\\
&\quad + 4[(1)^3 - 3(1)^2(3x) + 3(1)(3x)^2 - (3x)^3]x^2\\
&\quad + 6(1 - 6x + 9x^2)x^4 + 4(1 - 3x)x^6 + x^8\\
&= 1 - 12x + 54x^2 - 108x^3 + 81x^4\\
&\quad + 4x^2(1 - 9x + 27x^2 - 27x^3) + 6x^4 - 36x^5 + 54x^6\\
&\quad + 4x^6 - 12x^7 + x^8\\
&= 1 - 12x + 54x^2 - 108x^3 + 81x^4 + 4x^2 - 36x^3 + 108x^4\\
&\quad - 108x^5 + 6x^4 - 36x^5 + 54x^6 + 4x^6 - 12x^7 + x^8\\
&= 1 - 12x + 58x^2 - 144x^3 + 195x^4 - 144x^5 + 58x^6 - 12x^7 + x^8.
\end{aligned}$$

27. $\left(3a^2b^m - \dfrac{a^mb^2}{3}\right)^4$

$$\begin{aligned}
&= (3a^2b^m)^4 - 4(3a^2b^m)^3\left(\frac{a^mb^2}{3}\right) + 6(3a^2b^m)^2\left(\frac{a^mb^2}{3}\right)^2\\
&\quad - 4(3a^2b^m)\left(\frac{a^mb^2}{3}\right)^3 + \left(\frac{a^mb^2}{3}\right)^4\\
&= 81a^8b^{4m} - 36a^{m+6}b^{3m+2} + 6a^{2m+4}b^{2m+4}\\
&\quad - \frac{4a^{3m+2}b^{m+6}}{9} + \frac{a^{4m}b^8}{81}.
\end{aligned}$$

28. $(a^4 - a^3b + a^2b^2 - ab^3 + b^4)^2$

$$\begin{aligned}
&= [(a^4 - a^3b) + (a^2b^2 - ab^3 + b^4)]^2\\
&= (a^4 - a^3b)^2 + 2(a^4 - a^3b)(a^2b^2 - ab^3 + b^4) + (a^2b^2 - ab^3 + b^4)^2\\
&= a^8 - 2a^7b + a^6b^2 + 2a^6b^2 - 4a^5b^3 + 4a^4b^4 - 2a^3b^5 + a^4b^4\\
&\quad + a^2b^6 + b^8 - 2a^3b^5 + 2a^2b^6 - 2ab^7\\
&= a^8 - 2a^7b + 3a^6b^2 - 4a^5b^3 + 5a^4b^4 - 4a^3b^5 + 3a^2b^6\\
&\quad - 2ab^7 + b^8.
\end{aligned}$$

29. $\left(a + \frac{a^2}{2} - \frac{a^3}{3}\right)^3$

$$= \left[\left(a + \frac{a^2}{2}\right) - \frac{a^3}{3}\right]^3$$

$$= \left(a + \frac{a^2}{2}\right)^3 - 3\left(a + \frac{a^2}{2}\right)^2\left(\frac{a^3}{3}\right) + 3\left(a + \frac{a^2}{2}\right)\left(\frac{a^3}{3}\right)^2 - \left(\frac{a^3}{3}\right)^3$$

$$= a^3 + \frac{3a^4}{2} + \frac{3a^5}{4} + \frac{a^6}{8} - a^3\left(a^2 + a^3 + \frac{a^4}{4}\right) + \frac{a^6}{3}\left(a + \frac{a^2}{2}\right) - \frac{a^9}{27}$$

$$= a^3 + \frac{3a^4}{2} + \frac{3a^5}{4} + \frac{a^6}{8} - a^5 - a^6 - \frac{a^7}{4} + \frac{a^7}{3} + \frac{a^8}{6} - \frac{a^9}{27}$$

$$= a^3 + \frac{3a^4}{2} - \frac{a^5}{4} - \frac{7a^6}{8} + \frac{a^7}{12} + \frac{a^8}{6} - \frac{a^9}{27}.$$

30. $(a^mb^n - x^py^q)^4$

$$= (a^mb^n)^4 - 4(a^mb^n)^3(x^py^q) + 6(a^mb^n)^2(x^py^q)^2 - 4(a^mb^n)(x^py^q)^3 + (x^py^q)^4$$

$$= a^{4m}b^{4n} - 4a^{3m}b^{3n}x^py^q + 6a^{2m}b^{2n}x^{2p}y^{2q} - 4a^mb^nx^{3p}y^{3q} + x^{4p}y^{4q}.$$

Exercise 86. Page 220

Simplify:

1. $\sqrt{4a^4b^2} = \pm 2a^2b.$
2. $\sqrt[3]{27a^6b^9} = 3a^2b^3.$
3. $\sqrt[4]{16x^8y^{12}} = \pm 2x^2y^3.$
4. $\sqrt[5]{32a^{10}b^{15}} = 2a^2b^3.$
5. $\sqrt[3]{-27a^3x^6} = -3ax^2.$
6. $\sqrt{625a^4b^4} = \pm 25a^2b^2.$
7. $\sqrt[4]{625a^8b^8} = \pm 5a^2b^2.$
8. $\sqrt[6]{64x^{18}} = \pm 2x^3.$
9. $\sqrt[5]{-32a^{15}b^{20}} = -2a^3b^4.$
10. $\sqrt[3]{512a^{12}b^{18}} = 8a^4b^6.$
11. $\sqrt[6]{729a^{12}b^{18}} = \pm 3a^2b^3.$
12. $\sqrt[3]{-216x^3y^6} = -6xy^2.$
13. $\sqrt[3]{1728c^6} = 12c^2.$
14. $\sqrt[4]{81a^{16}y^8} = \pm 3a^4y^2.$
15. $\sqrt[3]{-1331x^9y^{12}} = -11x^3y^4.$
16. $\sqrt[3]{x^{3n}y^{6n}} = x^ny^{2n}.$
17. $\sqrt[4]{a^{4m}b^{8n}} = \pm a^mb^{2n}.$
18. $\sqrt[3]{-27a^{3p}c^{3q}} = -3a^pc^q.$
19. $\sqrt[5]{-\frac{32x^5}{243a^{10}}} = -\frac{2x}{3a^2}.$
20. $\sqrt[3]{-\frac{8x^3y^9}{27a^6}} = -\frac{2xy^3}{3a^2}.$
21. $\sqrt[4]{\frac{16a^8}{81b^{12}c^{16}}} = \pm\frac{2a^2}{3b^3c^4}.$
22. $\frac{\sqrt[3]{216a^6x^3}}{\sqrt[5]{32a^{15}x^{10}}} = \frac{6a^2x}{2a^3x^2} = \frac{3}{ax}.$
23. $\sqrt[3]{\frac{125a^3b^6c^9}{216x^9y^6z^3}} = \frac{5ab^2c^3}{6x^3y^2z}.$
24. $\sqrt[6]{\frac{729x^6y^{12}}{64a^{12}b^{12}}} = \pm\frac{3xy^2}{2a^2b^2}.$

Exercise 87. Page 222

1. Find the square root of $x^4 - 6x^3 + 13x^2 - 12x + 4$.

$$
\begin{array}{rl}
 & x^4 - 6x^3 + 13x^2 - 12x + 4 \,\big|\, \underline{x^2 - 3x + 2} \\
 & x^4 \\
2x^2 - 3x & \big|\, -6x^3 + 13x^2 \\
 & \big|\, \underline{-6x^3 + 9x^2} \\
2x^2 - 6x + 2 & \big|\, 4x^2 - 12x + 4 \\
 & \big|\, \underline{4x^2 - 12x + 4}
\end{array}
$$

2. Find the square root of $1 - 2a^3 + a^4 - 2a + 3a^2$.
Arrange in descending powers of a.

$$
\begin{array}{rl}
 & a^4 - 2a^3 + 3a^2 - 2a + 1 \,\big|\, \underline{a^2 - a + 1} \\
 & a^4 \\
2a^2 - a & \big|\, -2a^3 + 3a^2 \\
 & \big|\, \underline{-2a^3 + a^2} \\
2a^2 - 2a + 1 & \big|\, 2a^2 - 2a + 1 \\
 & \big|\, \underline{2a^2 - 2a + 1}
\end{array}
$$

3. Find the square root of $9x^4 - 12x^3y + 34x^2y^2 - 20xy^3 + 25y^4$.

$$
\begin{array}{rl}
 & 9x^4 - 12x^3y + 34x^2y^2 - 20xy^3 + 25y^4 \,\big|\, \underline{3x^2 - 2xy + 5y^2} \\
 & 9x^4 \\
6x^2 - 2xy & \big|\, -12x^3y + 34x^2y^2 \\
 & \big|\, \underline{-12x^3y + 4x^2y^2} \\
6x^2 - 4xy + 5y^2 & \big|\, 30x^2y^2 - 20xy^3 + 25y^4 \\
 & \big|\, \underline{30x^2y^2 - 20xy^3 + 25y^4}
\end{array}
$$

4. Find the square root of $49a^4 - 42a^3b + 37a^2b^2 - 12ab^3 + 4b^4$.

$$
\begin{array}{rl}
 & 49a^4 - 42a^3b + 37a^2b^2 - 12ab^3 + 4b^4 \,\big|\, \underline{7a^2 - 3ab + 2b^2} \\
 & 49a^4 \\
14a^2 - 3ab & \big|\, -42a^3b + 37a^2b^2 \\
 & \big|\, \underline{-42a^3b + 9a^2b^2} \\
14a^2 - 6ab + 2b^2 & \big|\, 28a^2b^2 - 12ab^3 + 4b^4 \\
 & \big|\, \underline{28a^2b^2 - 12ab^3 + 4b^4}
\end{array}
$$

5. Find the square root of $x^4 + 10x^3y + 33x^2y^2 + 40xy^3 + 16y^4$.

$$
\begin{array}{rl}
 & x^4 + 10x^3y + 33x^2y^2 + 40xy^3 + 16y^4 \,\big|\, \underline{x^2 + 5xy + 4y^2} \\
 & x^4 \\
2x^2 + 5xy & \big|\, 10x^3y + 33x^2y^2 \\
 & \big|\, \underline{10x^3y + 25x^2y^2} \\
2x^2 + 10xy + 4y^2 & \big|\, 8x^2y^2 + 40xy^3 + 16y^4 \\
 & \big|\, \underline{8x^2y^2 + 40xy^3 + 16y^4}
\end{array}
$$

6. Find the square root of $4x^4 + 37x^2y^2 - 30xy^3 - 20x^3y + 9y^4$. Arrange in descending powers of x.

$$
\begin{array}{r|l}
 & 4x^4 - 20x^3y + 37x^2y^2 - 30xy^3 + 9y^4 \;\underline{|\,2x^2 - 5xy + 3y^2} \\
 & 4x^4 \\ \hline
4x^2 - 5xy & -20x^3y + 37x^2y^2 \\
 & -20x^3y + 25x^2y^2 \\ \hline
4x^2 - 10xy + 3y^2 & 12x^2y^2 - 30xy^3 + 9y^4 \\
 & 12x^2y^2 - 30xy^3 + 9y^4 \\ \hline
\end{array}
$$

7. Find the square root of $16a^4 - 40a^3x + a^2x^2 + 30ax^3 + 9x^4$.

$$
\begin{array}{r|l}
 & 16a^4 - 40a^3x + a^2x^2 + 30ax^3 + 9x^4 \;\underline{|\,4a^2 - 5ax - 3x^2} \\
 & 16a^4 \\ \hline
8a^2 - 5ax & -40a^3x + a^2x^2 \\
 & -40a^3x + 25a^2x^2 \\ \hline
8a^2 - 10ax - 3x^2 & -24a^2x^2 + 30ax^3 + 9x^4 \\
 & -24a^2x^2 + 30ax^3 + 9x^4 \\ \hline
\end{array}
$$

8. Find the square root of $16a^4 + 56a^3x + a^2x^2 - 84ax^3 + 36x^4$.

$$
\begin{array}{r|l}
 & 16a^4 + 56a^3x + a^2x^2 - 84ax^3 + 36x^4 \;\underline{|\,4a^2 + 7ax - 6x^2} \\
 & 16a^4 \\ \hline
8a^2 + 7ax & 56a^3x + a^2x^2 \\
 & 56a^3x + 49a^2x^2 \\ \hline
8a^2 + 14ax - 6x^2 & -48a^2x^2 - 84ax^3 + 36x^4 \\
 & -48a^2x^2 - 84ax^3 + 36x^4 \\ \hline
\end{array}
$$

9. Find the square root of $9c^4 - 48c^3d + 34c^2d^2 + 80cd^3 + 25d^4$.

$$
\begin{array}{r|l}
 & 9c^4 - 48c^3d + 34c^2d^2 + 80cd^3 + 25d^4 \;\underline{|\,3c^2 - 8cd - 5d^2} \\
 & 9c^4 \\ \hline
6c^2 - 8cd & -48c^3d + 34c^2d^2 \\
 & -48c^3d + 64c^2d^2 \\ \hline
6c^2 - 16cd - 5d^2 & -30c^2d^2 + 80cd^3 + 25d^4 \\
 & -30c^2d^2 + 80cd^3 + 25d^4 \\ \hline
\end{array}
$$

10. Find the square root of $13x^4 + 13x^2 + 4x^6 - 14x^3 + 4 - 4x - 12x^5$. Arrange in descending powers of x.

$$
\begin{array}{r|l}
 & 4x^6 - 12x^5 + 13x^4 - 14x^3 + 13x^2 - 4x + 4 \;\underline{|\,2x^3 - 3x^2 + x - 2} \\
 & 4x^6 \\ \hline
4x^3 - 3x^2 & -12x^5 + 13x^4 \\
 & -12x^5 + 9x^4 \\ \hline
4x^3 - 6x^2 + x & 4x^4 - 14x^3 + 13x^2 \\
 & 4x^4 - 6x^3 + x^2 \\ \hline
4x^3 - 6x^2 + 2x - 2 & -8x^3 + 12x^2 - 4x + 4 \\
 & -8x^3 + 12x^2 - 4x + 4 \\ \hline
\end{array}
$$

11. Find the square root of $4x^6-14x^2y^4+25y^6-7x^4y^2+12x^5y+40xy^5-44x^3y^3$.

Arrange in descending powers of x.

$$\begin{array}{r|l}
 & 4x^6+12x^5y-7x^4y^2-44x^3y^3-14x^2y^4+40xy^5+25y^6 \;\big|\; \underline{2x^3+3x^2y-4xy^2-5y^3} \\
 & 4x^6 \\
\hline
4x^3+3x^2y & 12x^5y-7x^4y^2 \\
 & 12x^5y+9x^4y^2 \\
\hline
4x^3+6x^2y-4xy^2 & -16x^4y^2-44x^3y^3-14x^2y^4 \\
 & -16x^4y^2-24x^3y^3+16x^2y^4 \\
\hline
4x^3+6x^2y-8xy^2-5y^3 & -20x^3y^3-30x^2y^4+40xy^5+25y^6 \\
 & -20x^3y^3-30x^2y^4+40xy^5+25y^6 \\
\hline
\end{array}$$

12. Find the square root of $4a^6-20a^5b-3a^4b^2+82a^3b^3+19a^2b^4-42ab^5+9b^6$.

$$\begin{array}{r|l}
 & 4a^6-20a^5b-3a^4b^2+82a^3b^3+19a^2b^4-42ab^5+9b^6 \;\big|\; \underline{2a^3-5a^2b-7ab^2+3b^3} \\
 & 4a^6 \\
\hline
4a^3-5a^2b & -20a^5b-3a^4b^2 \\
 & -20a^5b+25a^4b^2 \\
\hline
4a^3-10a^2b-7ab^2 & -28a^4b^2+82a^3b^3+19a^2b^4 \\
 & -28a^4b^2+70a^3b^3+49a^2b^4 \\
\hline
4a^3-10a^2b-14ab^2+3b^3 & 12a^3b^3-30a^2b^4-42ab^5+9b^6 \\
 & 12a^3b^3-30a^2b^4-42ab^5+9b^6 \\
\hline
\end{array}$$

Exercise 88. Page 224

1. Find the square root of $\frac{a^4}{4}-\frac{a^2b^3}{3}+\frac{b^6}{9}+\frac{a^2c^4}{4}+\frac{c^8}{16}-\frac{b^3c^4}{6}$.

$$
\begin{array}{l|l|l}
 & \frac{a^4}{4}-\frac{a^2b^3}{3}+\frac{b^6}{9}+\frac{a^2c^4}{4}-\frac{b^3c^4}{6}+\frac{c^8}{16} & \frac{a^2}{2}-\frac{b^3}{3}+\frac{c^4}{4} \\
 & \frac{a^4}{4} & \\
a^2-\frac{b^3}{3} & -\frac{a^2b^3}{3}+\frac{b^6}{9} & \\
 & -\frac{a^2b^3}{3}+\frac{b^6}{9} & \\
a^2-\frac{2b^3}{3}+\frac{c^4}{4} & \frac{a^2c^4}{4}-\frac{b^3c^4}{6}+\frac{c^8}{16} & \\
 & \frac{a^2c^4}{4}-\frac{b^3c^4}{6}+\frac{c^8}{16} &
\end{array}
$$

2. Find the square root of $\frac{x^2}{y^2}-\frac{4xz}{ay}+\frac{4z^2}{a^2}+\frac{6qx}{by}+\frac{9q^2}{b^2}-\frac{12qz}{ab}$.

$$
\begin{array}{l|l|l}
 & \frac{x^2}{y^2}-\frac{4xz}{ay}+\frac{4z^2}{a^2}+\frac{6qx}{by}-\frac{12qz}{ab}+\frac{9q^2}{b^2} & \frac{x}{y}-\frac{2z}{a}+\frac{3q}{b} \\
 & \frac{x^2}{y^2} & \\
\frac{2x}{y}-\frac{2z}{a} & -\frac{4xz}{ay}+\frac{4z^2}{a^2} & \\
 & -\frac{4xz}{ay}+\frac{4z^2}{a^2} & \\
\frac{2x}{y}-\frac{4z}{a}+\frac{3q}{b} & \frac{6qx}{by}-\frac{12qz}{ab}+\frac{9q^2}{b^2} & \\
 & \frac{6qx}{by}-\frac{12qz}{ab}+\frac{9q^2}{b^2} &
\end{array}
$$

3. Find the square root of

$$\frac{9a^6b^4}{25x^6y^8}-\frac{12a^5b^5}{35x^7y^9}-\frac{332a^4b^6}{735x^8y^{10}}+\frac{16a^3b^7}{63x^9y^{11}}+\frac{16a^2b^8}{81x^{10}y^{12}}.$$

$$\begin{array}{l|l}
 & \frac{9a^6b^4}{25x^6y^8}-\frac{12a^5b^5}{35x^7y^9}-\frac{332a^4b^6}{735x^8y^{10}}+\frac{16a^3b^7}{63x^9y^{11}}+\frac{16a^2b^8}{81x^{10}y^{12}} \quad \Big| \; \frac{3a^3b^2}{5x^3y^4}-\frac{2a^2b^3}{7x^4y^5}-\frac{4ab^4}{9x^5y^6} \\
 & \frac{9a^6b^4}{25x^6y^8} \\ \hline
\frac{6a^3b^2}{5x^3y^4}-\frac{2a^2b^3}{7x^4y^5} & -\frac{12a^5b^5}{35x^7y^9}-\frac{332a^4b^6}{735x^8y^{10}} \\
 & -\frac{12a^5b^5}{35x^7y^9}+\frac{4a^4b^6}{49x^8y^{10}} \\ \hline
\frac{6a^3b^2}{5x^3y^4}-\frac{4a^2b^3}{7x^4y^5}-\frac{4ab^4}{9x^5y^6} & -\frac{8a^4b^6}{15x^8y^{10}}+\frac{16a^3b^7}{63x^9y^{11}}+\frac{16a^2b^8}{81x^{10}y^{12}} \\
 & -\frac{8a^4b^6}{15x^8y^{10}}+\frac{16a^3b^7}{63x^9y^{11}}+\frac{16a^2b^8}{81x^{10}y^{12}} \\ \hline
\end{array}$$

4. Find the square root of $x^6y^6-\frac{2x^3y^{12}}{3a^3}+\frac{y^{18}}{9a^6}+\frac{4x^3y^{18}}{a^9}-\frac{4y^{24}}{3a^{12}}+\frac{4y^{30}}{a^{18}}$.

$$\begin{array}{l|l}
 & x^6y^6-\frac{2x^3y^{12}}{3a^3}+\frac{y^{18}}{9a^6}+\frac{4x^3y^{18}}{a^9}-\frac{4y^{24}}{3a^{12}}+\frac{4y^{30}}{a^{18}} \quad \Big| \; x^3y^3-\frac{y^9}{3a^3}+\frac{2y^{15}}{a^9} \\
 & x^6y^6 \\ \hline
2x^3y^3-\frac{y^9}{3a^3} & -\frac{2x^3y^{12}}{3a^3}+\frac{y^{18}}{9a^6} \\
 & -\frac{2x^3y^{12}}{3a^3}+\frac{y^{18}}{9a^6} \\ \hline
2x^3y^3-\frac{2y^9}{3a^3}+\frac{2y^{15}}{a^9} & \frac{4x^3y^{18}}{a^9}-\frac{4y^{24}}{3a^{12}}+\frac{4y^{30}}{a^{18}} \\
 & \frac{4x^3y^{18}}{a^9}-\frac{4y^{24}}{3a^{12}}+\frac{4y^{30}}{a^{18}} \\ \hline
\end{array}$$

5. Find the square root of $\frac{9y^2}{16x^2} + 25\frac{3}{7} - \frac{20x}{7y} - \frac{15y}{2x} + \frac{4x^2}{49y^2}$.

Arrange in descending powers of y.

$$
\begin{array}{r|l}
 & \frac{9y^2}{16x^2} - \frac{15y}{2x} + 25\frac{3}{7} - \frac{20x}{7y} + \frac{4x^2}{49y^2} \;\Big|\; \frac{3y}{4x} - 5 + \frac{2x}{7y} \\
 & \frac{9y^2}{16x^2} \\ \hline
\frac{3y}{2x} - 5 & -\frac{15y}{2x} + 25\frac{3}{7} \\
 & -\frac{15y}{2x} + 25 \\ \hline
\frac{3y}{2x} - 10 + \frac{2x}{7y} & \frac{3}{7} - \frac{20x}{7y} + \frac{4x^2}{49y^2} \\
 & \frac{3}{7} - \frac{20x}{7y} + \frac{4x^2}{49y^2} \\ \hline
\end{array}
$$

6. Find the square root of $\frac{a^4}{4} + \frac{a^3b}{3} - \frac{5a^2b^2}{36} - \frac{ab^3}{6} + \frac{b^4}{16}$.

$$
\begin{array}{r|l}
 & \frac{a^4}{4} + \frac{a^3b}{3} - \frac{5a^2b^2}{36} - \frac{ab^3}{6} + \frac{b^4}{16} \;\Big|\; \frac{a^2}{2} + \frac{ab}{3} - \frac{b^2}{4} \\
 & \frac{a^4}{4} \\ \hline
a^2 + \frac{ab}{3} & \frac{a^3b}{3} - \frac{5a^2b^2}{36} \\
 & \frac{a^3b}{3} + \frac{a^2b^2}{9} \\ \hline
a^2 + \frac{2ab}{3} - \frac{b^2}{4} & -\frac{a^2b^2}{4} - \frac{ab^3}{6} + \frac{b^4}{16} \\
 & -\frac{a^2b^2}{4} - \frac{ab^3}{6} + \frac{b^4}{16} \\ \hline
\end{array}
$$

7. Find the square root of $\frac{9a^2b^2}{49x^2y^2} - \frac{15ab}{28xy} + \frac{203}{192} - \frac{35xy}{36ab} + \frac{49x^2y^2}{81a^2b^2}$.

$$\begin{array}{r|l|l}
 & \frac{9a^2b^2}{49x^2y^2} - \frac{15ab}{28xy} + \frac{203}{192} - \frac{35xy}{36ab} + \frac{49x^2y^2}{81a^2b^2} & \frac{3ab}{7xy} - \frac{5}{8} + \frac{7xy}{9ab} \\
 & \frac{9a^2b^2}{49x^2y^2} & \\
\frac{6ab}{7xy} - \frac{5}{8} & -\frac{15ab}{28xy} + \frac{203}{192} & \\
 & -\frac{15ab}{28xy} + \frac{25}{64} & \\
\frac{6ab}{7xy} - \frac{5}{4} + \frac{7xy}{9ab} & \frac{2}{3} - \frac{35xy}{36ab} + \frac{49x^2y^2}{81a^2b^2} & \\
 & \frac{2}{3} - \frac{35xy}{36ab} + \frac{49x^2y^2}{81a^2b^2} &
\end{array}$$

8. Find the square root of $\frac{9a^2b^2}{x^2y^2} + \frac{4x^2y^2}{9a^2b^2} - \frac{4xy}{15ab} + \frac{101}{25} - \frac{6ab}{5xy}$.

Arrange in descending powers of a.

$$\begin{array}{r|l|l}
 & \frac{9a^2b^2}{x^2y^2} - \frac{6ab}{5xy} + \frac{101}{25} - \frac{4xy}{15ab} + \frac{4x^2y^2}{9a^2b^2} & \frac{3ab}{xy} - \frac{1}{5} + \frac{2xy}{3ab} \\
 & \frac{9a^2b^2}{x^2y^2} & \\
\frac{6ab}{xy} - \frac{1}{5} & -\frac{6ab}{5xy} + \frac{101}{25} & \\
 & -\frac{6ab}{5xy} + \frac{1}{25} & \\
\frac{6ab}{xy} - \frac{2}{5} + \frac{2xy}{3ab} & 4 - \frac{4xy}{15ab} + \frac{4x^2y^2}{9a^2b^2} & \\
 & 4 - \frac{4xy}{15ab} + \frac{4x^2y^2}{9a^2b^2} &
\end{array}$$

9. Find the square root of $\frac{9}{25}+\frac{a^8}{25}+\frac{m^6}{36}-\frac{6a^4}{25}-\frac{m^3}{5}+\frac{4n^8}{49}+\frac{12n^4}{35}+\frac{a^4m^3}{15}-\frac{4a^4n^4}{35}-\frac{2m^3n^4}{21}$.

$$\begin{array}{l|l}
 & \frac{9}{25}-\frac{6a^4}{25}+\frac{a^8}{25}-\frac{m^3}{5}+\frac{a^4m^3}{15}+\frac{m^6}{36}+\frac{12n^4}{35}-\frac{4a^4n^4}{35}-\frac{2m^3n^4}{21}+\frac{4n^8}{49} \quad \Big| \; \frac{3}{5}-\frac{a^4}{5}-\frac{m^3}{6}+\frac{2n^4}{7} \\
 & \frac{9}{25} \\
\frac{6}{5}-\frac{a^4}{5} & -\frac{6a^4}{25}+\frac{a^8}{25} \\
 & -\frac{6a^4}{25}+\frac{a^8}{25} \\
\frac{6}{5}-\frac{2a^4}{5}-\frac{m^3}{6} & -\frac{m^3}{5}+\frac{a^4m^3}{15}+\frac{m^6}{36} \\
 & -\frac{m^3}{5}+\frac{a^4m^3}{15}+\frac{m^6}{36} \\
\frac{6}{5}-\frac{2a^4}{5}-\frac{m^3}{3}+\frac{2n^4}{7} & \frac{12n^4}{35}-\frac{4a^4n^4}{35}-\frac{2m^3n^4}{21}+\frac{4n^8}{49} \\
 & \frac{12n^4}{35}-\frac{4a^4n^4}{35}-\frac{2m^3n^4}{21}+\frac{4n^8}{49}
\end{array}$$

10. Find the square root of $\frac{4a^4}{9x^4}-\frac{2a^3}{3x^3}+\frac{25a^2}{36x^2}-\frac{11a}{9x}+\frac{13}{9}-\frac{17x}{18a}+\frac{7x^2}{9a^2}-\frac{2x^3}{3a^3}+\frac{x^4}{4a^4}$.

$$\frac{4a^4}{9x^4}-\frac{2a^3}{3x^3}+\frac{25a^2}{36x^2}-\frac{11a}{9x}+\frac{13}{9}-\frac{17x}{18a}+\frac{7x^2}{9a^2}-\frac{2x^3}{3a^3}+\frac{x^4}{4a^4}\,\Big|\,\frac{2a^2}{3x^2}-\frac{a}{2x}+\frac{1}{3}-\frac{2x}{3a}+\frac{x^2}{2a^2}$$

$$\frac{4a^4}{9x^4}$$

$$\frac{4a^2}{3x^2}-\frac{a}{2x}\,\Big|\,-\frac{2a^3}{3x^3}+\frac{25a^2}{36x^2}$$

$$-\frac{2a^3}{3x^3}+\frac{a^2}{4x^2}$$

$$\frac{4a^2}{3x^2}-\frac{a}{x}+\frac{1}{3}\,\Big|\,\frac{4a^2}{9x^2}-\frac{11a}{9x}+\frac{13}{9}$$

$$\frac{4a^2}{9x^2}-\frac{a}{3x}+\frac{1}{9}$$

$$\frac{4a^2}{3x^2}-\frac{a}{x}+\frac{2}{3}-\frac{2x}{3a}\,\Big|\,-\frac{8a}{9x}+\frac{4}{3}-\frac{17x}{18a}+\frac{7x^2}{9a^2}$$

$$-\frac{8a}{9x}+\frac{2}{3}-\frac{4x}{9a}+\frac{4x^2}{9a^2}$$

$$\frac{4a^2}{3x^2}-\frac{a}{x}+\frac{2}{3}-\frac{4x}{3a}+\frac{x^2}{2a^2}\,\Big|\,\frac{2}{3}-\frac{x}{2a}+\frac{x^2}{3a^2}-\frac{2x^3}{3a^3}+\frac{x^4}{4a^4}$$

$$\frac{2}{3}-\frac{x}{2a}+\frac{x^2}{3a^2}-\frac{2x^3}{3a^3}+\frac{x^4}{4a^4}$$

11. Find the square root of $\frac{1189\,a^2b^2}{2205\,x^2y^2}+\frac{3662}{6237}+\frac{25\,x^2y^2}{121\,a^2b^2}+\frac{4\,a^4b^4}{25\,x^4y^4}+\frac{172\,ab}{231\,xy}+\frac{40\,xy}{99\,ab}+\frac{12\,a^3b^3}{35\,x^3y^3}.$

$$
\begin{array}{r|l}
 & \frac{4\,a^4b^4}{25\,x^4y^4}+\frac{12\,a^3b^3}{35\,x^3y^3}+\frac{1189\,a^2b^2}{2205\,x^2y^2}+\frac{172\,ab}{231\,xy}+\frac{3662}{6237}+\frac{40\,xy}{99\,ab}+\frac{25\,x^2y^2}{121\,a^2b^2}\ \Big|\ \underline{\frac{2\,a^2b^2}{5\,x^2y^2}+\frac{3\,ab}{7\,xy}+\frac{4}{9}+\frac{5\,xy}{11\,ab}} \\
 & \underline{\frac{4\,a^4b^4}{25\,x^4y^4}} \\
\frac{4\,a^2b^2}{5\,x^2y^2}+\frac{3\,ab}{7\,xy} & \frac{12\,a^3b^3}{35\,x^3y^3}+\frac{1189\,a^2b^2}{2205\,x^2y^2} \\
 & \underline{\frac{12\,a^3b^3}{35\,x^3y^3}+\frac{9\,a^2b^2}{49\,x^2y^2}} \\
\frac{4\,a^2b^2}{5\,x^2y^2}+\frac{6\,ab}{7\,xy}+\frac{4}{9} & \frac{16\,a^2b^2}{45\,x^2y^2}+\frac{172\,ab}{231\,xy}+\frac{3662}{6237} \\
 & \underline{\frac{16\,a^2b^2}{45\,x^2y^2}+\frac{8\,ab}{21\,xy}+\frac{16}{81}} \\
\frac{4\,a^2b^2}{5\,x^2y^2}+\frac{6\,ab}{7\,xy}+\frac{8}{9}+\frac{5\,xy}{11\,ab} & \frac{4\,ab}{11\,xy}+\frac{30}{77}+\frac{40\,xy}{99\,ab}+\frac{25\,x^2y^2}{121\,a^2b^2} \\
 & \underline{\frac{4\,ab}{11\,xy}+\frac{30}{77}+\frac{40\,xy}{99\,ab}+\frac{25\,x^2y^2}{121\,a^2b^2}}
\end{array}
$$

12. Find to three terms the square root of $a^2 + b$.

$$
\begin{array}{r|l}
 & a^2 + b \qquad\qquad \Big|\; a + \frac{b}{2a} - \frac{b^2}{8a^3} + \cdots \\
 & a^2 \\
\hline
2a + \frac{b}{2a} & b \\
 & b + \frac{b^2}{4a^2} \\
\hline
2a + \frac{b}{a} - \frac{b^2}{8a^3} & -\frac{b^2}{4a^2} \\
 & -\frac{b^2}{4a^2} - \frac{b^3}{8a^4} + \frac{b^4}{64a^6} \\
\hline
 & \frac{b^3}{8a^4} - \frac{b^4}{64a^6}
\end{array}
$$

13. Find to three terms the square root of $x^4 - \frac{1}{4}y^2$.

$$
\begin{array}{r|l}
 & x^4 - \frac{1}{4}y^2 \qquad\qquad \Big|\; x^2 - \frac{y^2}{8x^2} - \frac{y^4}{128x^6} - \cdots \\
 & x^4 \\
\hline
2x^2 - \frac{y^2}{8x^2} & -\frac{1}{4}y^2 \\
 & -\frac{1}{4}y^2 + \frac{y^4}{64x^4} \\
\hline
2x^2 - \frac{y^2}{4x^2} - \frac{y^4}{128x^6} & -\frac{y^4}{64x^4} \\
 & -\frac{y^4}{64x^4} + \frac{y^6}{512x^8} + \frac{y^8}{16384x^{12}} \\
\hline
 & -\frac{y^6}{512x^8} - \frac{y^8}{16384x^{12}}
\end{array}
$$

14. Find to three terms the square root of $b^2 - c^2$.

$$
\begin{array}{r|l}
 & b^2 - c^2 \qquad\qquad \Big|\; b - \frac{c^2}{2b} - \frac{c^4}{8b^3} - \cdots \\
 & b^2 \\
\hline
2b - \frac{c^2}{2b} & -c^2 \\
 & -c^2 + \frac{c^4}{4b^2} \\
\hline
2b - \frac{c^2}{b} - \frac{c^4}{8b^3} & -\frac{c^4}{4b^2} \\
 & -\frac{c^4}{4b^2} + \frac{c^6}{8b^4} + \frac{c^8}{64b^6} \\
\hline
 & -\frac{c^6}{8b^4} - \frac{c^8}{64b^6}
\end{array}
$$

15. Find to three terms the square root of $x^2 - 2a$.

$$
\begin{array}{r|l}
\multicolumn{2}{l}{x^2 - 2a \qquad \left|\, x - \dfrac{a}{x} - \dfrac{a^2}{2x^3} - \cdots\right.} \\
\multicolumn{2}{l}{x^2} \\ \hline
2x - \dfrac{a}{x} & -2a \\
 & -2a + \dfrac{a^2}{x^2} \\ \hline
2x - \dfrac{2a}{x} - \dfrac{a^2}{2x^3} & -\dfrac{a^2}{x^2} \\
 & -\dfrac{a^2}{x^2} + \dfrac{a^3}{x^4} + \dfrac{a^4}{4x^6} \\ \hline
\multicolumn{2}{r}{-\dfrac{a^3}{x^4} - \dfrac{a^4}{4x^6}}
\end{array}
$$

16. Find to three terms the square root of $x^2 + 4b$.

$$
\begin{array}{r|l}
\multicolumn{2}{l}{x^2 + 4b \qquad \left|\, x + \dfrac{2b}{x} - \dfrac{2b^2}{x^3} + \cdots\right.} \\
\multicolumn{2}{l}{x^2} \\ \hline
2x + \dfrac{2b}{x} & 4b \\
 & 4b + \dfrac{4b^2}{x^2} \\ \hline
2x + \dfrac{4b}{x} - \dfrac{2b^2}{x^3} & -\dfrac{4b^2}{x^2} \\
 & -\dfrac{4b^2}{x^2} - \dfrac{8b^3}{x^4} + \dfrac{4b^4}{x^6} \\ \hline
\multicolumn{2}{r}{\dfrac{8b^3}{x^4} - \dfrac{4b^4}{x^6}}
\end{array}
$$

17. Find to three terms the square root of $x^2 + 5$.

$$
\begin{array}{r|l}
\multicolumn{2}{l}{x^2 + 5 \qquad \left|\, x + \dfrac{5}{2x} - \dfrac{25}{8x^3} + \cdots\right.} \\
\multicolumn{2}{l}{x^2} \\ \hline
2x + \dfrac{5}{2x} & 5 \\
 & 5 + \dfrac{25}{4x^2} \\ \hline
2x + \dfrac{5}{x} - \dfrac{25}{8x^3} & -\dfrac{25}{4x^2} \\
 & -\dfrac{25}{4x^2} - \dfrac{125}{8x^4} + \dfrac{625}{64x^6} \\ \hline
\multicolumn{2}{r}{\dfrac{125}{8x^4} - \dfrac{625}{64x^6}}
\end{array}
$$

18. Find to three terms the square root of $4x^2+2$.

$$\begin{array}{r|l}
 & 4x^2+2 \qquad \left|\underline{\,2x+\frac{1}{2x}-\frac{1}{16x^3}+\cdots}\right. \\
 & \underline{4x^2} \\
4x+\frac{1}{2x} & 2 \\
 & \underline{2+\frac{1}{4x^2}} \\
4x+\frac{1}{x}-\frac{1}{16x^3} & -\frac{1}{4x^2} \\
 & \underline{-\frac{1}{4x^2}-\frac{1}{16x^4}+\frac{1}{256x^6}} \\
 & \qquad\frac{1}{16x^4}-\frac{1}{256x^6}
\end{array}$$

19. Find to three terms the square root of $16a^4-3a^2$.

$$\begin{array}{r|l}
 & 16a^4-3a^2 \qquad \left|\underline{\,4a^2-\frac{3}{8}-\frac{9}{512a^2}-\cdots}\right. \\
 & \underline{16a^4} \\
8a^2-\frac{3}{8} & -3a^2 \\
 & \underline{-3a^2+\frac{9}{64}} \\
8a^2-\frac{3}{4}-\frac{9}{512a^2} & -\frac{9}{64} \\
 & \underline{-\frac{9}{64}+\frac{27}{2048a^2}+\frac{81}{262144a^4}} \\
 & \qquad-\frac{27}{2048a^2}-\frac{81}{262144a^4}
\end{array}$$

20. Find to three terms the square root of $25x^2-10x$.

$$\begin{array}{r|l}
 & 25x^2-10x \qquad \left|\underline{\,5x-1-\frac{1}{10x}-\cdots}\right. \\
 & \underline{25x^2} \\
10x-1 & -10x \\
 & \underline{-10x+1} \\
10x-2-\frac{1}{10x} & -1 \\
 & \underline{-1+\frac{1}{5x}+\frac{1}{100x^2}} \\
 & \qquad-\frac{1}{5x}-\frac{1}{100x^2}
\end{array}$$

Exercise 89. Page 227

1. Find the square root of 576.

5 76(24
4
44)1 76
1 76

2. Find the square root of 18,769.

1 87 69(137
1
23)87
69
267)18 69
18 69

3. Find the square root of 494,209.

49 42 09(703
49
1403)42 09
42 09

4. Find the square root of 755,161.

75 51 61(869
64
166)11 51
9 96
1729)1 55 61
1 55 61

5. Find the square root of 18,090.25.

1 80 90.25(134.5
1
23)80
69
264)11 90
10 56
2685)1 34 25
1 34 25

6. Find the square root of 150.0625.

1 50.06 25(12.25
1
22)50
44
242)6 06
4 84
2445)1 22 25
1 22 25

7. Find the square root of 4076.8225.

40 76.82 25(63.85
36
123)4 76
3 69
1268)1 07 82
1 01 44
12765)6 38 25
6 38 25

8. Find the square root of 432.2241.

4 32.22 41(20.79
4
407)32 22
28 49
4149)3 73 41
3 73 41

9. Find the square root of 709,469.29.

70 94 69.29(842.3
64
164)6 94
6 56
1682)38 69
33 64
16843)5 05 29
5 05 29

10. Find the square root of 35.545444.

```
      35.54 54 44(5.962
      25
  109)10 54
       9 81
   1186)73 54
        71 16
   11922)2 38 44
         2 38 44
```

11. Find the square root of 2,611,456.

```
      2 61 14 56(1616
      1
   26)1 61
      1 56
   321)5 14
       3 21
   3226)1 93 56
        1 93 56
```

12. Find the square root of 279,100.89.

```
      27 91 00.89(528.3
      25
  102)2 91
      2 04
   1048)87 00
        83 84
   10563)3 16 89
         3 16 89
```

13. Find the square root of 351.112644.

```
      3 51.11 26 44(18.738
      1
   28)2 51
      2 24
   367)27 11
       25 69
   3743)1 42 26
        1 12 29
   37468)29 97 44
         29 97 44
```

14. Find the square root of 95,765,796.

```
      95 76 57 96(9786
      81
  187)14 76
      13 09
   1948)1 67 57
        1 55 84
    19566)11 73 96
          11 73 96
```

15. Find the square root of 69,384.8281.

```
      6 93 84.82 81(263.41
      4
   46)2 93
      2 76
   523)17 84
       15 69
   5264)2 15 82
        2 10 56
    52681)5 26 81
          5 26 81
```

16. Find the square root of 1590.015625.

```
      15 90.01 56 25(39.875
      9
   69)6 90
      6 21
   788)69 01
       63 04
   7967)5 97 56
        5 57 69
    79745)39 87 25
          39 87 25
```

17. Find the square root of 179,301.4336.

17 93 01.43 36(423.44
16
82) 1 93
1 64
843) 29 01
25 29
8464) 3 72 43
3 38 56
84684) 33 87 36
33 87 36

nd the square root of
21.

24 99.05 21 (567.89
25
) 7 24
6 36
1127) 88 99
78 89
11348) 10 10 05
9 07 84
113569) 1 [illegible] 21 21
1 02 21 21

19. Find to four decimal places the square root of 2.

2.00 (1.4142 · · ·
1
24) 1 00
96
281) 4 00
2 81
2824) 1 19 00
1 12 96
28282) 6 04 00
5 65 64
38 36

20. Find to four decimal places the square root of 3.

3.00 (1.7320 · · ·
1
27) 2 00
1 89
343) 11 00
10 29
3462) 71 00
69 24
34640) 1 76 00

21. Find to four decimal places the square root of 5.

5.00 (2.2360 · · ·
4
42) 1 00
84
443) 16 00
13 29
4466) 2 71 00
2 67 96
44720) 3 04 00

22. Find ~~to~~ four decimal places the square root of 6.

6.00 (2.4494 · · ·
4
44) 2 00
1 76
484) 24 00
19 36
4889) 4 64 00
4 40 01
48984) 23 99 00
19 59 36
4 39 64

23. Find to four decimal places the square root of 7.

7.00 (2.6457 · · ·
4
46) 3 00
2 76
524) 24 00
20 96
5285) 3 04 00
2 64 25
52907) 39 75 00
37 03 49
2 71 51

24. Find to four decimal places the square root of 8.

8.00 (2.8284 · · ·
4
48) 4 00
3 84
562) 16 00
11 24
5648) 4 76 00
4 51 84
56564) 24 16 00
22 62 56
1 53 44

25. Find to four decimal places the square root of 0.3.

0.30 (0.5477 · · ·
25
104) 5 00
4 16
1087) 84 00
76 09
10947) 7 91 00
7 66 29
24 71

26. Find to four decimal places the square root of 0.4.

0.40 (0.6324 · · ·
36
123) 4 00
3 69
1262) 31 00
25 24
12644) 5 76 00
5 05 76
70 24

27. Find to four decimal places the square root of 0.5.

0.50 (0.7071 · · ·
49
1407) 1 00 00
98 49
14141) 1 51 00
1 41 41
9 59

28. Find to four decimal places the square root of 0.6.

0.60 (0.7745 · · ·
49
147) 11 00
10 29
1544) 71 00
61 76
15485) 9 24 00
7 74 25
1 49 75

29. Find to four decimal places the square root of 0.8.

0.80 (0.8944 · · ·
64
169) 16 00
15 21
1784) 79 00
71 36
17884) 7 64 00
7 15 36
48 64

30. Find to four decimal places the square root of 0.9.

$$\begin{array}{r}
0.90\,(0.9486\cdots\\
\underline{81}\\
184)\,9\ 00\\
\underline{7\ 36}\\
1888)\,1\ 64\ 00\\
\underline{1\ 51\ 04}\\
18966)\,12\ 96\ 00\\
\underline{11\ 37\ 96}\\
1\ 58\ 04
\end{array}$$

31. Find to four decimal places the square root of 0.307.

$$\begin{array}{r}
0.30\ 70\,(0.5540\cdots\\
\underline{25}\\
105)\,5\ 70\\
\underline{5\ 25}\\
1104)\,45\ 00\\
\underline{44\ 16}\\
11080)\,84\ 00
\end{array}$$

32. Find to four decimal places the square root of 0.635.

$$\begin{array}{r}
0.63\ 50\,(0.7968\cdots\\
\underline{49}\\
149)\,14\ 50\\
\underline{13\ 41}\\
1586)\,1\ 09\ 00\\
\underline{95\ 16}\\
15928)\,13\ 84\ 00\\
\underline{12\ 74\ 24}\\
1\ 09\ 76
\end{array}$$

33. Find to four decimal places the square root of 0.375.

$$\begin{array}{r}
0.37\ 50\,(0.6123\cdots\\
\underline{36}\\
121)\,1\ 50\\
\underline{1\ 21}\\
1222)\,29\ 00\\
\underline{24\ 44}\\
12243)\,4\ 56\ 00\\
\underline{3\ 67\ 29}\\
88\ 71
\end{array}$$

34. Find to four decimal places the square root of 0.869.

$$\begin{array}{r}
0.86\ 90\,(0.9322\cdots\\
\underline{81}\\
183)\,5\ 90\\
\underline{5\ 49}\\
1862)\,41\ 00\\
\underline{37\ 24}\\
18642)\,3\ 76\ 00\\
\underline{3\ 72\ 84}\\
3\ 16
\end{array}$$

35. Find to four decimal places the square root of $\frac{7}{15}$.

$$\sqrt{\frac{7}{15}}=\sqrt{\frac{105}{15^2}}=\frac{\sqrt{105}}{\sqrt{15^2}}$$

$$=\frac{10.2469\cdots}{15}=0.6831\cdots$$

$$\begin{array}{r}
1\ 05\,(10.2469\cdots\\
\underline{1}\\
202)\,05\ 00\\
\underline{4\ 04}\\
2044)\,96\ 00\\
\underline{81\ 76}\\
20486)\,14\ 24\ 00\\
\underline{12\ 29\ 16}\\
204929)\,1\ 94\ 84\ 00\\
\underline{1\ 84\ 43\ 61}\\
10\ 40\ 39
\end{array}$$

36. Find to four decimal places the square root of $\frac{8}{19}$.

$$\sqrt{\frac{8}{19}}=\sqrt{\frac{152}{19^2}}=\frac{\sqrt{152}}{\sqrt{19^2}}$$

$$=\frac{12.3288\cdots}{19}=0.6488\cdots$$

$$\begin{array}{rl}
 & 1\ 52\,(12.3288\cdots\\
 & \underline{1}\\
22) & 52\\
 & \underline{44}\\
243) & 8\ 00\\
 & \underline{7\ 29}\\
2462) & 71\ 00\\
 & \underline{49\ 24}\\
24648) & 21\ 76\ 00\\
 & \underline{19\ 71\ 84}\\
246568) & 2\ 04\ 16\ 00\\
 & \underline{1\ 97\ 25\ 44}\\
 & 6\ 90\ 56
\end{array}$$

37. Find to four decimal places the square root of $\frac{2}{3}$.

$\frac{2}{3}=0.66666667.$

$$\begin{array}{rl}
 & 0.66\ 66\ 66\ 67\,(0.8164\cdots\\
 & \underline{64}\\
161) & 2\ 66\\
 & \underline{1\ 61}\\
1626) & 1\ 05\ 66\\
 & \underline{97\ 56}\\
16324) & 8\ 10\ 67\\
 & \underline{6\ 52\ 96}\\
 & 1\ 57\ 71
\end{array}$$

38. Find to four decimal places the square root of $\frac{5}{8}$.

$\frac{5}{8}=0.625.$

$$\begin{array}{rl}
 & 0.62\ 50\,(0.7905\cdots\\
 & \underline{49}\\
149) & 13\ 50\\
 & \underline{13\ 41}\\
15805) & 9\ 00\ 00\\
 & \underline{7\ 90\ 25}\\
 & 1\ 09\ 75
\end{array}$$

Exercise 90. Page 229

1. Find the cube root of $27x^3-189x^2y+441xy^2-343y^3$.

$$\begin{array}{rr|l}
 & & \underline{|\,3x-7y}\\
 & & 27x^3-189x^2y+441xy^2-343y^3\\
 & & \underline{27x^3}\\
3(3x)^2=27x^2 & & -189x^2y+441xy^2-343y^3\\
(9x-7y)(-7y)= & \underline{-63xy+49y^2} & \\
 & 27x^2-63xy+49y^2 & \underline{-189x^2y+441xy^2-343y^3}
\end{array}$$

2. Find the cube root of $300ab^2-240a^2b-125b^3+64a^3$.

Arrange in descending powers of a.

$$\begin{array}{rr|l}
 & & \underline{|\,4a-5b}\\
 & & 64a^3-240a^2b+300ab^2-125b^3\\
 & & \underline{64a^3}\\
3(4a)^2=48a^2 & & -240a^2b+300ab^2-125b^3\\
(12a-5b)(-5b)= & \underline{-60ab+25b^2} & \\
 & 48a^2-60ab+25b^2 & \underline{-240a^2b+300ab^2-125b^3}
\end{array}$$

3. Find the cube root of $a^6 - 3a^5b + 6a^4b^2 - 7a^3b^3 + 6a^2b^4 - 3ab^5 + b^6$.

$$\begin{array}{rr|l}
 & & a^6 - 3a^5b + 6a^4b^2 - 7a^3b^3 + 6a^2b^4 - 3ab^5 + b^6 \;\underline{\big|\, a^2 - ab + b^2} \\
 & & a^6 \\
3(a^2)^2 = & 3a^4 & -3a^5b + 6a^4b^2 - 7a^3b^3 \\
(3a^2 - ab)(-ab) = & \underline{-3a^3b + a^2b^2} & \\
 & 3a^4 - 3a^3b + a^2b^2 & \underline{-3a^5b + 3a^4b^2 - a^3b^3} \\
3(a^2 - ab)^2 = & 3a^4 - 6a^3b + 3a^2b^2 & 3a^4b^2 - 6a^3b^3 + 6a^2b^4 - 3ab^5 + b^6 \\
(3a^2 - 3ab + b^2)(b^2) = & \underline{+3a^2b^2 - 3ab^3 + b^4} & \\
 & 3a^4 - 6a^3b + 6a^2b^2 - 3ab^3 + b^4 & \underline{3a^4b^2 - 6a^3b^3 + 6a^2b^4 - 3ab^5 + b^6}
\end{array}$$

4. Find the cube root of $8x^6 - 12x^5y + 30x^4y^2 - 25x^3y^3 + 30x^2y^4 - 12xy^5 + 8y^6$.

$$\begin{array}{rr|l}
 & & 8x^6 - 12x^5y + 30x^4y^2 - 25x^3y^3 + 30x^2y^4 - 12xy^5 + 8y^6 \;\underline{\big|\, 2x^2 - xy + 2y^2} \\
 & & 8x^6 \\
3(2x^2)^2 = & 12x^4 & -12x^5y + 30x^4y^2 - 25x^3y^3 \\
(6x^2 - xy)(-xy) = & \underline{-6x^3y + x^2y^2} & \\
 & 12x^4 - 6x^3y + x^2y^2 & \underline{-12x^5y + 6x^4y^2 - x^3y^3} \\
3(2x^2 - xy)^2 = & 12x^4 - 12x^3y + 3x^2y^2 & 24x^4y^2 - 24x^3y^3 + 30x^2y^4 - 12xy^5 + 8y^6 \\
(6x^2 - 3xy + 2y^2)(2y^2) = & \underline{+12x^2y^2 - 6xy^3 + 4y^4} & \\
 & 12x^4 - 12x^3y + 15x^2y^2 - 6xy^3 + 4y^4 & \underline{24x^4y^2 - 24x^3y^3 + 30x^2y^4 - 12xy^5 + 8y^6}
\end{array}$$

5. Find the cube root of $a^6 - 6a^5b + 21a^4b^2 - 44a^3b^3 + 63a^2b^4 - 54ab^5 + 27b^6$.

$$\begin{array}{r|l}
 & a^6 - 6a^5b + 21a^4b^2 - 44a^3b^3 + 63a^2b^4 - 54ab^5 + 27b^6 \;\underline{|\,a^2 - 2ab + 3b^2} \\
 & a^6 \\
3(a^2)^2 = 3a^4 & -6a^5b + 21a^4b^2 - 44a^3b^3 \\
(3a^2 - 2ab)(-2ab) = \underline{\quad -6a^3b + 4a^2b^2} & \\
3a^4 - 6a^3b + 4a^2b^2 & -6a^5b + 12a^4b^2 - 8a^3b^3 \\
3(a^2 - 2ab)^2 = 3a^4 - 12a^3b + 12a^2b^2 & 9a^4b^2 - 36a^3b^3 + 63a^2b^4 - 54ab^5 + 27b^6 \\
(3a^2 - 6ab + 3b^2)(3b^2) = \underline{\quad + 9a^2b^2 - 18ab^3 + 9b^4} & \\
3a^4 - 12a^3b + 21a^2b^2 - 18ab^3 + 9b^4 & 9a^4b^2 - 36a^3b^3 + 63a^2b^4 - 54ab^5 + 27b^6
\end{array}$$

6. Find the cube root of $8c^6 - 60c^5 + 114c^4 + 55c^3 - 171c^2 - 135c - 27$.

$$\begin{array}{r|l}
 & 8c^6 - 60c^5 + 114c^4 + 55c^3 - 171c^2 - 135c - 27 \;\underline{|\,2c^2 - 5c - 3} \\
 & 8c^6 \\
3(2c^2)^2 = 12c^4 & -60c^5 + 114c^4 + 55c^3 \\
(6c^2 - 5c)(-5c) = \underline{\quad -30c^3 + 25c^2} & \\
12c^4 - 30c^3 + 25c^2 & -60c^5 + 150c^4 - 125c^3 \\
3(2c^2 - 5c)^2 = 12c^4 - 60c^3 + 75c^2 & -36c^4 + 180c^3 - 171c^2 - 135c - 27 \\
(6c^2 - 15c - 3)(-3) = \underline{\quad -18c^2 + 45c + 9} & \\
12c^4 - 60c^3 + 57c^2 + 45c + 9 & -36c^4 + 180c^3 - 171c^2 - 135c - 27
\end{array}$$

7. Find the cube root of $27a^6 + 108a^5 + 198a^4 + 208a^3 + 132a^2 + 48a + 8$.

$$
\begin{array}{rl|l}
 & 27a^6 + 108a^5 + 198a^4 + 208a^3 + 132a^2 + 48a + 8 & \underline{3a^2 + 4a + 2} \\
 & \underline{27a^6} & \\
3(3a^2)^2 = & 27a^4 & 108a^5 + 198a^4 + 208a^3 \\
(9a^2 + 4a)(4a) = & \underline{\qquad + 36a^3 + 16a^2} & \\
 & 27a^4 + 36a^3 + 16a^2 & \underline{108a^5 + 144a^4 + 64a^3} \\
3(3a^2 + 4a)^2 = & 27a^4 + 72a^3 + 48a^2 & 54a^4 + 144a^3 + 132a^2 + 48a + 8 \\
(9a^2 + 12a + 2)(2) = & \underline{\qquad + 18a^2 + 24a + 4} & \\
 & 27a^4 + 72a^3 + 60a^2 + 24a + 4 & 54a^4 + 144a^3 + 132a^2 + 48a + 8
\end{array}
$$

8. Find the cube root of $64a^6 - 144a^5b + 60a^4b^2 + 45a^3b^3 - 15a^2b^4 - 9ab^5 - b^6$.

$$
\begin{array}{rl|l}
 & 64a^6 - 144a^5b + 60a^4b^2 + 45a^3b^3 - 15a^2b^4 - 9ab^5 - b^6 & \underline{4a^2 - 3ab - b^2} \\
 & \underline{64a^6} & \\
3(4a^2)^2 = & 48a^4 & -144a^5b + 60a^4b^2 + 45a^3b^3 \\
(12a^2 - 3ab)(-3ab) = & \underline{\qquad - 36a^3b + 9a^2b^2} & \\
 & 48a^4 - 36a^3b + 9a^2b^2 & \underline{-144a^5b + 108a^4b^2 - 27a^3b^3} \\
3(4a^2 - 3ab)^2 = & 48a^4 - 72a^3b + 27a^2b^2 & -48a^4b^2 + 72a^3b^3 - 15a^2b^4 - 9ab^5 - b^6 \\
(12a^2 - 9ab - b^2)(-b^2) = & \underline{\qquad - 12a^2b^2 + 9ab^3 + b^4} & \\
 & 48a^4 - 72a^3b + 15a^2b^2 + 9ab^3 + b^4 & \underline{-48a^4b^2 + 72a^3b^3 - 15a^2b^4 - 9ab^5 - b^6}
\end{array}
$$

9. Find the cube root of $125x^9-300x^8+465x^7-424x^6+279x^5-108x^4+27x^3$.

$$\begin{array}{rrl}
 & & 125x^9-300x^8+465x^7-424x^6+279x^5-108x^4+27x^3 \;\Big|\; \underline{5x^3-4x^2+3x} \\
 & & \underline{125x^9} \\
3(5x^3)^2= & 75x^6 & -300x^8+465x^7-424x^6 \\
(15x^3-4x^2)(-4x^2)= & \underline{-60x^5+16x^4} & \\
 & 75x^6-60x^5+16x^4 & \underline{-300x^8+240x^7-64x^6} \\
3(5x^3-4x^2)^2= & 75x^6-120x^5+48x^4 & 225x^7-360x^6+279x^5-108x^4+27x^3 \\
(15x^3-12x^2+3x)(3x)= & \underline{+45x^4-36x^3+9x^2} & \\
 & 75x^6-120x^5+93x^4-36x^3+9x^2 & \underline{225x^7-360x^6+279x^5-108x^4+27x^3}
\end{array}$$

10. Find the cube root of $27c^{12}+189c^{11}+198c^{10}-791c^9-594c^8+1701c^7-729c^6$.

$$\begin{array}{rrl}
 & & \underline{3c^4+7c^3-9c^2} \\
 & & 27c^{12}+189c^{11}+198c^{10}-791c^9-594c^8+1701c^7-729c^6 \\
 & & \underline{27c^{12}} \\
3(3c^4)^2= & 27c^8 & 189c^{11}+198c^{10}-791c^9 \\
(9c^4+7c^3)(7c^3)= & \underline{+63c^7+49c^6} & \\
 & 27c^8+63c^7+49c^6 & \underline{189c^{11}+441c^{10}+343c^9} \\
3(3c^4+7c^3)^2= & 27c^8+126c^7+147c^6 & -243c^{10}-1134c^9-594c^8+1701c^7-729c^6 \\
(9c^4+21c^3-9c^2)(-9c^2)= & \underline{-81c^6-189c^5+81c^4} & \\
 & 27c^8+126c^7+66c^6-189c^5+81c^4 & \underline{-243c^{10}-1134c^9-594c^8+1701c^7-729c^6}
\end{array}$$

11. Find the cube root of $a^6 - 9a^5b + 18a^4b^2 + 27a^3b^3 - 54a^2b^4 - 81ab^5 - 27b^6$.

$$\begin{array}{rl|l}
 & & \underline{a^2 - 3ab - 3b^2} \\
 & & a^6 - 9a^5b + 18a^4b^2 + 27a^3b^3 - 54a^2b^4 - 81ab^5 - 27b^6 \\
 & & a^6 \\
3(a^2)^2 = & 3a^4 & - 9a^5b + 18a^4b^2 + 27a^3b^3 \\
(3a^2 - 3ab)(-3ab) = & \underline{\quad - 9a^3b + 9a^2b^2} & \\
 & 3a^4 - 9a^3b + 9a^2b^2 & - 9a^5b + 27a^4b^2 - 27a^3b^3 \\
3(a^2 - 3ab)^2 = & 3a^4 - 18a^3b + 27a^2b^2 & - 9a^4b^2 + 54a^3b^3 - 54a^2b^4 - 81ab^5 - 27b^6 \\
(3a^2 - 9ab - 3b^2)(-3b^2) = & \underline{\qquad\qquad - 9a^2b^2 + 27ab^3 + 9b^4} & \\
 & 3a^4 - 18a^3b + 18a^2b^2 + 27ab^3 + 9b^4 & \underline{- 9a^4b^2 + 54a^3b^3 - 54a^2b^4 - 81ab^5 - 27b^6}
\end{array}$$

12. Find the cube root of $\frac{27a^{12}}{x^9}-\frac{135a^{11}}{x^8}+\frac{198a^{10}}{x^7}-\frac{35a^9}{x^6}-\frac{66a^8}{x^5}-\frac{15a^7}{x^4}-\frac{a^6}{x^3}$.

$$\frac{27a^{12}}{x^9}-\frac{135a^{11}}{x^8}+\frac{198a^{10}}{x^7}-\frac{35a^9}{x^6}-\frac{66a^8}{x^5}-\frac{15a^7}{x^4}-\frac{a^6}{x^3}\,\Big|\,\frac{3a^4}{x^3}-\frac{5a^3}{x^2}-\frac{a^2}{x}$$

$$\frac{27a^{12}}{x^9}$$

$$3\left(\frac{3a^4}{x^3}\right)^2=\frac{27a^8}{x^6}\qquad -\frac{135a^{11}}{x^8}+\frac{198a^{10}}{x^7}-\frac{35a^9}{x^6}$$

$$\left(\frac{9a^4}{x^3}-\frac{5a^3}{x^2}\right)\left(-\frac{5a^3}{x^2}\right)=\quad -\frac{45a^7}{x^5}+\frac{25a^6}{x^4}$$

$$\frac{27a^8}{x^6}-\frac{45a^7}{x^5}+\frac{25a^6}{x^4}\qquad -\frac{135a^{11}}{x^8}+\frac{225a^{10}}{x^7}-\frac{125a^9}{x^6}$$

$$3\left(\frac{3a^4}{x^3}-\frac{5a^3}{x^2}\right)^2=\frac{27a^8}{x^6}-\frac{90a^7}{x^5}+\frac{75a^6}{x^4}\qquad -\frac{27a^{10}}{x^7}+\frac{90a^9}{x^6}-\frac{66a^8}{x^5}-\frac{15a^7}{x^4}-\frac{a^6}{x^3}$$

$$\left(\frac{9a^4}{x^3}-\frac{15a^3}{x^2}-\frac{a^2}{x}\right)\left(-\frac{a^2}{x}\right)=\quad -\frac{9a^6}{x^4}+\frac{15a^5}{x^3}+\frac{a^4}{x^2}$$

$$\frac{27a^8}{x^6}-\frac{90a^7}{x^5}+\frac{66a^6}{x^4}+\frac{15a^5}{x^3}+\frac{a^4}{x^2}\qquad -\frac{27a^{10}}{x^7}+\frac{90a^9}{x^6}-\frac{66a^8}{x^5}-\frac{15a^7}{x^4}-\frac{a^6}{x^3}$$

13. Find the cube root of $\frac{8a^3}{27x^3}+\frac{4a^2}{3x^2}+\frac{3a}{x}+4+\frac{27x}{8a}+\frac{27x^2}{16a^2}+\frac{27x^3}{64a^3}$.

$$\begin{array}{rr|l}
 & \frac{8a^3}{27x^3}+\frac{4a^2}{3x^2}+\frac{3a}{x}+4+\frac{27x}{8a}+\frac{27x^2}{16a^2}+\frac{27x^3}{64a^3} & \frac{2a}{3x}+1+\frac{3x}{4a} \\
 & \frac{8a^3}{27x^3} & \\ \hline
3\left(\frac{2a}{3x}\right)^2=\frac{4a^2}{3x^2} & \frac{4a^2}{3x^2}+\frac{3a}{x}+4 & \\
\left(\frac{2a}{x}+1\right)(1)=\quad+\frac{2a}{x}+1 & & \\ \cline{1-1}
\frac{4a^2}{3x^2}+\frac{2a}{x}+1 & \frac{4a^2}{3x^2}+\frac{2a}{x}+1 & \\ \hline
3\left(\frac{2a}{3x}+1\right)^2=\frac{4a^2}{3x^2}+\frac{4a}{x}+3 & \frac{a}{x}+3+\frac{27x}{8a}+\frac{27x^2}{16a^2}+\frac{27x^3}{64a^3} & \\
\left(\frac{2a}{x}+3+\frac{3x}{4a}\right)\left(\frac{3x}{4a}\right)=\quad+\frac{3}{2}+\frac{9x}{4a}+\frac{9x^2}{16a^2} & & \\ \cline{1-1}
\frac{4a^2}{3x^2}+\frac{4a}{x}+\frac{9}{2}+\frac{9x}{4a}+\frac{9x^2}{16a^2} & \frac{a}{x}+3+\frac{27x}{8a}+\frac{27x^2}{16a^2}+\frac{27x^3}{64a^3} & \\ \hline
\end{array}$$

14. Find the cube root of $8x^9 - 36x^8y + 6x^7y^2 + 141x^6y^3 - 84x^5y^4 - 186x^4y^5 + 104x^3y^6 + 60x^2y^7 - 48xy^8 + 8y^9$.

$$\underline{|\,2x^3-3x^2y-4xy^2+2y^3}$$

$$8x^9-36x^8y+6x^7y^2+141x^6y^3-84x^5y^4-186x^4y^5+104x^3y^6+60x^2y^7-48xy^8+8y^9$$

$$\underline{8x^9}$$

$$3(2x^3)^2=12x^6 \qquad -36x^8y+6x^7y^2+141x^6y^3$$

$$(6x^3-3x^2y)(-3x^2y)=\underline{-18x^5y+9x^4y^2}$$

$$12x^6-18x^5y+9x^4y^2 \qquad \underline{-36x^8y+54x^7y^2-27x^6y^3}$$

$$3(2x^3-3x^2y)^2=12x^6-36x^5y+27x^4y^2 \qquad -48x^7y^2+168x^6y^3-84x^5y^4-186x^4y^5+104x^3y^6$$

$$(6x^3-9x^2y-4xy^2)(-4xy^2)=\underline{-24x^4y^2+36x^3y^3+16x^2y^4}$$

$$12x^6-36x^5y+3x^4y^2+36x^3y^3+16x^2y^4 \qquad \underline{-48x^7y^2+144x^6y^3-12x^5y^4-144x^4y^5-64x^3y^6}$$

$$3(2x^3-3x^2y-4xy^2)^2=12x^6-36x^5y-21x^4y^2+72x^3y^3+48x^2y^4 \qquad 24x^6y^3-72x^5y^4-42x^4y^5+168x^3y^6+60x^2y^7-48xy^8+8y^9$$

$$(6x^3-9x^2y-12xy^2+2y^3)(2y^3)=\underline{+12x^3y^3-18x^2y^4-24xy^5+4y^6}$$

$$12x^6-36x^5y-21x^4y^2+84x^3y^3+30x^2y^4-24xy^5+4y^6 \qquad \underline{24x^6y^3-72x^5y^4-42x^4y^5+168x^3y^6+60x^2y^7-48xy^8+8y^9}$$

Exercise 91. Page 232

1. Find the cube root of 103,823.

$$\begin{array}{rr|l}
\multicolumn{2}{r}{} & \multicolumn{1}{l}{103\ 823\,\underline{|47}} \\
\multicolumn{2}{r}{} & \multicolumn{1}{l}{64} \\ \cline{3-3}
3 \times 40^2 = & 4800 & 39\ 823 \\
3(40 \times 7) = & 840 & \\
7^2 = & 49 & \\ \cline{2-2}
 & 5689 & 39\ 823 \\ \cline{3-3}
\end{array}$$

2. Find the cube root of 262,144.

$$\begin{array}{rr|l}
\multicolumn{2}{r}{} & \multicolumn{1}{l}{262\ 144\,\underline{|64}} \\
\multicolumn{2}{r}{} & \multicolumn{1}{l}{216} \\ \cline{3-3}
3 \times 60^2 = & 10800 & 46\ 144 \\
3(60 \times 4) = & 720 & \\
4^2 = & 16 & \\ \cline{2-2}
 & 11536 & 46\ 144 \\ \cline{3-3}
\end{array}$$

3. Find the cube root of 8741.816.

$$\begin{array}{rr|l}
\multicolumn{2}{r}{} & \multicolumn{1}{l}{8\ 741.816\,\underline{|20.6}} \\
\multicolumn{2}{r}{} & \multicolumn{1}{l}{8} \\ \cline{3-3}
3 \times 200^2 = & 120000 & 741\ 816 \\
3(200 \times 6) = & 3600 & \\
6^2 = & 36 & \\
 & 123636 & 741\ 816 \\ \cline{3-3}
\end{array}$$

4. Find the cube root of 410.172407.

$$\begin{array}{rr|l}
\multicolumn{2}{r}{} & \multicolumn{1}{l}{410.172\ 407\,\underline{|7.43}} \\
\multicolumn{2}{r}{} & \multicolumn{1}{l}{343} \\ \cline{3-3}
3 \times 70^2 = & 14700 & 67\ 172 \\
3(70 \times 4) = & 840 & \\
4^2 = & \left.\begin{array}{r} 16 \\ \hline 15556 \\ 856 \end{array}\right\} & \begin{array}{l} \\ 62\ 224 \\ \hline 4\ 948\ 407 \end{array} \\ \cline{2-2}
3 \times 740^2 = & 1642800 & \\
3(740 \times 3) = & 6660 & \\
3^2 = & 9 & \\ \cline{2-2}
 & 1649469 & 4\ 948\ 407 \\ \cline{3-3}
\end{array}$$

5. Find the cube root of 6,148,602.368.

		6 148 602.368	183.2
		1	
$3 \times 10^2 =$	300	5 148	
$3(10 \times 8) =$	240		
$8^2 =$	64		
	604	4 832	
	304	316 602	
$3 \times 180^2 =$	97200		
$3(180 \times 3) =$	1620		
$3^2 =$	9		
	98829	296 487	
	1629	20 115 368	
$3 \times 1830^2 =$	10046700		
$3(1830 \times 2) =$	10980		
$2^2 =$	4		
	10057684	20 115 368	

6. Find the cube root of 634,725.648584.

		634 725.648 584	85.94
		512	
$3 \times 80^2 =$	19200	122 725	
$3(80 \times 5) =$	1200		
$5^2 =$	25		
	20425	102 125	
	1225	20 600 648	
$3 \times 850^2 =$	2167500		
$3(850 \times 9) =$	22950		
$9^2 =$	81		
	2190531	19 714 779	
	23031	885 869 584	
$3 \times 8590^2 =$	221364300		
$3(8590 \times 4) =$	103080		
$4^2 =$	16		
	221467396	885 869 584	

7. **Find to four decimal places the cube root of 100.**

		100.000 | 4.6415 · · ·
		64
$3 \times 40^2 =$	4800	36 000
$3(40 \times 6) =$	720	
$6^2 =$	36	
	5556	33 336
	756	2 664 000
$3 \times 460^2 =$	634800	
$3(460 \times 4) =$	5520	
$4^2 =$	16	
	640336	2 561 344
	5536	102 656 000
$3 \times 4640^2 =$	64588800	
$3(4640 \times 1) =$	13920	
$1^2 =$	1	
	64602721	64 602 721
	13921	38 053 2790
$3 \times 4641^2 =$	64616643	32 308 3215
		5 744 9575

8. **Find to four decimal places the cube root of 206.**

		206.000 | 5.9059 · · ·
		125
$3 \times 50^2 =$	7500	81 000
$3(50 \times 9) =$	1350	
$9^2 =$	81	
	8931	80 379
	1431	621 000 000
$3 \times 5900^2 =$	104430000	
$3(5900 \times 5) =$	88500	
$5^2 =$	25	
	104518525	522 592 625
	88525	98 407 3750
$3 \times 5905^2 =$	104607075	94 146 3675
		4 261 0075

9. Find to four decimal places the cube root of 6.3.

		6.300	1.8469⋯
		1	
$3 \times 10^2 =$	300	5 300	
$3(10 \times 8) =$	240		
$8^2 =$	64		
	604	4 832	
	304	468 000	
$3 \times 180^2 =$	97200		
$3(180 \times 4) =$	2160		
$4^2 =$	16		
	99376	397 504	
	2176	70 496 000	
$3 \times 1840^2 =$	10156800		
$3(1840 \times 6) =$	33120		
$6^2 =$	36		
	10189956	61 139 736	
	33156	9 356 2640	
$3 \times 1846^2 =$	10223148	9 200 8332	
		155 4308	

10. Find to four decimal places the cube root of 0.375.

		0.375 000	0.7211⋯
		343	
$3 \times 70^2 =$	14700	32 000	
$3(70 \times 2) =$	420		
$2^2 =$	4		
	15124	30 248	
	424	1 752 000	
$3 \times 720^2 =$	1555200		
$3(720 \times 1) =$	2160		
$1^2 =$	1		
	1557361	1 557 361	
	2161	194 6390	
$3 \times 721^2 =$	1559523	155 9523	
		38 6867	

11. Find to four decimal places the cube root of 0.5.

$$
\begin{array}{rr|l}
 & & 0.500\,000\;\underline{|\,0.7937\cdots} \\
 & & \underline{343} \\
3\times 70^2 = & 14700 & 157\,000 \\
3(70\times 9) = & 1890 & \\
\begin{array}{r}9^2 = \\ \\ \\ \end{array} & \left.\begin{array}{r}\underline{81} \\ 16671 \\ \underline{1971}\end{array}\right\} & \begin{array}{r} \\ \underline{150\,039} \\ 6\,961\,000\end{array} \\
3\times 790^2 = & 1872300 & \\
3(790\times 3) = & 7110 & \\
\begin{array}{r}3^2 = \\ \\ \\ \end{array} & \left.\begin{array}{r}\underline{9} \\ 1879419 \\ \underline{7119}\end{array}\right\} & \begin{array}{r} \\ \underline{5\,638\,257} \\ 1\,322\,7430\end{array} \\
3\times 793^2 = & 1886547 & \underline{1\,320\,5829} \\
 & & 2\,1601
\end{array}
$$

12. Find to four decimal places the cube root of 0.98.

$$
\begin{array}{rr|l}
 & & 0.980\,000\;\underline{|\,0.9932\cdots} \\
 & & \underline{729} \\
3\times 90^2 = & 24300 & 251\,000 \\
3(90\times 9) = & 2430 & \\
\begin{array}{r}9^2 = \\ \\ \\ \end{array} & \left.\begin{array}{r}\underline{81} \\ 26811 \\ \underline{2511}\end{array}\right\} & \begin{array}{r} \\ \underline{241\,299} \\ 9\,701\,000\end{array} \\
3\times 990^2 = & 2940300 & \\
3(990\times 3) = & 8910 & \\
\begin{array}{r}3^2 = \\ \\ \\ \end{array} & \left.\begin{array}{r}\underline{9} \\ 2949219 \\ \underline{8919}\end{array}\right\} & \begin{array}{r} \\ \underline{8\,847\,657} \\ 853\,3430\end{array} \\
3\times 993^2 = & 2958147 & \underline{591\,6294} \\
 & & 261\,7136
\end{array}
$$

13. Find to four decimal places the cube root of $\frac{2}{3}$.

$$\tfrac{2}{3} = 0.6666666667.$$

$$\begin{array}{rcrl}
 & & & 0.666\ 666\ 666\ \underline{|0.8735\cdots} \\
 & & & \underline{512} \\
3 \times 80^2 & = & 19200 & \vert\ 154\ 666 \\
3(80 \times 7) & = & 1680 & \vert \\
7^2 & = & \underline{49} & \vert \\
 & & 20929 & \vert\ \underline{146\ 503} \\
 & & \underline{1729} & \vert\ 8\ 163\ 666 \\
3 \times 870^2 & = & 2270700 & \vert \\
3(870 \times 3) & = & 7830 & \vert \\
3^2 & = & \underline{9} & \vert \\
 & & 2278539 & \vert\ \underline{6\ 835\ 617} \\
 & & \underline{7839} & \vert\ 1\ 328\ 0497 \\
3 \times 873^2 & = & 2286387 & \vert\ \underline{1\ 143\ 1935} \\
 & & & \ \ 184\ 8562
\end{array}$$

14. Find to four decimal places the cube root of $\frac{4}{5}$.

$$\tfrac{4}{5} = 0.8.$$

$$\begin{array}{rcrl}
 & & & 0.800\ 000\ \underline{|0.9283\cdots} \\
 & & & \underline{729} \\
3 \times 90^2 & = & 24300 & \vert\ 71\ 000 \\
3(90 \times 2) & = & 540 & \vert \\
2^2 & = & \underline{4} & \vert \\
 & & 24844 & \vert\ \underline{49\ 688} \\
 & & \underline{544} & \vert\ 21\ 312\ 000 \\
3 \times 920^2 & = & 2539200 & \vert \\
3(920 \times 8) & = & 22080 & \vert \\
8^2 & = & \underline{64} & \vert \\
 & & 2561344 & \vert\ \underline{20\ 490\ 752} \\
 & & \underline{22144} & \vert\ 821\ 2480 \\
3 \times 928^2 & = & 2583552 & \vert\ \underline{775\ 0656} \\
 & & & \ \ 46\ 1824
\end{array}$$

15. **Find to four decimal places the cube root of $\frac{3}{7}$.**

$$\tfrac{3}{7} = 0.428571428571.$$

$$\begin{array}{rcr|r}
 & & & 0.428\ 571\ 428\ \underline{|0.7539\cdots} \\
 & & & \underline{343} \\
3 \times 70^2 & = & 14700 & 85\ 571 \\
3(70 \times 5) & = & 1050 & \\
5^2 & = & \underline{25} & \\
 & & 15775 & \underline{78\ 875} \\
 & & \underline{1075} & 6\ 696\ 428 \\
3 \times 750^2 & = & 1687500 & \\
3(750 \times 3) & = & 6750 & \\
3^2 & = & \underline{9} & \\
 & & 1694259 & \underline{5\ 082\ 777} \\
 & & \underline{6759} & 1\ 613\ 6515 \\
3 \times 753^2 & = & 1701027 & \underline{1\ 530\ 9243} \\
 & & & 82\ 7272
\end{array}$$

16. Find to four decimal places the cube root of $\frac{5}{16}$.

$$\tfrac{5}{16} = 0.3125.$$

$$\begin{array}{rcr|r}
 & & & 0.312\ 500\ \underline{|0.6786\cdots} \\
 & & & \underline{216} \\
3 \times 60^2 & = & 10800 & 96\ 500 \\
3(60 \times 7) & = & 1260 & \\
7^2 & = & \underline{49} & \\
 & & 12109 & \underline{84\ 763} \\
 & & \underline{1309} & 11\ 737\ 000 \\
3 \times 670^2 & = & 1346700 & \\
3(670 \times 8) & = & 16080 & \\
8^2 & = & \underline{64} & \\
 & & 1362844 & \underline{10\ 902\ 752} \\
 & & \underline{16144} & 834\ 2480 \\
3 \times 678^2 & = & 1379052 & \underline{827.\ 4312} \\
 & & & 6\ 8168
\end{array}$$

Exercise 92. Page 232

1. Find the fourth root of

$$256c^{16} - 768ac^{13}x^5 + 864a^2c^{10}x^{10} - 432a^3c^7x^{15} + 81a^4c^4x^{20}.$$

The fourth root is the square root of the square root.

$$\begin{array}{r|l}
 & \multicolumn{1}{r}{\underline{|16c^8 - 24ac^5x^5 + 9a^2c^2x^{10}}} \\
 & 256c^{16} - 768ac^{13}x^5 + 864a^2c^{10}x^{10} - 432a^3c^7x^{15} + 81a^4c^4x^{20} \\
 & 256c^{16} \\ \hline
32c^8 - 24ac^5x^5 & -768ac^{13}x^5 + 864a^2c^{10}x^{10} \\
 & \underline{-768ac^{13}x^5 + 576a^2c^{10}x^{10}} \\
32c^8 - 48ac^5x^5 + 9a^2c^2x^{10} & 288a^2c^{10}x^{10} - 432a^3c^7x^{15} + 81a^4c^4x^{20} \\
 & \underline{288a^2c^{10}x^{10} - 432a^3c^7x^{15} + 81a^4c^4x^{20}}
\end{array}$$

$$\begin{array}{r|l}
 & 16c^8 - 24ac^5x^5 + 9a^2c^2x^{10} \;\underline{|4c^4 - 3acx^5} \\
 & 16c^8 \\ \hline
8c^4 - 3acx^5 & -24ac^5x^5 + 9a^2c^2x^{10} \\
 & \underline{-24ac^5x^5 + 9a^2c^2x^{10}}
\end{array}$$

Therefore, the required fourth root is $4c^4 - 3acx^5$.

2. Find the fourth root of

$$1 + 16x + 108x^2 + 400x^3 + 886x^4 + 1200x^5 + 972x^6 + 432x^7 + 81x^8.$$

The fourth root is the square root of the square root.

$$\begin{array}{r|l}
 & \multicolumn{1}{r}{\underline{|1 + 8x + 22x^2 + 24x^3 + 9x^4}} \\
 & 1 + 16x + 108x^2 + 400x^3 + 886x^4 + 1200x^5 + 972x^6 + 432x^7 + 81x^8 \\
 & 1 \\ \hline
2 + 8x & 16x + 108x^2 \\
 & \underline{16x + 64x^2} \\
2 + 16x + 22x^2 & 44x^2 + 400x^3 + 886x^4 \\
 & \underline{44x^2 + 352x^3 + 484x^4} \\
2 + 16x + 44x^2 + 24x^3 & 48x^3 + 402x^4 + 1200x^5 + 972x^6 \\
 & \underline{48x^3 + 384x^4 + 1056x^5 + 576x^6} \\
2 + 16x + 44x^2 + 48x^3 + 9x^4 & 18x^4 + 144x^5 + 396x^6 + 432x^7 + 81x^8 \\
 & \underline{18x^4 + 144x^5 + 396x^6 + 432x^7 + 81x^8}
\end{array}$$

$$\begin{array}{r|l}
 & 1 + 8x + 22x^2 + 24x^3 + 9x^4 \;\underline{|1 + 4x + 3x^2} \\
 & 1 \\ \hline
2 + 4x & 8x + 22x^2 \\
 & \underline{8x + 16x^2} \\
2 + 8x + 3x^2 & 6x^2 + 24x^3 + 9x^4 \\
 & \underline{6x^2 + 24x^3 + 9x^4}
\end{array}$$

Therefore, the required fourth root is $1 + 4x + 3x^2$.

3. Find the sixth root of $729\,x^6 - 2916\,x^5y + 4860\,x^4y^2 - 4320\,x^3y^3 + 2160\,x^2y^4 - 576\,xy^5 + 64\,y^6$.

The sixth root is the cube root of the square root.

$$\begin{array}{r|l}
 & 729\,x^6 - 2916\,x^5y + 4860\,x^4y^2 - 4320\,x^3y^3 + 2160\,x^2y^4 - 576\,xy^5 + 64\,y^6 \;\underline{|\,27\,x^3 - 54\,x^2y + 36\,xy^2 - 8\,y^3} \\
 & \underline{729\,x^6} \\
54\,x^3 - 54\,x^2y & - 2916\,x^5y + 4860\,x^4y^2 \\
 & \underline{- 2916\,x^5y + 2916\,x^4y^2} \\
54\,x^3 - 108\,x^2y + 36\,xy^2 & 1944\,x^4y^2 - 4320\,x^3y^3 + 2160\,x^2y^4 \\
 & \underline{1944\,x^4y^2 - 3888\,x^3y^3 + 1296\,x^2y^4} \\
54\,x^3 - 108\,x^2y + 72\,xy^2 - 8\,y^3 & - 432\,x^3y^3 + 864\,x^2y^4 - 576\,xy^5 + 64\,y^6 \\
 & \underline{- 432\,x^3y^3 + 864\,x^2y^4 - 576\,xy^5 + 64\,y^6}
\end{array}$$

$$\begin{array}{rr|l}
 & & 27\,x^3 - 54\,x^2y + 36\,xy^2 - 8\,y^3 \;\underline{|\,3\,x - 2\,y} \\
 & & \underline{27\,x^3} \\
3\,(3\,x)^2 = & 27\,x^2 & - 54\,x^2y + 36\,xy^2 - 8\,y^3 \\
(9\,x - 2\,y)\,(-2\,y) = & \underline{- 18\,xy + 4\,y^2} & \\
 & 27\,x^2 - 18\,xy + 4\,y^2 & \underline{- 54\,x^2y + 36\,xy^2 - 8\,y^3}
\end{array}$$

Therefore, the required sixth root is $3\,x - 2\,y$.

4. Find the sixth root of $\frac{64x^6}{15625} - \frac{576x^5y}{21875} + \frac{432x^4y^2}{6125} - \frac{864x^3y^3}{8575} + \frac{972x^2y^4}{12005} - \frac{2916xy^5}{84035} + \frac{729y^6}{117649}$.

The sixth root is the cube root of the square root.

$$\begin{array}{l|l}
 & \frac{8x^3}{125} - \frac{36x^2y}{175} + \frac{54xy^2}{245} - \frac{27y^3}{343} \\
 & \frac{64x^6}{15625} - \frac{576x^5y}{21875} + \frac{432x^4y^2}{6125} - \frac{864x^3y^3}{8575} + \frac{972x^2y^4}{12005} - \frac{2916xy^5}{84035} + \frac{729y^6}{117649} \\
 & \frac{64x^6}{15625} \\
\frac{16x^3}{125} - \frac{36x^2y}{175} & - \frac{576x^5y}{21875} + \frac{432x^4y^2}{6125} \\
 & - \frac{576x^5y}{21875} + \frac{1296x^4y^2}{30625} \\
\frac{16x^3}{125} - \frac{72x^2y}{175} + \frac{54xy^2}{245} & \frac{864x^4y^2}{30625} - \frac{864x^3y^3}{8575} + \frac{972x^2y^4}{12005} \\
 & \frac{864x^4y^2}{30625} - \frac{3888x^3y^3}{42875} + \frac{2916x^2y^4}{60025} \\
\frac{16x^3}{125} - \frac{72x^2y}{175} + \frac{108xy^2}{245} - \frac{27y^3}{343} & - \frac{432x^3y^3}{42875} + \frac{1944x^2y^4}{60025} - \frac{2916xy^5}{84035} + \frac{729y^6}{117649} \\
 & - \frac{432x^3y^3}{42875} + \frac{1944x^2y^4}{60025} - \frac{2916xy^5}{84035} + \frac{729y^6}{117649}
\end{array}$$

$$\begin{array}{l|l}
 & \frac{2x}{5} - \frac{3y}{7} \\
 & \frac{8x^3}{125} - \frac{36x^2y}{175} + \frac{54xy^2}{245} - \frac{27y^3}{343} \\
 & \frac{8x^3}{125} \\
\frac{12x^2}{25} - \frac{18xy}{35} + \frac{9y^2}{49} & - \frac{36x^2y}{175} + \frac{54xy^2}{245} - \frac{27y^3}{343} \\
 & - \frac{36x^2y}{175} + \frac{54xy^2}{245} - \frac{27y^3}{343}
\end{array}$$

Therefore, the required sixth root is $\frac{2x}{5} - \frac{3y}{7}$.

5. Find the eighth root of $6561x^{16}+17.496x^{14}+20,412x^{12}+13.608x^{10}+5670x^8+1512x^6+252x^4+24x^2+1$.

The eighth root is the square root of the square root of the square root.

$$\begin{array}{r|l}
 & 6561x^{16}+17496x^{14}+20412x^{12}+13608x^{10}+5670x^8+1512x^6+252x^4+24x^2+1\;\underline{|\,81x^8+108x^6+54x^4+12x^2+1}\\
 & 6561x^{16}\\
\hline
162x^8+108x^6 & 17496x^{14}+20412x^{12}\\
 & 17496x^{14}+11664x^{12}\\
\hline
162x^8+216x^6+54x^4 & 8748x^{12}+13608x^{10}+5670x^8\\
 & 8748x^{12}+11664x^{10}+2916x^8\\
\hline
162x^8+216x^6+108x^4+12x^2 & 1944x^{10}+2754x^8+1512x^6+252x^4\\
 & 1944x^{10}+2592x^8+1296x^6+144x^4\\
\hline
162x^8+216x^6+108x^4+24x^2+1 & 162x^8+216x^6+108x^4+24x^2+1\\
 & 162x^8+216x^6+108x^4+24x^2+1\\
\hline
\end{array}$$

$$\begin{array}{r|l}
 & 81x^8+108x^6+54x^4+12x^2+1\;\underline{|\,9x^4+6x^2+1}\\
 & 81x^8\\
\hline
18x^4+6x^2 & 108x^6+54x^4\\
 & 108x^6+36x^4\\
\hline
18x^4+12x^2+1 & 18x^4+12x^2+1\\
 & 18x^4+12x^2+1\\
\hline
\end{array}
\qquad
\begin{array}{r|l}
 & 9x^4+6x^2+1\;\underline{|\,3x^2+1}\\
 & 9x^4\\
\hline
6x^2+1 & 6x^2+1\\
 & 6x^2+1\\
\hline
\end{array}$$

Therefore, the required eighth root is $3x^2+1$.

6. Find the ninth root of $a^{27}-9a^{24}y^2+36a^{21}y^4-84a^{18}y^6+126a^{15}y^8-126a^{12}y^{10}+84a^9y^{12}-36a^6y^{14}+9a^3y^{16}-y^{18}$.

The ninth root is the cube root of the cube root.

$$
\begin{array}{rl|l}
 & & \underline{a^9-3a^6y^2+3a^3y^4-y^6} \\
 & & a^{27}-9a^{24}y^2+36a^{21}y^4-84a^{18}y^6+126a^{15}y^8-126a^{12}y^{10}+84a^9y^{12}-36a^6y^{14}+9a^3y^{16}-y^{18} \\
 & & a^{27} \\
3(a^9)^2= & 3a^{18} & -9a^{24}y^2+36a^{21}y^4-84a^{18}y^6 \\
(3a^9-3a^6y^2)(-3a^6y^2)= & \underline{\quad -9a^{15}y^2+9a^{12}y^4} & \\
 & 3a^{18}-9a^{15}y^2+9a^{12}y^4 & \underline{-9a^{24}y^2+27a^{21}y^4-27a^{18}y^6} \\
3(a^9-3a^6y^2)^2= & 3a^{18}-18a^{15}y^2+27a^{12}y^4 & 9a^{21}y^4-57a^{18}y^6+126a^{15}y^8-126a^{12}y^{10}+84a^9y^{12} \\
(3a^9-9a^6y^2+3a^3y^4)(3a^3y^4)= & \underline{\quad +9a^{12}y^4-27a^9y^6+9a^6y^8} & \\
 & 3a^{18}-18a^{15}y^2+36a^{12}y^4-27a^9y^6+9a^6y^8 & \underline{9a^{21}y^4-54a^{18}y^6+108a^{15}y^8-81a^{12}y^{10}+27a^9y^{12}} \\
3(a^9-3a^6y^2+3a^3y^4)^2= & 3a^{18}-18a^{15}y^2+45a^{12}y^4-54a^9y^6+27a^6y^8 & -3a^{18}y^6+18a^{15}y^8-45a^{12}y^{10}+57a^9y^{12}-36a^6y^{14}+9a^3y^{16}-y^{18} \\
(3a^9-9a^6y^2+9a^3y^4-y^6)(-y^6)= & \underline{\quad -3a^9y^6+9a^6y^8-9a^3y^{10}+y^{12}} & \\
 & 3a^{18}-18a^{15}y^2+45a^{12}y^4-57a^9y^6+36a^6y^8-9a^3y^{10}+y^{12} & \underline{-3a^{18}y^6+18a^{15}y^8-45a^{12}y^{10}+57a^9y^{12}-36a^6y^{14}+9a^3y^{16}-y^{18}}
\end{array}
$$

$$
\begin{array}{rl|l}
 & & a^9-3a^6y^2+3a^3y^4-y^6 \;\underline{|\,a^3-y^2} \\
 & & a^9 \\
3(a^3)^2= & 3a^6 & -3a^6y^2+3a^3y^4-y^6 \\
(3a^3-y^2)(-y^2)= & \underline{\quad -3a^3y^2+y^4} & \\
 & 3a^6-3a^3y^2+y^4 & \underline{-3a^6y^2+3a^3y^4-y^6}
\end{array}
$$

Therefore, the required ninth root is a^3-y^2.

Exercise 93. Page 237

Express with positive exponents:

1. $a^{-2}b^{-3}c^2 = \dfrac{c^2}{a^2b^3}$.

2. $x^{-\frac{2}{3}}y^{-\frac{1}{3}}z^{-2} = \dfrac{1}{x^{\frac{2}{3}}y^{\frac{1}{3}}z^2}$.

3. $a^{\frac{1}{3}}b^{-\frac{1}{3}}c^{-\frac{2}{3}} = \dfrac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}c^{\frac{2}{3}}}$.

4. $8\,a^{-4}b^{\frac{3}{2}}c^{-\frac{1}{2}} = \dfrac{8\,b^{\frac{3}{2}}}{a^4c^{\frac{1}{2}}}$.

5. $4^{-1}a^{-3}c^{-2} = \dfrac{1}{4\,a^3c^2}$.

6. $2\,axy^{-5} = \dfrac{2\,ax}{y^5}$.

7. $\dfrac{a^{-4}b^{-5}c^{-6}}{a^{-2}b^{-7}c^{-5}} = \dfrac{b^2}{a^2c}$.

8. $\dfrac{3\,a^{-1}b^{-2}c^{-3}}{5\,x^{-2}a^{-3}y^{-5}} = \dfrac{3\,a^2x^2y^5}{5\,b^2c^3}$.

Write without denominators:

9. $\dfrac{3\,x^2y^3}{a^{-2}b^{-3}} = 3\,a^2b^3x^2y^3$.

10. $\dfrac{5\,a^2b^5c^3}{a^{-1}b^{-2}c^{-3}} = 5\,a^3b^7c^6$.

11. $\dfrac{a^{-1}b^{-2}c^{-3}}{a^{-2}b^{-2}c^{-2}} = ac^{-1}$.

Express with fractional exponents:

12. $\sqrt[4]{\dfrac{1}{a^3}} = \dfrac{1}{\sqrt[4]{a^3}} = \dfrac{1}{a^{\frac{3}{4}}}$.

13. $\sqrt[3]{(x-a^2)^2} = (x-a^2)^{\frac{2}{3}}$.

14. $\sqrt{\left(\dfrac{a^2-b^2}{a^2+b^2}\right)^3} = \left(\dfrac{a^2-b^2}{a^2+b^2}\right)^{\frac{3}{2}}$.

Express with radical signs:

15. $a^{-\frac{1}{3}}x^{-3} = \dfrac{1}{a^{\frac{1}{3}}x^3} = \dfrac{1}{x^3\sqrt[3]{a}}$.

16. $b^{-\frac{2}{3}}c^{\frac{3}{4}}d^{-\frac{3}{10}} = \dfrac{c^{\frac{3}{4}}}{b^{\frac{2}{3}}d^{\frac{3}{10}}} = \dfrac{\sqrt[4]{c^3}}{\sqrt[3]{b^2}\sqrt[10]{d^3}}$.

17. $a^{-2}b^{\frac{2}{3}}c^{-3} = \dfrac{b^{\frac{2}{3}}}{a^2c^3} = \dfrac{\sqrt[3]{b^2}}{a^2c^3}$.

Perform the indicated operations:

18. $\left(\dfrac{25}{36}\right)^{-\frac{1}{2}} = \dfrac{1}{(\frac{25}{36})^{\frac{1}{2}}} = \dfrac{1}{\frac{5}{6}} = \dfrac{6}{5}$.

19. $\left(\dfrac{8}{27}\right)^{-\frac{1}{3}} = \dfrac{8^{-\frac{1}{3}}}{27^{-\frac{1}{3}}} = \dfrac{27^{\frac{1}{3}}}{8^{\frac{1}{3}}} = \dfrac{3}{2}$.

20. $(3\frac{3}{8})^{-\frac{2}{3}} = \left(\dfrac{27}{8}\right)^{-\frac{2}{3}} = \dfrac{27^{-\frac{2}{3}}}{8^{-\frac{2}{3}}} = \dfrac{8^{\frac{2}{3}}}{27^{\frac{2}{3}}} = \dfrac{2^2}{3^2} = \dfrac{4}{9}$.

21. $81^{-\frac{1}{2}} = \dfrac{1}{81^{\frac{1}{2}}} = \dfrac{1}{9}$.

22. $16^{\frac{5}{4}} = (2^4)^{\frac{5}{4}} = 2^5 = 32$.

23. $(-0.064)^{\frac{2}{3}} = [(-0.4)^3]^{\frac{2}{3}} = (-0.4)^2 = 0.16$.

24. $(7\tfrac{19}{32})^{\frac{3}{5}}=\left(\frac{243}{32}\right)^{\frac{3}{5}}=\left(\frac{3^5}{2^5}\right)^{\frac{3}{5}}=\frac{3^3}{2^3}=\frac{27}{8}.$

25. $\left(-\frac{32}{243}\right)^{-\frac{1}{5}}=\frac{1}{\left(-\frac{32}{243}\right)^{\frac{1}{5}}}=\frac{1}{-\frac{32^{\frac{1}{5}}}{243^{\frac{1}{5}}}}=\frac{1}{-\frac{2}{3}}=-\frac{3}{2}.$

26. $(-32)^{\frac{3}{5}}=[(-2)^5]^{\frac{3}{5}}=(-2)^3=-8.$

27. $81^{\frac{3}{4}}=(3^4)^{\frac{3}{4}}=3^3=27.$

28. $(5\tfrac{1}{16})^{-\frac{3}{4}}=\left(\frac{81}{16}\right)^{-\frac{3}{4}}=\frac{1}{\left(\frac{81}{16}\right)^{\frac{3}{4}}}=\frac{1}{\frac{81^{\frac{3}{4}}}{16^{\frac{3}{4}}}}=\frac{16^{\frac{3}{4}}}{81^{\frac{3}{4}}}=\frac{(2^4)^{\frac{3}{4}}}{(3^4)^{\frac{3}{4}}}=\frac{2^3}{3^3}=\frac{8}{27}.$

29. $\left(\frac{16}{81}\right)^{\frac{3}{4}}=\frac{16^{\frac{3}{4}}}{81^{\frac{3}{4}}}=\frac{(2^4)^{\frac{3}{4}}}{(3^4)^{\frac{3}{4}}}=\frac{2^3}{3^3}=\frac{8}{27}.$

30. $\left(\frac{9}{16}\right)^{-\frac{3}{2}}=\frac{1}{\left(\frac{9}{16}\right)^{\frac{3}{2}}}=\frac{1}{\frac{9^{\frac{3}{2}}}{16^{\frac{3}{2}}}}=\frac{16^{\frac{3}{2}}}{9^{\frac{3}{2}}}=\frac{(4^2)^{\frac{3}{2}}}{(3^2)^{\frac{3}{2}}}=\frac{4^3}{3^3}=\frac{64}{27}.$

31. $(0.16)^{-\frac{1}{2}}=\frac{1}{0.16^{\frac{1}{2}}}=\frac{1}{0.4}=2.5.$

32. $\left(\frac{1}{64}\right)^{-\frac{2}{3}}=\frac{1^{-\frac{2}{3}}}{64^{-\frac{2}{3}}}=\frac{64^{\frac{2}{3}}}{1^{\frac{2}{3}}}=\frac{(4^3)^{\frac{2}{3}}}{1}=4^2=16.$

33. $(-3\tfrac{3}{8})^{-\frac{1}{3}}=\left(-\frac{27}{8}\right)^{-\frac{1}{3}}=\left(-\frac{8}{27}\right)^{\frac{1}{3}}=-\frac{2}{3}.$

34. $64^{\frac{2}{3}}\times 27^{\frac{4}{3}}=(4^3)^{\frac{2}{3}}\times(3^3)^{\frac{4}{3}}=4^2\times 3^4=16\times 81=1296.$

35. $32^{\frac{1}{5}}\times 81^{\frac{3}{4}}=(2^5)^{\frac{1}{5}}\times(3^4)^{\frac{3}{4}}=2\times 3^3=2\times 27=54.$

36. $\left(\frac{1}{4}\right)^{\frac{1}{2}}\times\left(\frac{1}{8}\right)^{-\frac{1}{3}}=\frac{1}{4^{\frac{1}{2}}}\times\frac{1}{8^{-\frac{1}{3}}}=\frac{1}{2}\times 8^{\frac{1}{3}}=\frac{1}{2}\times 2=1.$

37. $\left(\frac{8}{27}\right)^{\frac{1}{3}}\times 64^{-\frac{1}{6}}=\frac{8^{\frac{1}{3}}}{27^{\frac{1}{3}}}\times\frac{1}{64^{\frac{1}{6}}}=\frac{2}{3}\times\frac{1}{2}=\frac{1}{3}.$

38. $32^{\frac{4}{5}}\times 81^{\frac{5}{4}}=(2^5)^{\frac{4}{5}}\times(3^4)^{\frac{5}{4}}=2^4\times 3^5=16\times 243=3888.$

39. $(a^{-\frac{1}{3}}b^{-\frac{2}{3}}c^{-1})^{-\frac{1}{3}}=a^{\frac{1}{9}}b^{\frac{2}{9}}c^{\frac{1}{3}}.$

40. $(x^{-\frac{3}{4}}y^{-\frac{1}{2}}z^{-\frac{1}{4}})^{\frac{1}{4}}=x^{-\frac{3}{16}}y^{-\frac{1}{8}}z^{-\frac{1}{16}}=\frac{1}{x^{\frac{3}{16}}y^{\frac{1}{8}}z^{\frac{1}{16}}}.$

41. $(a^{\frac{3}{2}}b^{\frac{2}{3}}c^{-\frac{1}{3}})^{-\frac{2}{3}}=a^{-1}b^{-\frac{4}{9}}c^{\frac{2}{9}}=\frac{c^{\frac{2}{9}}}{ab^{\frac{4}{9}}}.$

Exercise 94. Page 239

1. Multiply $8x^{\frac{3}{2}} - 4xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $2x^{\frac{1}{2}} - 3y^{\frac{1}{2}}$.

$$\begin{array}{l}
8x^{\frac{3}{2}} - 4xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y - y^{\frac{3}{2}} \\
2x^{\frac{1}{2}} - 3y^{\frac{1}{2}} \\
\hline
16x^2 - 8x^{\frac{3}{2}}y^{\frac{1}{2}} + 4xy - 2x^{\frac{1}{2}}y^{\frac{3}{2}} \\
\qquad - 24x^{\frac{3}{2}}y^{\frac{1}{2}} + 12xy - 6x^{\frac{1}{2}}y^{\frac{3}{2}} + 3y^2 \\
\hline
16x^2 - 32x^{\frac{3}{2}}y^{\frac{1}{2}} + 16xy - 8x^{\frac{1}{2}}y^{\frac{3}{2}} + 3y^2
\end{array}$$

2. Multiply $x - 6a^{\frac{1}{3}}x^{\frac{2}{3}} + 12a^{\frac{2}{3}}x^{\frac{1}{3}} - 8a$ by $x^{\frac{2}{3}} - 4a^{\frac{1}{3}}x^{\frac{1}{3}} + 4a^{\frac{2}{3}}$.

$$\begin{array}{l}
x - 6a^{\frac{1}{3}}x^{\frac{2}{3}} + 12a^{\frac{2}{3}}x^{\frac{1}{3}} - 8a \\
x^{\frac{2}{3}} - 4a^{\frac{1}{3}}x^{\frac{1}{3}} + 4a^{\frac{2}{3}} \\
\hline
x^{\frac{5}{3}} - 6a^{\frac{1}{3}}x^{\frac{4}{3}} + 12a^{\frac{2}{3}}x - 8ax^{\frac{2}{3}} \\
\qquad - 4a^{\frac{1}{3}}x^{\frac{4}{3}} + 24a^{\frac{2}{3}}x - 48ax^{\frac{2}{3}} + 32a^{\frac{4}{3}}x^{\frac{1}{3}} \\
\qquad\qquad + 4a^{\frac{2}{3}}x - 24ax^{\frac{2}{3}} + 48a^{\frac{4}{3}}x^{\frac{1}{3}} - 32a^{\frac{5}{3}} \\
\hline
x^{\frac{5}{3}} - 10a^{\frac{1}{3}}x^{\frac{4}{3}} + 40a^{\frac{2}{3}}x - 80ax^{\frac{2}{3}} + 80a^{\frac{4}{3}}x^{\frac{1}{3}} - 32a^{\frac{5}{3}}
\end{array}$$

3. Multiply $7a - 2 + 8a^{-1} - 4a^{-2}$ by $5a + 3 - 7a^{-1}$.

$$\begin{array}{l}
7a - 2 + 8a^{-1} - 4a^{-2} \\
5a + 3 - 7a^{-1} \\
\hline
35a^2 - 10a + 40 - 20a^{-1} \\
\qquad + 21a - 6 + 24a^{-1} - 12a^{-2} \\
\qquad\qquad - 49 + 14a^{-1} - 56a^{-2} + 28a^{-3} \\
\hline
35a^2 + 11a - 15 + 18a^{-1} - 68a^{-2} + 28a^{-3}
\end{array}$$

4. Multiply $x^{\frac{1}{2}} - 4y^{\frac{1}{2}} + 6x^{-\frac{1}{2}}y - 4x^{-1}y^{\frac{3}{2}} + x^{-\frac{3}{2}}y^2$ by $x^{\frac{1}{2}}y^{-\frac{1}{2}} + 2 + x^{-\frac{1}{2}}y^{\frac{1}{2}}$.

$$\begin{array}{l}
x^{\frac{1}{2}} - 4y^{\frac{1}{2}} + 6x^{-\frac{1}{2}}y - 4x^{-1}y^{\frac{3}{2}} + x^{-\frac{3}{2}}y^2 \\
x^{\frac{1}{2}}y^{-\frac{1}{2}} + 2 + x^{-\frac{1}{2}}y^{\frac{1}{2}} \\
\hline
xy^{-\frac{1}{2}} - 4x^{\frac{1}{2}} + 6y^{\frac{1}{2}} - 4x^{-\frac{1}{2}}y + x^{-1}y^{\frac{3}{2}} \\
\qquad + 2x^{\frac{1}{2}} - 8y^{\frac{1}{2}} + 12x^{-\frac{1}{2}}y - 8x^{-1}y^{\frac{3}{2}} + 2x^{-\frac{3}{2}}y^2 \\
\qquad\qquad + y^{\frac{1}{2}} - 4x^{-\frac{1}{2}}y + 6x^{-1}y^{\frac{3}{2}} - 4x^{-\frac{3}{2}}y^2 + x^{-2}y^{\frac{5}{2}} \\
\hline
xy^{-\frac{1}{2}} - 2x^{\frac{1}{2}} - y^{\frac{1}{2}} + 4x^{-\frac{1}{2}}y - x^{-1}y^{\frac{3}{2}} - 2x^{-\frac{3}{2}}y^2 + x^{-2}y^{\frac{5}{2}}
\end{array}$$

5. Multiply $2x - 4x^{-1} - 4x^{-2} - x^{-3}$ by $2x - 4x^{-1} - 4x^{-2} - x^{-3}$.

$$\begin{array}{rrrrrrrr}
2x & -\ 4x^{-1} & -\ 4x^{-2} & -\ x^{-3} & & & & \\
2x & -\ 4x^{-1} & -\ 4x^{-2} & -\ x^{-3} & & & & \\
\hline
4x^2 & -\ 8 & -\ 8x^{-1} & -\ 2x^{-2} & & & & \\
 & -\ 8 & & +\ 16x^{-2} & +\ 16x^{-3} & +\ 4x^{-4} & & \\
 & & -\ 8x^{-1} & & +\ 16x^{-3} & +\ 16x^{-4} & +\ 4x^{-5} & \\
 & & & -\ 2x^{-2} & & +\ 4x^{-4} & +\ 4x^{-5} & +\ x^{-6} \\
\hline
4x^2 & -\ 16 & -\ 16x^{-1} & +\ 12x^{-2} & +\ 32x^{-3} & +\ 24x^{-4} & +\ 8x^{-5} & +\ x^{-6}
\end{array}$$

6. Multiply $x^{\frac{3}{2}} - 6a^{\frac{1}{2}}x + 12ax^{\frac{1}{2}} - 8a^{\frac{3}{2}}$ by $x - 4a^{\frac{1}{2}}x^{\frac{1}{2}} - 4a$.

$$\begin{array}{llllll}
x^{\frac{3}{2}} - 6a^{\frac{1}{2}}x + 12ax^{\frac{1}{2}} - 8a^{\frac{3}{2}} & & & & & \\
x - 4a^{\frac{1}{2}}x^{\frac{1}{2}} - 4a & & & & & \\
\hline
x^{\frac{5}{2}} & -\ 6a^{\frac{1}{2}}x^2 & +\ 12ax^{\frac{3}{2}} & -\ 8a^{\frac{3}{2}}x & & \\
 & -\ 4a^{\frac{1}{2}}x^2 & +\ 24ax^{\frac{3}{2}} & -\ 48a^{\frac{3}{2}}x & +\ 32a^2x^{\frac{1}{2}} & \\
 & & -\ 4ax^{\frac{3}{2}} & +\ 24a^{\frac{3}{2}}x & -\ 48a^2x^{\frac{1}{2}} & +\ 32a^{\frac{5}{2}} \\
\hline
x^{\frac{5}{2}} & -\ 10a^{\frac{1}{2}}x^2 & +\ 32ax^{\frac{3}{2}} & -\ 32a^{\frac{3}{2}}x & -\ 16a^2x^{\frac{1}{2}} & +\ 32a^{\frac{5}{2}}
\end{array}$$

7. Divide $a^2 - 3a^{\frac{5}{3}} + 6a^{\frac{4}{3}} - 7a + 6a^{\frac{2}{3}} - 3a^{\frac{1}{3}} + 1$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1$.

$$\begin{array}{l|l}
a^2 - 3a^{\frac{5}{3}} + 6a^{\frac{4}{3}} - 7a + 6a^{\frac{2}{3}} - 3a^{\frac{1}{3}} + 1 & a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 \\
\cline{2-2}
a^2 - a^{\frac{5}{3}} + a^{\frac{4}{3}} & a^{\frac{4}{3}} - 2a + 3a^{\frac{2}{3}} - 2a^{\frac{1}{3}} + 1 \\
\cline{1-1}
\quad - 2a^{\frac{5}{3}} + 5a^{\frac{4}{3}} - 7a & \\
\quad - 2a^{\frac{5}{3}} + 2a^{\frac{4}{3}} - 2a & \\
\cline{1-1}
\qquad + 3a^{\frac{4}{3}} - 5a + 6a^{\frac{2}{3}} & \\
\qquad + 3a^{\frac{4}{3}} - 3a + 3a^{\frac{2}{3}} & \\
\cline{1-1}
\qquad\quad - 2a + 3a^{\frac{2}{3}} - 3a^{\frac{1}{3}} & \\
\qquad\quad - 2a + 2a^{\frac{2}{3}} - 2a^{\frac{1}{3}} & \\
\cline{1-1}
\qquad\qquad + a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 & \\
\qquad\qquad + a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 & \\
\cline{1-1}
\end{array}$$

8. Divide $\frac{9}{16}a^2 - \frac{7}{8}a^{\frac{3}{2}}b^{\frac{1}{2}} + \frac{19}{36}ab + \frac{1}{6}a^{\frac{1}{2}}b^{\frac{3}{2}}$ by $\frac{3}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{2}}$.

$$\begin{array}{l|l}
\frac{9}{16}a^2 - \frac{7}{8}a^{\frac{3}{2}}b^{\frac{1}{2}} + \frac{19}{36}ab + \frac{1}{6}a^{\frac{1}{2}}b^{\frac{3}{2}} & \frac{3}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{2}} \\ \cline{2-2}
\underline{\frac{9}{16}a^2 + \frac{1}{8}a^{\frac{3}{2}}b^{\frac{1}{2}}\qquad\qquad} & \frac{3}{8}a^{\frac{3}{2}} - \frac{2}{3}ab^{\frac{1}{2}} + \frac{1}{2}a^{\frac{1}{2}}b \\
\multicolumn{1}{l}{\qquad - a^{\frac{3}{2}}b^{\frac{1}{2}} + \frac{19}{36}ab} & \\
\multicolumn{1}{l}{\qquad \underline{- a^{\frac{3}{2}}b^{\frac{1}{2}} - \frac{2}{9}ab\qquad\qquad}} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad + \frac{3}{4}ab + \frac{1}{6}a^{\frac{1}{2}}b^{\frac{3}{2}}} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad \underline{+ \frac{3}{4}ab + \frac{1}{6}a^{\frac{1}{2}}b^{\frac{3}{2}}}} &
\end{array}$$

9. Divide $x^{\frac{7}{3}} - 3x^2 + x^{\frac{5}{3}} - 4 + 12x^{\frac{1}{3}} - 4x^{\frac{2}{3}}$ by $x^{\frac{5}{3}} - 4$.

Arrange in descending powers of x.

$$\begin{array}{l|l}
x^{\frac{7}{3}} - 3x^2 + x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 12x^{\frac{1}{3}} - 4 & x^{\frac{5}{3}} - 4 \\ \cline{2-2}
\underline{x^{\frac{7}{3}} \qquad\qquad\quad - 4x^{\frac{2}{3}} \qquad\qquad\quad} & x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 1 \\
\multicolumn{1}{l}{\quad - 3x^2 + x^{\frac{5}{3}} \qquad\quad + 12x^{\frac{1}{3}} - 4} & \\
\multicolumn{1}{l}{\quad \underline{- 3x^2 \qquad\qquad\quad + 12x^{\frac{1}{3}}\quad\;\;}} & \\
\multicolumn{1}{l}{\qquad\qquad + x^{\frac{5}{3}} \qquad\qquad\qquad - 4} & \\
\multicolumn{1}{l}{\qquad\qquad \underline{+ x^{\frac{5}{3}} \qquad\qquad\qquad - 4}} &
\end{array}$$

10. Divide $6x - 13a^{\frac{1}{4}}x^{\frac{3}{4}} + 13a^{\frac{1}{2}}x^{\frac{1}{2}} - 13a^{\frac{3}{4}}x^{\frac{1}{4}} - 5a$ by $2x^{\frac{1}{2}} - 3a^{\frac{1}{4}}x^{\frac{1}{4}} - a^{\frac{1}{2}}$.

$$\begin{array}{l|l}
6x - 13a^{\frac{1}{4}}x^{\frac{3}{4}} + 13a^{\frac{1}{2}}x^{\frac{1}{2}} - 13a^{\frac{3}{4}}x^{\frac{1}{4}} - 5a & 2x^{\frac{1}{2}} - 3a^{\frac{1}{4}}x^{\frac{1}{4}} - a^{\frac{1}{2}} \\ \cline{2-2}
\underline{6x - \;\;9a^{\frac{1}{4}}x^{\frac{3}{4}} - \;\;3a^{\frac{1}{2}}x^{\frac{1}{2}}\qquad\qquad} & 3x^{\frac{1}{2}} - 2a^{\frac{1}{4}}x^{\frac{1}{4}} + 5a^{\frac{1}{2}} \\
\multicolumn{1}{l}{\quad - \;\;4a^{\frac{1}{4}}x^{\frac{3}{4}} + 16a^{\frac{1}{2}}x^{\frac{1}{2}} - 13a^{\frac{3}{4}}x^{\frac{1}{4}}} & \\
\multicolumn{1}{l}{\quad \underline{- \;\;4a^{\frac{1}{4}}x^{\frac{3}{4}} + \;\;6a^{\frac{1}{2}}x^{\frac{1}{2}} + \;\;2a^{\frac{3}{4}}x^{\frac{1}{4}}\qquad\;\;}} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad + 10a^{\frac{1}{2}}x^{\frac{1}{2}} - 15a^{\frac{3}{4}}x^{\frac{1}{4}} - 5a} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad \underline{+ 10a^{\frac{1}{2}}x^{\frac{1}{2}} - 15a^{\frac{3}{4}}x^{\frac{1}{4}} - 5a}} &
\end{array}$$

11. Divide $62x^{-5}y^{-3} + 43x^{-7}y^{-5} - 55x^{-6}y^{-4} - 55x^{-4}y^{-2} - 6x^{-2} - 21x^{-8}y^{-6} + 28x^{-3}y^{-1}$ by $4x^{-2}y^{-1} - 2x^{-1} - 3x^{-3}y^{-2}$.

Arrange both dividend and divisor in descending powers of x.

$$\begin{array}{l}
\underline{|\,-2x^{-1} + 4x^{-2}y^{-1} - 3x^{-3}y^{-2}} \\
\quad 3x^{-1} - 8x^{-2}y^{-1} + 7x^{-3}y^{-2} - 5x^{-4}y^{-3} + 7x^{-5}y^{-4}
\end{array}$$

$$\begin{array}{l}
-6x^{-2} + 28x^{-3}y^{-1} - 55x^{-4}y^{-2} + 62x^{-5}y^{-3} - 55x^{-6}y^{-4} + 43x^{-7}y^{-5} - 21x^{-8}y^{-6} \\
\underline{-6x^{-2} + 12x^{-3}y^{-1} - 9x^{-4}y^{-2}} \\
\qquad + 16x^{-3}y^{-1} - 46x^{-4}y^{-2} + 62x^{-5}y^{-3} \\
\qquad \underline{+ 16x^{-3}y^{-1} - 32x^{-4}y^{-2} + 24x^{-5}y^{-3}} \\
\qquad\qquad - 14x^{-4}y^{-2} + 38x^{-5}y^{-3} - 55x^{-6}y^{-4} \\
\qquad\qquad \underline{- 14x^{-4}y^{-2} + 28x^{-5}y^{-3} - 21x^{-6}y^{-4}} \\
\qquad\qquad\qquad + 10x^{-5}y^{-3} - 34x^{-6}y^{-4} + 43x^{-7}y^{-5} \\
\qquad\qquad\qquad \underline{+ 10x^{-5}y^{-3} - 20x^{-6}y^{-4} + 15x^{-7}y^{-5}} \\
\qquad\qquad\qquad\qquad - 14x^{-6}y^{-4} + 28x^{-7}y^{-5} - 21x^{-8}y^{-6} \\
\qquad\qquad\qquad\qquad \underline{- 14x^{-6}y^{-4} + 28x^{-7}y^{-5} - 21x^{-8}y^{-6}}
\end{array}$$

12. Find the square root of $a^{\frac{4}{3}}b^{-1} + 4\,ab^{-\frac{1}{2}} - 2\,a^{\frac{2}{3}} - 12\,a^{\frac{1}{3}}b^{\frac{1}{2}} + 9\,b$.

$$\begin{array}{r|l}
 & a^{\frac{4}{3}}b^{-1} + 4\,ab^{-\frac{1}{2}} - 2\,a^{\frac{2}{3}} - 12\,a^{\frac{1}{3}}b^{\frac{1}{2}} + 9\,b \;\underline{|\,a^{\frac{2}{3}}b^{-\frac{1}{2}} + 2\,a^{\frac{1}{3}} - 3\,b^{\frac{1}{2}}} \\
 & \underline{a^{\frac{4}{3}}b^{-1}} \\
2\,a^{\frac{2}{3}}b^{-\frac{1}{2}} + 2\,a^{\frac{1}{3}} & 4\,ab^{-\frac{1}{2}} - 2\,a^{\frac{2}{3}} \\
 & \underline{4\,ab^{-\frac{1}{2}} + 4\,a^{\frac{2}{3}}} \\
2\,a^{\frac{2}{3}}b^{-\frac{1}{2}} + 4\,a^{\frac{1}{3}} - 3\,b^{\frac{1}{2}} & -6\,a^{\frac{2}{3}} - 12\,a^{\frac{1}{3}}b^{\frac{1}{2}} + 9\,b \\
 & \underline{-6\,a^{\frac{2}{3}} - 12\,a^{\frac{1}{3}}b^{\frac{1}{2}} + 9\,b}
\end{array}$$

13. Find the square root of

$$60\,mn^{\frac{5}{4}} - 4\,m^{\frac{4}{3}}n^{\frac{3}{2}} - 48\,m^{\frac{5}{3}}n^{\frac{7}{4}} + 16\,m^2n^2 + 25\,m^{\frac{2}{3}}n.$$

Arrange in descending powers of m.

$$\begin{array}{r|l}
 & \qquad\qquad\quad \underline{|\,4\,mn - 6\,m^{\frac{2}{3}}n^{\frac{3}{4}} - 5\,m^{\frac{1}{3}}n^{\frac{1}{2}}} \\
 & 16\,m^2n^2 - 48\,m^{\frac{5}{3}}n^{\frac{7}{4}} - 4\,m^{\frac{4}{3}}n^{\frac{3}{2}} + 60\,mn^{\frac{5}{4}} + 25\,m^{\frac{2}{3}}n \\
 & \underline{16\,m^2n^2} \\
8\,mn - 6\,m^{\frac{2}{3}}n^{\frac{3}{4}} & -48\,m^{\frac{5}{3}}n^{\frac{7}{4}} - 4\,m^{\frac{4}{3}}n^{\frac{3}{2}} \\
 & \underline{-48\,m^{\frac{5}{3}}n^{\frac{7}{4}} + 36\,m^{\frac{4}{3}}n^{\frac{3}{2}}} \\
8\,mn - 12\,m^{\frac{2}{3}}n^{\frac{3}{4}} - 5\,m^{\frac{1}{3}}n^{\frac{1}{2}} & -40\,m^{\frac{4}{3}}n^{\frac{3}{2}} + 60\,mn^{\frac{5}{4}} + 25\,m^{\frac{2}{3}}n \\
 & \underline{-40\,m^{\frac{4}{3}}n^{\frac{3}{2}} + 60\,mn^{\frac{5}{4}} + 25\,m^{\frac{2}{3}}n}
\end{array}$$

14. Find the square root of

$$9\,x^{-\frac{4}{3}} + 24\,x^{-1}y^{-\frac{1}{3}} + 46\,x^{-\frac{2}{3}}y^{-\frac{2}{3}} + 40\,x^{-\frac{1}{3}}y^{-1} + 25\,y^{-\frac{4}{3}}.$$

$$\begin{array}{r|l}
 & \qquad\qquad\quad \underline{|\,3\,x^{-\frac{2}{3}} + 4\,x^{-\frac{1}{3}}y^{-\frac{1}{3}} + 5\,y^{-\frac{2}{3}}} \\
 & 9\,x^{-\frac{4}{3}} + 24\,x^{-1}y^{-\frac{1}{3}} + 46\,x^{-\frac{2}{3}}y^{-\frac{2}{3}} + 40\,x^{-\frac{1}{3}}y^{-1} + 25\,y^{-\frac{4}{3}} \\
 & \underline{9\,x^{-\frac{4}{3}}} \\
6\,x^{-\frac{2}{3}} + 4\,x^{-\frac{1}{3}}y^{-\frac{1}{3}} & 24\,x^{-1}y^{-\frac{1}{3}} + 46\,x^{-\frac{2}{3}}y^{-\frac{2}{3}} \\
 & \underline{24\,x^{-1}y^{-\frac{1}{3}} + 16\,x^{-\frac{2}{3}}y^{-\frac{2}{3}}} \\
6\,x^{-\frac{2}{3}} + 8\,x^{-\frac{1}{3}}y^{-\frac{1}{3}} + 5\,y^{-\frac{2}{3}} & 30\,x^{-\frac{2}{3}}y^{-\frac{2}{3}} + 40\,x^{-\frac{1}{3}}y^{-1} + 25\,y^{-\frac{4}{3}} \\
 & \underline{30\,x^{-\frac{2}{3}}y^{-\frac{2}{3}} + 40\,x^{-\frac{1}{3}}y^{-1} + 25\,y^{-\frac{4}{3}}}
\end{array}$$

15. Find the square root of $4x^{-\frac{6}{5}} - 20x^{-1}y^{-1} + 41x^{-\frac{4}{5}}y^{-2} - 52x^{-\frac{3}{5}}y^{-3} + 46x^{-\frac{2}{5}}y^{-4} - 24x^{-\frac{1}{5}}y^{-5} + 9y^{-6}$.

$$
\begin{array}{r|l}
 & \underline{2x^{-\frac{3}{5}} - 5x^{-\frac{2}{5}}y^{-1} + 4x^{-\frac{1}{5}}y^{-2} - 3y^{-3}} \\
 & 4x^{-\frac{6}{5}} - 20x^{-1}y^{-1} + 41x^{-\frac{4}{5}}y^{-2} - 52x^{-\frac{3}{5}}y^{-3} + 46x^{-\frac{2}{5}}y^{-4} - 24x^{-\frac{1}{5}}y^{-5} + 9y^{-6} \\
 & \underline{4x^{-\frac{6}{5}}} \\
4x^{-\frac{3}{5}} - 5x^{-\frac{2}{5}}y^{-1} & -20x^{-1}y^{-1} + 41x^{-\frac{4}{5}}y^{-2} \\
 & \underline{-20x^{-1}y^{-1} + 25x^{-\frac{4}{5}}y^{-2}} \\
4x^{-\frac{3}{5}} - 10x^{-\frac{2}{5}}y^{-1} + 4x^{-\frac{1}{5}}y^{-2} & 16x^{-\frac{4}{5}}y^{-2} - 52x^{-\frac{3}{5}}y^{-3} + 46x^{-\frac{2}{5}}y^{-4} \\
 & \underline{16x^{-\frac{4}{5}}y^{-2} - 40x^{-\frac{3}{5}}y^{-3} + 16x^{-\frac{2}{5}}y^{-4}} \\
4x^{-\frac{3}{5}} - 10x^{-\frac{2}{5}}y^{-1} + 8x^{-\frac{1}{5}}y^{-2} - 3y^{-3} & -12x^{-\frac{3}{5}}y^{-3} + 30x^{-\frac{2}{5}}y^{-4} - 24x^{-\frac{1}{5}}y^{-5} + 9y^{-6} \\
 & \underline{-12x^{-\frac{3}{5}}y^{-3} + 30x^{-\frac{2}{5}}y^{-4} - 24x^{-\frac{1}{5}}y^{-5} + 9y^{-6}}
\end{array}
$$

Exercise 95. Page 242

Reduce to simplest form:

1. $\sqrt[4]{36\,x^2y^2} = \sqrt[4]{(6\,xy)^2} = (6\,xy)^{\frac{2}{4}} = (6\,xy)^{\frac{1}{2}} = \sqrt{6\,xy}.$

2. $\sqrt[4]{49\,a^2b^2} = \sqrt[4]{(7\,ab)^2} = (7\,ab)^{\frac{2}{4}} = (7\,ab)^{\frac{1}{2}} = \sqrt{7\,ab}.$

3. $\sqrt[6]{27\,a^3b^3} = \sqrt[6]{(3\,ab)^3} = (3\,ab)^{\frac{3}{6}} = (3\,ab)^{\frac{1}{2}} = \sqrt{3\,ab}.$

4. $\sqrt[6]{64\,a^3b^3} = \sqrt[6]{2^6\,(ab)^3} = 2^{\frac{6}{6}}(ab)^{\frac{3}{6}} = 2\,(ab)^{\frac{1}{2}} = 2\,\sqrt{ab}.$

5. $\sqrt[8]{16\,x^4y^4} = \sqrt[8]{(2\,xy)^4} = (2\,xy)^{\frac{4}{8}} = (2\,xy)^{\frac{1}{2}} = \sqrt{2\,xy}.$

6. $\sqrt{72\,a^3b^4} = \sqrt{36\,a^2b^4 \times 2\,a} = 6\,ab^2\,\sqrt{2\,a}.$

7. $\sqrt[3]{72\,a^3b^4} = \sqrt[3]{8\,a^3b^3 \times 9\,b} = 2\,ab\,\sqrt[3]{9\,b}.$

8. $4\,\sqrt[3]{27\,a^2b^3} = 4\,\sqrt[3]{27\,b^3 \times a^2} = 4 \times 3\,b\,\sqrt[3]{a^2} = 12\,b\,\sqrt[3]{a^2}.$

9. $\sqrt[4]{112\,x^5y^6} = \sqrt[4]{16\,x^4y^4 \times 7\,xy^2} = 2\,xy\,\sqrt[4]{7\,xy^2}.$

10. $2\,\sqrt[4]{81\,a^6b^2} = 2\,\sqrt[4]{81\,a^4 \times a^2b^2} = 2 \times 3\,a\,\sqrt[4]{(ab)^2} = 6\,a\,\sqrt{ab}.$

11. $5\,a\,\sqrt{4\,a^2b} = 5\,a\,\sqrt{4\,a^2 \times b} = 5\,a \times 2\,a\,\sqrt{b} = 10\,a^2\,\sqrt{b}.$

12. $2\,\sqrt[3]{56\,x^3y^2} = 2\,\sqrt[3]{8\,x^3 \times 7\,y^2} = 2 \times 2\,x\,\sqrt[3]{7\,y^2} = 4\,x\,\sqrt[3]{7\,y^2}.$

13. $\sqrt[4]{567\,c^4d} = \sqrt[4]{81\,c^4 \times 7\,d} = 3\,c\,\sqrt[4]{7\,d}.$

14. $2\,\sqrt{405\,y^4z^3} = 2\,\sqrt{81\,y^4z^2 \times 5\,z} = 2 \times 9\,y^2z\,\sqrt{5\,z} = 18\,y^2z\,\sqrt{5\,z}.$

15. $5\,\sqrt{80\,a^4b^3} = 5\,\sqrt{16\,a^4b^2 \times 5\,b} = 5 \times 4\,a^2b\,\sqrt{5\,b} = 20\,a^2b\,\sqrt{5\,b}.$

16. $5\,\sqrt[4]{144\,c^5d^9} = 5\,\sqrt[4]{16\,c^4d^8 \times 9\,cd} = 5 \times 2\,cd^2\,\sqrt[4]{9\,cd} = 10\,cd^2\,\sqrt[4]{9\,cd}.$

17. $\sqrt[5]{2048\,a^5b^7} = \sqrt[5]{2^{10}a^5b^5 \times 2\,b^2} = 4\,ab\,\sqrt[5]{2\,b^2}.$

18. $\sqrt[4]{1024\,a^5x^3} = \sqrt[4]{2^8\,a^4 \times 2^2\,ax^3} = 4\,a\,\sqrt[4]{4\,ax^3}.$

19. $2\sqrt[3]{7290\,a^5} = 2\sqrt[3]{27 \times 27\,a^3 \times 10\,a^2} = 2 \times 3 \times 3a\sqrt[3]{10\,a^2} = 18\,a\sqrt[3]{10\,a^2}.$

20. $3\,\sqrt[5]{160\,b^6y^7} = 3\,\sqrt[5]{32\,b^5y^5 \times 5\,by^2} = 3 \times 2\,by\,\sqrt[5]{5\,by^2} = 6\,by\,\sqrt[5]{5\,by^2}.$

21. $5\,\sqrt[3]{500\,a^4b^5} = 5\,\sqrt[3]{125\,a^3b^3 \times 4\,ab^2} = 5 \times 5\,ab\,\sqrt[3]{4\,ab^2} = 25\,ab\,\sqrt[3]{4\,ab^2}.$

22. $2\,ab\,\sqrt[3]{108\,ab^2} = 2\,ab\,\sqrt[3]{27 \times 4\,ab^2} = 2\,ab \times 3\,\sqrt[3]{4\,ab^2} = 6\,ab\,\sqrt[3]{4\,ab^2}.$

23. $3\,ab\,\sqrt[3]{192\,x^5y^7} = 3\,ab\,\sqrt[3]{64\,x^3y^6 \times 3\,x^2y} = 3\,ab \times 4\,xy^2\,\sqrt[3]{3\,x^2y}$
$= 12\,abxy^2\,\sqrt[3]{3\,x^2y}.$

24. $2\,\sqrt[3]{-1029\,a^4} = 2\,\sqrt[3]{-343\,a^3 \times 3\,a} = 2 \times (-7\,a)\sqrt[3]{3\,a} = -14\,a\,\sqrt[3]{3\,a}.$

25. $3\,a^2\,\sqrt[3]{-648\,a^7} = 3\,a^2\,\sqrt[3]{-216\,a^6 \times 3\,a} = 3\,a^2 \times (-6\,a^2)\sqrt[3]{3\,a}$
$= -18\,a^4\,\sqrt[3]{3\,a}.$

26. $2\sqrt[3]{\frac{3a^2b^4}{4x^4y}}=2\sqrt[3]{\frac{3a^2b^4\times 2x^2y^2}{4x^4y\times 2x^2y^2}}=2\sqrt[3]{\frac{6a^2b^4x^2y^2}{8x^6y^3}}=2\sqrt[3]{\frac{b^3}{8x^6y^3}\times 6a^2bx^2y^2}$
$=2\times\frac{b}{2x^2y}\sqrt[3]{6a^2bx^2y^2}=\frac{b}{x^2y}\sqrt[3]{6a^2bx^2y^2}.$

27. $2a\sqrt[3]{\frac{5a^3x^2}{18b^4}}=2a\sqrt[3]{\frac{5a^3x^2\times 12b^2}{18b^4\times 12b^2}}=2a\sqrt[3]{\frac{60a^3b^2x^2}{216b^6}}$
$=2a\sqrt[3]{\frac{a^3}{216b^6}\times 60b^2x^2}=2a\times\frac{a}{6b^2}\sqrt[3]{60b^2x^2}=\frac{a^2}{3b^2}\sqrt[3]{60b^2x^2}.$

28. $3c\sqrt[4]{\frac{9x^2y^5}{8c^4d^2}}=3c\sqrt[4]{\frac{9x^2y^5\times 2d^2}{8c^4d^2\times 2d^2}}=3c\sqrt[4]{\frac{18d^2x^2y^5}{16c^4d^4}}$
$=3c\sqrt[4]{\frac{y^4}{16c^4d^4}\times 18d^2x^2y}=3c\times\frac{y}{2cd}\sqrt[4]{18d^2x^2y}=\frac{3y}{2d}\sqrt[4]{18d^2x^2y}.$

29. $3m\sqrt[3]{\frac{125m^2}{81n^5}}=3m\sqrt[3]{\frac{125m^2\times 9n}{81n^5\times 9n}}=3m\sqrt[3]{\frac{125\times 9m^2n}{729n^6}}$
$=3m\sqrt[3]{\frac{125}{729n^6}\times 9m^2n}=3m\times\frac{5}{9n^2}\sqrt[3]{9m^2n}=\frac{5m}{3n^2}\sqrt[3]{9m^2n}.$

30. $2a^2\sqrt{\frac{49c^3d^4}{50a^4b}}=2a^2\sqrt{\frac{49c^3d^4\times 2b}{50a^4b\times 2b}}=2a^2\sqrt{\frac{49\times 2bc^3d^4}{100a^4b^2}}$
$=2a^2\sqrt{\frac{49c^2d^4}{100a^4b^2}\times 2bc}=2a^2\times\frac{7cd^2}{10a^2b}\sqrt{2bc}=\frac{7cd^2}{5b}\sqrt{2bc}.$

31. $3ab\sqrt[3]{\frac{a^2b^3}{5x^4}}=3ab\sqrt[3]{\frac{a^2b^3\times 25x^2}{5x^4\times 25x^2}}=3ab\sqrt[3]{\frac{25a^2b^3x^2}{125x^6}}$
$=3ab\sqrt[3]{\frac{b^3}{125x^6}\times 25a^2x^2}=3ab\times\frac{b}{5x^2}\sqrt[3]{25a^2x^2}=\frac{3ab^2}{5x^2}\sqrt[3]{25a^2x^2}.$

32. $2a\sqrt[5]{\frac{18x^5y^4}{3125a^7}}=2a\sqrt[5]{\frac{18x^5y^4\times a^3}{3125a^7\times a^3}}=2a\sqrt[5]{\frac{18a^3x^5y^4}{3125a^{10}}}$
$=2a\sqrt[5]{\frac{x^5}{3125a^{10}}\times 18a^3y^4}=2a\times\frac{x}{5a^2}\sqrt[5]{18a^3y^4}=\frac{2x}{5a}\sqrt[5]{18a^3y^4}.$

33. $2\sqrt[5]{-\frac{3a^7x^4}{512c^8d^9}}=2\sqrt[5]{-\frac{3a^7x^4\times 2c^2d}{512c^8d^9\times 2c^2d}}=2\sqrt[5]{-\frac{6a^7c^2dx^4}{1024c^{10}d^{10}}}$
$=2\sqrt[5]{-\frac{a^5}{1024c^{10}d^{10}}\times 6a^2c^2dx^4}=2\times\left(-\frac{a}{4c^2d^2}\right)\sqrt[5]{6a^2c^2dx^4}$
$=-\frac{a}{2c^2d^2}\sqrt[5]{6a^2c^2dx^4}.$

34. $\sqrt[3]{\dfrac{(x+y)^4z^3}{512}}=\sqrt[3]{\dfrac{(x+y)^3z^3}{512}\times(x+y)}=\dfrac{(x+y)z}{8}\sqrt[3]{x+y}.$

35. $3x^2y\sqrt[3]{\dfrac{9x^5z^4}{80y^7}}=3x^2y\sqrt[3]{\dfrac{9x^5z^4\times 100y^2}{80y^7\times 100y^2}}=3x^2y\sqrt[3]{\dfrac{900x^5y^2z^4}{8000y^9}}$

$=3x^2y\sqrt[3]{\dfrac{x^3z^3}{8000y^9}\times 900x^2y^2z}=3x^2y\times\dfrac{xz}{20y^3}\sqrt[3]{900x^2y^2z}$

$=\dfrac{3x^3z}{20y^2}\sqrt[3]{900x^2y^2z}.$

36. $3a\sqrt[6]{3645a^7b^6c^5}=3a\sqrt[6]{3^6\times 5a^7b^6c^5}=3a\sqrt[6]{3^6a^6b^6\times 5ac^5}$

$=3a\times 3ab\sqrt[6]{5ac^5}=9a^2b\sqrt[6]{5ac^5}$

37. $\sqrt[3]{\dfrac{(a-b)^5(c-d)^4}{(x-y)^2}}=\sqrt[3]{\dfrac{(a-b)^5(c-d)^4(x-y)}{(x-y)^3}}$

$=\sqrt[3]{\dfrac{(a-b)^3(c-d)^3}{(x-y)^3}\times(a-b)^2(c-d)(x-y)}$

$=\dfrac{(a-b)(c-d)}{x-y}\sqrt[3]{(a-b)^2(c-d)(x-y)}.$

38. $\sqrt[4]{\dfrac{(a-x)^5(b+y)}{243(a+b)^7}}=\sqrt[4]{\dfrac{(a-x)^5(b+y)}{3^5(a+b)^7}}=\sqrt[4]{\dfrac{(a-x)^5(b+y)\times 3^3(a+b)}{3^5(a+b)^7\times 3^3(a+b)}}$

$=\sqrt[4]{\dfrac{27(a-x)^5(b+y)(a+b)}{3^8(a+b)^8}}=\sqrt[4]{\dfrac{(a-x)^4}{3^8(a+b)^8}\times 27(a-x)(b+y)(a+b)}$

$=\dfrac{a-x}{9(a+b)^2}\sqrt[4]{27(a-x)(b+y)(a+b)}.$

Exercise 96. Page 243

Reduce to an entire surd:

1. $3a^3\sqrt{3a}=\sqrt{(3a^3)^2\times 3a}=\sqrt{9a^6\times 3a}=\sqrt{27a^7}.$

2. $-\frac{3}{4}\sqrt[3]{\frac{8}{9}a^2}=-\sqrt[3]{(\frac{3}{4})^3\times\frac{8}{9}a^2}=-\sqrt[3]{\frac{27}{64}\times\frac{8}{9}a^2}=-\sqrt[3]{\frac{3}{8}a^2}.$

3. $\frac{2}{3}b\sqrt[3]{\frac{9}{16}b^2c}=\sqrt[3]{(\frac{2}{3}b)^3\times\frac{9}{16}b^2c}=\sqrt[3]{\frac{8}{27}b^3\times\frac{9}{16}b^2c}=\sqrt[3]{\frac{1}{6}b^5c}.$

4. $-3a\sqrt[3]{bd}=-\sqrt[3]{(3a)^3\times bd}=-\sqrt[3]{27a^3\times bd}=-\sqrt[3]{27a^3bd}.$

5. $-5a\sqrt[4]{a^2c^3}=-\sqrt[4]{(5a)^4\times a^2c^3}=-\sqrt[4]{625a^4\times a^2c^3}=-\sqrt[4]{625a^6c^3}.$

6. $4x^3y^2\sqrt[4]{4x^3y^2} = \sqrt[4]{(4x^3y^2)^4 \times 4x^3y^2} = \sqrt[4]{256x^{12}y^8 \times 4x^3y^2}$
$= \sqrt[4]{1024x^{15}y^{10}}.$

7. $- \;\sqrt[4]{8x^3y^5} = -\sqrt[4]{(\frac{1}{2})^4 \times 8x^3y^5} = -\sqrt[4]{\frac{1}{16} \times 8x^3y^5} = -\sqrt[4]{\frac{1}{2}x^3y^5}.$

8. $8b\sqrt[3]{4a^2b} = \sqrt[3]{(8b)^3 \times 4a^2b} = \sqrt[3]{512b^3 \times 4a^2b} = \sqrt[3]{2048a^2b^4}.$

9. $-2a\sqrt[5]{\frac{27}{16}a^4b^2} = -\sqrt[5]{(2a)^5 \times \frac{27}{16}a^4b^2} = -\sqrt[5]{32a^5 \times \frac{27}{16}a^4b^2}$
$= -\sqrt[5]{54a^9b^2}.$

Exercise 97. Page 244

1. Reduce $\sqrt{3}$ and $\sqrt[3]{5}$ to surds of the same order.

$$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = \sqrt[6]{3^3} = \sqrt[6]{27}.$$
$$\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{5^2} = \sqrt[6]{25}.$$

2. Reduce $\sqrt[3]{7}$ and $\sqrt[5]{6}$ to surds of the same order.

$$\sqrt[3]{7} = 7^{\frac{1}{3}} = 7^{\frac{5}{15}} = \sqrt[15]{7^5} = \sqrt[15]{16807}.$$
$$\sqrt[5]{6} = 6^{\frac{1}{5}} = 6^{\frac{3}{15}} = \sqrt[15]{6^3} = \sqrt[15]{216}.$$

3. Reduce $\sqrt[3]{a^2}$ and $\sqrt[4]{b^3}$ to surds of the same order.

$$\sqrt[3]{a^2} = a^{\frac{2}{3}} = a^{\frac{8}{12}} = \sqrt[12]{a^8}.$$
$$\sqrt[4]{b^3} = b^{\frac{3}{4}} = b^{\frac{9}{12}} = \sqrt[12]{b^9}.$$

4. Reduce $\sqrt[3]{4}$ and $\sqrt[4]{8}$ to surds of the same order.

$$\sqrt[3]{4} = \sqrt[3]{2^2} = 2^{\frac{2}{3}} = 2^{\frac{8}{12}} = \sqrt[12]{2^8} = \sqrt[12]{256}.$$
$$\sqrt[4]{8} = \sqrt[4]{2^3} = 2^{\frac{3}{4}} = 2^{\frac{9}{12}} = \sqrt[12]{2^9} = \sqrt[12]{512}.$$

5. Reduce $2\sqrt[3]{6}$, $3\sqrt{5}$, and $4\sqrt[4]{3}$ to surds of the same order.

$$2\sqrt[3]{6} = 2 \times 6^{\frac{1}{3}} = 2 \times 6^{\frac{4}{12}} = 2\sqrt[12]{6^4} = 2\sqrt[12]{1296}.$$
$$3\sqrt{5} = 3 \times 5^{\frac{1}{2}} = 3 \times 5^{\frac{6}{12}} = 3\sqrt[12]{5^6} = 3\sqrt[12]{15625}.$$
$$4\sqrt[4]{3} = 4 \times 3^{\frac{1}{4}} = 4 \times 3^{\frac{3}{12}} = 4\sqrt[12]{3^3} = 4\sqrt[12]{27}.$$

6. Reduce $2\sqrt{2}$, $5\sqrt[3]{5}$, and $6\sqrt[4]{6}$ to surds of the same order.

$$2\sqrt{2} = 2 \times 2^{\frac{1}{2}} = 2 \times 2^{\frac{6}{12}} = 2\sqrt[12]{2^6} = 2\sqrt[12]{64}.$$
$$5\sqrt[3]{5} = 5 \times 5^{\frac{1}{3}} = 5 \times 5^{\frac{4}{12}} = 5\sqrt[12]{5^4} = 5\sqrt[12]{625}.$$
$$6\sqrt[4]{6} = 6 \times 6^{\frac{1}{4}} = 6 \times 6^{\frac{3}{12}} = 6\sqrt[12]{6^3} = 6\sqrt[12]{216}.$$

7. Reduce $\sqrt[4]{7}$, $2\sqrt[5]{5}$, and $\sqrt[10]{120}$ to surds of the same order.

$$\sqrt[4]{7} = 7^{\frac{1}{4}} = 7^{\frac{5}{20}} = \sqrt[20]{7^5} = \sqrt[20]{16807}.$$
$$2\sqrt[5]{5} = 2 \times 5^{\frac{1}{5}} = 2 \times 5^{\frac{4}{20}} = 2\sqrt[20]{5^4} = 2\sqrt[20]{625}.$$
$$\sqrt[10]{120} = 120^{\frac{1}{10}} = 120^{\frac{2}{20}} = \sqrt[20]{120^2} = \sqrt[20]{14400}.$$

8. Reduce $2\sqrt[3]{a^2}$, $a\sqrt[5]{b^2}$, and $b\sqrt[6]{a^3}$ to surds of the same order.

$$2\sqrt[3]{a^2} = 2a^{\frac{2}{3}} = 2a^{\frac{20}{30}} = 2\sqrt[30]{a^{20}}.$$
$$a\sqrt[5]{b^2} = ab^{\frac{2}{5}} = ab^{\frac{12}{30}} = a\sqrt[30]{b^{12}}.$$
$$b\sqrt[6]{a^3} = ba^{\frac{3}{6}} = ba^{\frac{15}{30}} = b\sqrt[30]{a^{15}}.$$

9. Arrange $\sqrt{18}$ and $\sqrt[3]{79}$ in order of magnitude.

$$\sqrt{18} = 18^{\frac{1}{2}} = 18^{\frac{3}{6}} = \sqrt[6]{18^3} = \sqrt[6]{5832}.$$
$$\sqrt[3]{79} = 79^{\frac{1}{3}} = 79^{\frac{2}{6}} = \sqrt[6]{79^2} = \sqrt[6]{6241}.$$

Therefore, the required order is $\sqrt[3]{79}$, $\sqrt{18}$.

10. Arrange $2\sqrt[3]{3}$ and $3\sqrt[4]{2}$ in order of magnitude.

$$2\sqrt[3]{3} = 2 \times 3^{\frac{1}{3}} = 2 \times 3^{\frac{4}{12}} = 2\sqrt[12]{3^4} = 2\sqrt[12]{81} = \sqrt[12]{2^{12} \times 81}$$
$$= \sqrt[12]{4096 \times 81} = \sqrt[12]{331776}.$$
$$3\sqrt[4]{2} = 3 \times 2^{\frac{1}{4}} = 3 \times 2^{\frac{3}{12}} = 3\sqrt[12]{2^3} = 3\sqrt[12]{8} = \sqrt[12]{3^{12} \times 8}$$
$$= \sqrt[12]{531441 \times 8} = \sqrt[12]{4251528}.$$

Therefore, the required order is $3\sqrt[4]{2}$, $2\sqrt[3]{3}$.

11. Arrange $\sqrt{33}$ and $\sqrt[3]{185}$ in order of magnitude.

$$\sqrt{33} = 33^{\frac{1}{2}} = 33^{\frac{3}{6}} = \sqrt[6]{33^3} = \sqrt[6]{35937}.$$
$$\sqrt[3]{185} = 185^{\frac{1}{3}} = 185^{\frac{2}{6}} = \sqrt[6]{185^2} = \sqrt[6]{34225}.$$

Therefore, the required order is $\sqrt{33}$, $\sqrt[3]{185}$.

12. Arrange $3\sqrt{\frac{2}{5}}$, $2\sqrt[3]{\frac{2}{3}}$, and $3\sqrt[3]{\frac{2}{9}}$ in order of magnitude.

$$3\sqrt{\tfrac{2}{5}} = \sqrt{3^2 \times \tfrac{2}{5}} = \sqrt{9 \times \tfrac{2}{5}} = \sqrt{\tfrac{18}{5}} = (\tfrac{18}{5})^{\frac{1}{2}} = (\tfrac{18}{5})^{\frac{3}{6}} = \sqrt[6]{(\tfrac{18}{5})^3}$$
$$= \sqrt[6]{\tfrac{5832}{125}} = \sqrt[6]{46\tfrac{82}{125}}.$$
$$2\sqrt[3]{\tfrac{2}{3}} = \sqrt[3]{2^3 \times \tfrac{2}{3}} = \sqrt[3]{8 \times \tfrac{2}{3}} = \sqrt[3]{\tfrac{16}{3}} = (\tfrac{16}{3})^{\frac{1}{3}} = (\tfrac{16}{3})^{\frac{2}{6}} = \sqrt[6]{(\tfrac{16}{3})^2}$$
$$= \sqrt[6]{\tfrac{256}{9}} = \sqrt[6]{28\tfrac{4}{9}}.$$
$$3\sqrt[3]{\tfrac{2}{9}} = \sqrt[3]{3^3 \times \tfrac{2}{9}} = \sqrt[3]{27 \times \tfrac{2}{9}} = \sqrt[3]{6} = 6^{\frac{1}{3}} = 6^{\frac{2}{6}} = \sqrt[6]{6^2} = \sqrt[6]{36}.$$

Therefore, the required order is $3\sqrt{\frac{2}{5}}$, $3\sqrt[3]{\frac{2}{9}}$, $2\sqrt[3]{\frac{2}{3}}$.

13. Arrange $\sqrt[6]{81}$, $\sqrt[3]{10}$, and $\sqrt{8}$ in order of magnitude.

$$\sqrt[3]{10} = 10^{\frac{1}{3}} = 10^{\frac{2}{6}} = \sqrt[6]{10^2} = \sqrt[6]{100}.$$
$$\sqrt{8} = \sqrt{2^3} = 2^{\frac{3}{2}} = 2^{\frac{9}{6}} = \sqrt[6]{2^9} = \sqrt[6]{512}.$$

Therefore, the required order is $\sqrt{8}$, $\sqrt[3]{10}$, $\sqrt[6]{81}$.

14. Arrange $2\sqrt{2}$, $2\sqrt[3]{3}$, and $2\sqrt[4]{5}$ in order of magnitude.

$$2\sqrt{2} = 2 \times 2^{\frac{1}{2}} = 2 \times 2^{\frac{6}{12}} = 2\sqrt[12]{2^6} = 2\sqrt[12]{64}.$$
$$2\sqrt[3]{3} = 2 \times 3^{\frac{1}{3}} = 2 \times 3^{\frac{4}{12}} = 2\sqrt[12]{3^4} = 2\sqrt[12]{81}.$$
$$2\sqrt[4]{5} = 2 \times 5^{\frac{1}{4}} = 2 \times 5^{\frac{3}{12}} = 2\sqrt[12]{5^3} = 2\sqrt[12]{125}.$$

Therefore, the required order is $2\sqrt[4]{5}$, $2\sqrt[3]{3}$, $2\sqrt{2}$.

Exercise 98. Page 245

Simplify:

1. $8\sqrt{a} + 5\sqrt{x} - 7\sqrt{a} + 4\sqrt{a} - 6\sqrt{x} - 3\sqrt{a}$
$$= (8 - 7 + 4 - 3)\sqrt{a} + (5 - 6)\sqrt{x}$$
$$= 2\sqrt{a} - \sqrt{x}.$$

2. $\sqrt{a} + 3\sqrt{2a} - 2\sqrt{3a} + \sqrt{4a} - \sqrt{8a} + \sqrt{12a}$
$$= \sqrt{a} + 3\sqrt{2a} - 2\sqrt{3a} + 2\sqrt{a} - 2\sqrt{2a} + 2\sqrt{3a}$$
$$= (1 + 2)\sqrt{a} + (3 - 2)\sqrt{2a} + (-2 + 2)\sqrt{3a}$$
$$= 3\sqrt{a} + \sqrt{2a}.$$

3. $7\sqrt{4a} + 4\sqrt{9a} + 3\sqrt{45a} - 5\sqrt{36a} - 2\sqrt{80a}$
$$= 14\sqrt{a} + 12\sqrt{a} + 9\sqrt{5a} - 30\sqrt{a} - 8\sqrt{5a}$$
$$= (14 + 12 - 30)\sqrt{a} + (9 - 8)\sqrt{5a}$$
$$= -4\sqrt{a} + \sqrt{5a}$$
$$= \sqrt{5a} - 4\sqrt{a}.$$

4. $3\sqrt{8} + 4\sqrt{32} - 5\sqrt{50} - 7\sqrt{72} + 6\sqrt{98} + \sqrt{18}$
$$= 3\sqrt{4 \times 2} + 4\sqrt{16 \times 2} - 5\sqrt{25 \times 2} - 7\sqrt{36 \times 2}$$
$$+ 6\sqrt{49 \times 2} + \sqrt{9 \times 2}$$
$$= 6\sqrt{2} + 16\sqrt{2} - 25\sqrt{2} - 42\sqrt{2} + 42\sqrt{2} + 3\sqrt{2}$$
$$= (6 + 16 - 25 - 42 + 42 + 3)\sqrt{2}$$
$$= 0.$$

5. $7\sqrt{12}-5\sqrt{27}+8\sqrt{48}-6\sqrt{75}+2\sqrt{108}+\sqrt{27}$

$$=7\sqrt{4\times 3}-5\sqrt{9\times 3}+8\sqrt{16\times 3}-6\sqrt{25\times 3}$$
$$+2\sqrt{36\times 3}+\sqrt{9\times 3}$$
$$=14\sqrt{3}-15\sqrt{3}+32\sqrt{3}-30\sqrt{3}+12\sqrt{3}+3\sqrt{3}$$
$$=(14-15+32-30+12+3)\sqrt{3}$$
$$=16\sqrt{3}.$$

6. $5\sqrt[3]{16}+3\sqrt[3]{-54}-6\sqrt[3]{-128}+7\sqrt[3]{-250}+2\sqrt[3]{432}$

$$=5\sqrt[3]{8\times 2}+3\sqrt[3]{-27\times 2}-6\sqrt[3]{-64\times 2}+7\sqrt[3]{-125\times 2}$$
$$+2\sqrt[3]{216\times 2}$$
$$=5\times 2\sqrt[3]{2}+3(-3)\sqrt[3]{2}-6(-4)\sqrt[3]{2}+7(-5)\sqrt[3]{2}+2\times 6\sqrt[3]{2}$$
$$=10\sqrt[3]{2}-9\sqrt[3]{2}+24\sqrt[3]{2}-35\sqrt[3]{2}+12\sqrt[3]{2}$$
$$=(10-9+24-35+12)\sqrt[3]{2}$$
$$=2\sqrt[3]{2}.$$

7. $7\sqrt[3]{24}+5\sqrt[3]{81}+4\sqrt[3]{-192}+2\sqrt[3]{-375}-\sqrt[3]{1029}$

$$=7\sqrt[3]{8\times 3}+5\sqrt[3]{27\times 3}+4\sqrt[3]{-64\times 3}+2\sqrt[3]{-125\times 3}$$
$$-\sqrt[3]{343\times 3}$$
$$=7\times 2\sqrt[3]{3}+5\times 3\sqrt[3]{3}+4(-4)\sqrt[3]{3}+2(-5)\sqrt[3]{3}-7\sqrt[3]{3}$$
$$=14\sqrt[3]{3}+15\sqrt[3]{3}-16\sqrt[3]{3}-10\sqrt[3]{3}-7\sqrt[3]{3}$$
$$=(14+15-16-10-7)\sqrt[3]{3}$$
$$=-4\sqrt[3]{3}.$$

8. $\sqrt{(a+b)^2x}+\sqrt{(a-b)^2x}-\sqrt{a^2x}+\sqrt{(1-a)^2x}-\sqrt{x}$

$$=(a+b)\sqrt{x}+(a-b)\sqrt{x}-a\sqrt{x}+(1-a)\sqrt{x}-\sqrt{x}$$
$$=(a+b+a-b-a+1-a-1)\sqrt{x}$$
$$=0.$$

9. $3\sqrt{7}+2\sqrt{54}-5\sqrt{63}+3\sqrt{150}+4\sqrt{252}+7\sqrt{24}$

$$=3\sqrt{7}+2\sqrt{9\times 6}-5\sqrt{9\times 7}+3\sqrt{25\times 6}+4\sqrt{36\times 7}$$
$$+7\sqrt{4\times 6}$$
$$=3\sqrt{7}+2\times 3\sqrt{6}-5\times 3\sqrt{7}+3\times 5\sqrt{6}+4\times 6\sqrt{7}$$
$$+7\times 2\sqrt{6}$$
$$=3\sqrt{7}+6\sqrt{6}-15\sqrt{7}+15\sqrt{6}+24\sqrt{7}+14\sqrt{6}$$
$$=(3-15+24)\sqrt{7}+(6+15+14)\sqrt{6}$$
$$=12\sqrt{7}+35\sqrt{6}.$$

10. $\sqrt{275} + \sqrt{1300} - \sqrt{44} + \sqrt{52} - 2\sqrt{99} - 2\sqrt{208}$

$$= \sqrt{25 \times 11} + \sqrt{100 \times 13} - \sqrt{4 \times 11} + \sqrt{4 \times 13} - 2\sqrt{9 \times 11} - 2\sqrt{16 \times 13}$$
$$= 5\sqrt{11} + 10\sqrt{13} - 2\sqrt{11} + 2\sqrt{13} - 2 \times 3\sqrt{11} - 2 \times 4\sqrt{13}$$
$$= 5\sqrt{11} + 10\sqrt{13} - 2\sqrt{11} + 2\sqrt{13} - 6\sqrt{11} - 8\sqrt{13}$$
$$= (5 - 2 - 6)\sqrt{11} + (10 + 2 - 8)\sqrt{13}$$
$$= 4\sqrt{13} - 3\sqrt{11}.$$

11. $3\tfrac{1}{2}\sqrt{24} - 5\tfrac{3}{4}\sqrt{54} + 13\tfrac{1}{3}\sqrt{99} + 2\tfrac{17}{24}\sqrt{216} - 21\sqrt{44}$

$$= \tfrac{7}{2}\sqrt{4 \times 6} - \tfrac{23}{4}\sqrt{9 \times 6} + \tfrac{40}{3}\sqrt{9 \times 11} + \tfrac{65}{24}\sqrt{36 \times 6} - 21\sqrt{4 \times 11}$$
$$= \tfrac{7}{2} \times 2\sqrt{6} - \tfrac{23}{4} \times 3\sqrt{6} + \tfrac{40}{3} \times 3\sqrt{11} + \tfrac{65}{24} \times 6\sqrt{6} - 21 \times 2\sqrt{11}$$
$$= 7\sqrt{6} - 17\tfrac{1}{4}\sqrt{6} + 40\sqrt{11} + 16\tfrac{1}{4}\sqrt{6} - 42\sqrt{11}$$
$$= (7 - 17\tfrac{1}{4} + 16\tfrac{1}{4})\sqrt{6} + (40 - 42)\sqrt{11}$$
$$= 6\sqrt{6} - 2\sqrt{11}.$$

12. $\sqrt[3]{\tfrac{4}{27}} - \sqrt[3]{5} - 3\sqrt[3]{256} + \sqrt[3]{625} + \sqrt[3]{\tfrac{32}{27}} - \sqrt[3]{\tfrac{125}{54}}$

$$= \sqrt[3]{\tfrac{1}{27} \times 4} - \sqrt[3]{5} - 3\sqrt[3]{64 \times 4} + \sqrt[3]{125 \times 5} + \sqrt[3]{\tfrac{8}{27} \times 4} - \sqrt[3]{\frac{125}{27 \times 8} \times 4}$$
$$= \frac{1}{3}\sqrt[3]{4} - \sqrt[3]{5} - 3 \times 4\sqrt[3]{4} + 5\sqrt[3]{5} + \frac{2}{3}\sqrt[3]{4} - \frac{5}{3 \times 2}\sqrt[3]{4}$$
$$= \tfrac{1}{3}\sqrt[3]{4} - \sqrt[3]{5} - 12\sqrt[3]{4} + 5\sqrt[3]{5} + \tfrac{2}{3}\sqrt[3]{4} - \tfrac{5}{6}\sqrt[3]{4}$$
$$= (-1 + 5)\sqrt[3]{5} + (\tfrac{1}{3} - 12 + \tfrac{2}{3} - \tfrac{5}{6})\sqrt[3]{4}$$
$$= 4\sqrt[3]{5} - 11\tfrac{5}{6}\sqrt[3]{4}.$$

13. $\sqrt{9} + \sqrt{20} - \sqrt{\tfrac{1}{16}} - \sqrt{\tfrac{9}{5}} + \sqrt{\tfrac{9}{16}} + \sqrt{\tfrac{25}{81}} + \sqrt{\tfrac{20}{9}}$

$$= \sqrt{9} + \sqrt{4 \times 5} - \sqrt{\tfrac{1}{16}} - \sqrt{\tfrac{9}{25} \times 5} + \sqrt{\tfrac{9}{16}} + \sqrt{\tfrac{25}{81}} + \sqrt{\tfrac{4}{9} \times 5}$$
$$= 3 + 2\sqrt{5} - \tfrac{1}{4} - \tfrac{3}{5}\sqrt{5} + \tfrac{3}{4} + \tfrac{5}{9} + \tfrac{2}{3}\sqrt{5}$$
$$= 3 - \tfrac{1}{4} + \tfrac{3}{4} + \tfrac{5}{9} + (2 - \tfrac{3}{5} + \tfrac{2}{3})\sqrt{5}$$
$$= 4\tfrac{1}{18} + 2\tfrac{1}{15}\sqrt{5}.$$

14. $\sqrt{11\frac{1}{4}} + 3\sqrt{8} - \sqrt{20} - \frac{1}{6}\sqrt{245} - 5\sqrt{\frac{1}{18}} - \sqrt{24\frac{1}{2}}$

$$= \sqrt{\tfrac{45}{4}} + 3\sqrt{4 \times 2} - \sqrt{4 \times 5} - \tfrac{1}{6}\sqrt{49 \times 5} - 5\sqrt{\tfrac{1}{36} \times 2} - \sqrt{\tfrac{49}{2}}$$
$$= \sqrt{\tfrac{9}{4} \times 5} + 3 \times 2\sqrt{2} - 2\sqrt{5} - \tfrac{7}{6}\sqrt{5} - \tfrac{5}{6}\sqrt{2} - \sqrt{\tfrac{49}{4} \times 2}$$
$$= \tfrac{3}{2}\sqrt{5} + 6\sqrt{2} - 2\sqrt{5} - \tfrac{7}{6}\sqrt{5} - \tfrac{5}{6}\sqrt{2} - \tfrac{7}{2}\sqrt{2}$$
$$= (6 - \tfrac{5}{6} - \tfrac{7}{2})\sqrt{2} + (\tfrac{3}{2} - 2 - \tfrac{7}{6})\sqrt{5}$$
$$= \tfrac{5}{3}\sqrt{2} - \tfrac{5}{3}\sqrt{5}.$$

15. $\frac{3}{2}\sqrt[3]{\frac{2}{49}} + 0.8\sqrt{\frac{8}{3}} - \frac{1}{15}\sqrt{96} + \frac{3}{2}\sqrt[3]{\frac{7}{4}} - \frac{11}{28}\sqrt[3]{1750} + 8\sqrt{\frac{3}{2}}$

$$= \frac{3}{2}\sqrt[3]{\frac{2 \times 7}{49 \times 7}} + \frac{4}{5}\sqrt{\frac{4}{9} \times 6} - \frac{1}{15}\sqrt{16 \times 6} + \frac{3}{2}\sqrt[3]{\frac{7 \times 2}{4 \times 2}}$$
$$- \tfrac{11}{28}\sqrt[3]{125 \times 14} + 8\sqrt{\tfrac{1}{4} \times 6}$$
$$= \tfrac{3}{2} \times \tfrac{1}{7}\sqrt[3]{14} + \tfrac{4}{5} \times \tfrac{2}{3}\sqrt{6} - \tfrac{4}{15}\sqrt{6} + \tfrac{3}{2} \times \tfrac{1}{2}\sqrt[3]{14} - \tfrac{11}{28} \times 5\sqrt[3]{14}$$
$$+ 8 \times \tfrac{1}{2}\sqrt{6}$$
$$= \tfrac{3}{14}\sqrt[3]{14} + \tfrac{8}{15}\sqrt{6} - \tfrac{4}{15}\sqrt{6} + \tfrac{3}{4}\sqrt[3]{14} - \tfrac{55}{28}\sqrt[3]{14} + 4\sqrt{6}$$
$$= (\tfrac{8}{15} - \tfrac{4}{15} + 4)\sqrt{6} + (\tfrac{3}{14} + \tfrac{3}{4} - \tfrac{55}{28})\sqrt[3]{14}$$
$$= 4\tfrac{4}{15}\sqrt{6} - \sqrt[3]{14}.$$

Exercise 99. Page 246

Find the product of:

1. $\sqrt{3} \times \sqrt{12} = \sqrt{36} = 6.$
2. $\sqrt{5} \times \sqrt{20} = \sqrt{100} = 10.$
3. $\sqrt{20} \times \sqrt{30} = \sqrt{600} = \sqrt{100 \times 6} = 10\sqrt{6}.$
4. $\sqrt[3]{5} \times \sqrt[3]{50} = \sqrt[3]{250} = \sqrt[3]{125 \times 2} = 5\sqrt[3]{2}.$
5. $\sqrt[3]{2a^2} \times \sqrt[3]{4a} = \sqrt[3]{8a^3} = 2a.$
6. $5\sqrt{3} \times 3\sqrt{5} = 15\sqrt{15}.$
7. $\sqrt[3]{3} \times \sqrt[3]{36} = \sqrt[3]{108} = \sqrt[3]{27 \times 4} = 3\sqrt[3]{4}.$
8. $\sqrt[3]{6} \times \sqrt{18} = \sqrt[3]{6} \times 3\sqrt{2} = \sqrt[6]{6^2} \times 3\sqrt[6]{2^3} = \sqrt[6]{36} \times 3\sqrt[6]{8} = 3\sqrt[6]{288}.$
9. $\sqrt[3]{9} \times 2\sqrt[3]{3} = 2\sqrt[3]{27} = 2 \times 3 = 6.$
10. $2\sqrt{\frac{2}{3}} \times 3\sqrt{\frac{5}{6}} = 6\sqrt{\frac{5}{9}} = 6 \times \frac{1}{3}\sqrt{5} = 2\sqrt{5}.$
11. $3\sqrt[3]{16} + 5\sqrt[3]{4} = 15\sqrt[3]{64} = 15 \times 4 = 60.$
12. $\frac{1}{2}\sqrt[4]{8} \times \frac{3}{4}\sqrt[4]{2} = \frac{1}{2} \times \frac{3}{4}\sqrt[4]{8 \times 2} = \frac{3}{8}\sqrt[4]{16} = \frac{3}{8} \times 2 = \frac{3}{4}.$

13. $\sqrt[3]{-108} \times \sqrt[3]{40} = \sqrt[3]{-27 \times 4} \times \sqrt[3]{8 \times 5} = -3\sqrt[3]{4} \times 2\sqrt[3]{5} = -6\sqrt[3]{20}.$

14. $\sqrt[5]{-343} \times \sqrt[5]{-1568} = \sqrt[5]{-(7)^3} \times \sqrt[5]{-(2)^5 \times 7^2} = \sqrt[5]{(-2)^5 \times (-7)^5}$
$= (-2)(-7) = 14.$

15. $\sqrt{6} \times \sqrt{12} \times \sqrt{18} = \sqrt{6} \times \sqrt{4 \times 3} \times \sqrt{9 \times 2} = \sqrt{6} \times 2\sqrt{3} \times 3\sqrt{2}$
$= 6\sqrt{36} = 6 \times 6 = 36.$

16. $\frac{2}{3}\sqrt[3]{4} \times \frac{3}{4}\sqrt[3]{-3} \times \frac{4}{5}\sqrt[3]{9} = \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}\sqrt[3]{4 \times (-3) \times 9}$
$= \frac{2}{5}\sqrt[3]{-27 \times 4} = \frac{2}{5} \times (-3)\sqrt[3]{4} = -\frac{6}{5}\sqrt[3]{4}.$

17. $\sqrt[3]{16} \times 5\sqrt[3]{4} \times 3\sqrt[3]{\tfrac{1}{2}} = 15\sqrt[3]{32} = 15\sqrt[3]{8 \times 4} = 15 \times 2\sqrt[3]{4} = 30\sqrt[3]{4}.$

18. $5\sqrt[3]{24} \times 3\sqrt[3]{32} \times 2\sqrt[3]{36} = 5\sqrt[3]{2^3 \times 3} \times 3\sqrt[3]{2^5} \times 2\sqrt[3]{2^2 \times 3^2}$
$= 5 \times 3 \times 2\sqrt[3]{2^{10} \times 3^3} = 30\sqrt[3]{2^9 \times 3^3 \times 2} = 30 \times 8 \times 3\sqrt[3]{2} = 720\sqrt[3]{2}.$

19. $$\sqrt[3]{\frac{5}{6}} \times \sqrt[4]{\frac{2}{3}} \times \sqrt[3]{\frac{1}{2}} = \sqrt[12]{\left(\frac{5}{6}\right)^4} \times \sqrt[12]{\left(\frac{2}{3}\right)^3} \times \sqrt[12]{\left(\frac{1}{2}\right)^4}$$
$$= \sqrt[12]{\frac{5^4}{2^4 \times 3^4}} \times \sqrt[12]{\frac{2^3}{3^3}} \times \sqrt[12]{\frac{1}{2^4}} = \sqrt[12]{\frac{5^4 \times 2^3}{2^8 \times 3^7}} = \sqrt[12]{\frac{5^4}{2^5 \times 3^7}}$$
$$= \sqrt[12]{\frac{5^4 \times 2^7 \times 3^5}{2^{12} \times 3^{12}}} = \frac{1}{2} \times \frac{1}{3}\sqrt[12]{5^4 \times 2^7 \times 3^5} = \frac{1}{6}\sqrt[12]{5^4 \times 2^7 \times 3^5}.$$

20. $5\sqrt{2} \times \frac{2}{5}\sqrt{72} \times \frac{2}{3}\sqrt{48} = 5\sqrt{2} \times \frac{2}{5}\sqrt{36 \times 2} \times \frac{2}{3}\sqrt{16 \times 3}$
$= 5\sqrt{2} \times \frac{12}{5}\sqrt{2} \times \frac{8}{3}\sqrt{3} = 5 \times \frac{12}{5} \times \frac{8}{3} \times 2 \times \sqrt{3} = 64\sqrt{3}.$

21. $(3\sqrt[3]{24} + 4\sqrt[3]{81} - 5\sqrt[3]{375} + \frac{1}{2}\sqrt[3]{192}) \times \sqrt[3]{2}$
$= (3\sqrt[3]{8 \times 3} + 4\sqrt[3]{27 \times 3} - 5\sqrt[3]{125 \times 3} + \frac{1}{2}\sqrt[3]{64 \times 3})\sqrt[3]{2}$
$= (3 \times 2\sqrt[3]{3} + 4 \times 3\sqrt[3]{3} - 5 \times 5\sqrt[3]{3} + \frac{1}{2} \times 4\sqrt[3]{3})\sqrt[3]{2}$
$= (6\sqrt[3]{3} + 12\sqrt[3]{3} - 25\sqrt[3]{3} + 2\sqrt[3]{3})\sqrt[3]{2}$
$= (-5\sqrt[3]{3})\sqrt[3]{2}$
$= -5\sqrt[3]{6}.$

22. $(2\sqrt{6} - \sqrt{12} - 3\sqrt{24} + \frac{1}{4}\sqrt{48}) \times 3\sqrt{2}$
$= 6\sqrt{12} - 3\sqrt{24} - 9\sqrt{48} + \frac{3}{4}\sqrt{96}$
$= 6\sqrt{4 \times 3} - 3\sqrt{4 \times 6} - 9\sqrt{16 \times 3} + \frac{3}{4}\sqrt{16 \times 6}$
$= 6 \times 2\sqrt{3} - 3 \times 2\sqrt{6} - 9 \times 4\sqrt{3} + \frac{3}{4} \times 4\sqrt{6}$
$= 12\sqrt{3} - 6\sqrt{6} - 36\sqrt{3} + 3\sqrt{6}$
$= -24\sqrt{3} - 3\sqrt{6},$

Exercise 100. Page 246

Find the product of:

1. $(\sqrt{7+\sqrt{18}})(\sqrt{7-\sqrt{18}})$.

$$\begin{aligned}&(\sqrt{7+\sqrt{18}})(\sqrt{7-\sqrt{18}})\\&\quad=\sqrt{(7+\sqrt{18})(7-\sqrt{18})}\\&\quad=\sqrt{49-18}=\sqrt{31}.\end{aligned}$$

2. $(\sqrt{3+2\sqrt{5}})(\sqrt{3+2\sqrt{5}})$.

$$\begin{aligned}&(\sqrt{3+2\sqrt{5}})(\sqrt{3+2\sqrt{5}})\\&\quad=\sqrt{(3+2\sqrt{5})(3+2\sqrt{5})}\\&\quad=\sqrt{(3+2\sqrt{5})^2}\\&\quad=3+2\sqrt{5}.\end{aligned}$$

3. $(\sqrt{8}+\sqrt{7})(\sqrt{7}-\sqrt{2})$.

$$\sqrt{8}+\sqrt{7}=2\sqrt{2}+\sqrt{7}$$

$$\begin{array}{rrr}\sqrt{7}+2\sqrt{2} & & \\ \sqrt{7}-\sqrt{2} & & \\ \hline 7+2\sqrt{14} & & \\ -\sqrt{14} & -4 & \\ \hline 3+\sqrt{14} & & \end{array}$$

4. $(7+3\sqrt{7})(2\sqrt{7}-7)$.

$$\begin{array}{rr}7+3\sqrt{7} & \\ -7+2\sqrt{7} & \\ \hline -49-21\sqrt{7} & \\ +14\sqrt{7} & +42 \\ \hline -7-7\sqrt{7} & \end{array}$$

5. $(8+3\sqrt{5})(2-\sqrt{5})$.

$$\begin{array}{rr}8+3\sqrt{5} & \\ 2-\sqrt{5} & \\ \hline 16+6\sqrt{5} & \\ -8\sqrt{5} & -15 \\ \hline 1-2\sqrt{5} & \end{array}$$

6. $(2a+3\sqrt{x})(3a-2\sqrt{x})$.

$$\begin{array}{rr}2a\ +\ \ 3\sqrt{x} & \\ 3a\ -\ \ 2\sqrt{x} & \\ \hline 6a^2+9a\sqrt{x} & \\ -4a\sqrt{x} & -6x \\ \hline 6a^2+5a\sqrt{x} & -6x\end{array}$$

7. $(5\sqrt{7}-2\sqrt{5})(3\sqrt{7}+10\sqrt{5})$.

$$\begin{array}{rr}5\sqrt{7}-\ \ 2\sqrt{5} & \\ 3\sqrt{7}+10\sqrt{5} & \\ \hline 105\ \ -\ \ 6\sqrt{35} & \\ +50\sqrt{35} & -100 \\ \hline 5+44\sqrt{35} & \end{array}$$

8. $(4\sqrt{a}-\sqrt{3x})(\sqrt{a}+2\sqrt{3x})$.

$$\begin{array}{rr}4\sqrt{a}-\ \ \sqrt{3x} & \\ \sqrt{a}+2\sqrt{3x} & \\ \hline 4a-\ \ \sqrt{3ax} & \\ +8\sqrt{3ax} & -6x \\ \hline 4a+7\sqrt{3ax} & -6x\end{array}$$

9. $(2\sqrt{30} - 3\sqrt{5} + 5\sqrt{3})(\sqrt{8} + \sqrt{3} - \sqrt{5})$.

$$\sqrt{8} + \sqrt{3} - \sqrt{5} = 2\sqrt{2} + \sqrt{3} - \sqrt{5}.$$

$$\begin{array}{rrrrr}
5\sqrt{3} & -\ 3\sqrt{5} & +\ 2\sqrt{30} & & \\
2\sqrt{2} & +\ \sqrt{3} & -\ \sqrt{5} & & \\
\cline{1-3}
10\sqrt{6} & -\ 6\sqrt{10} & +\ 8\sqrt{15} & & \\
 & +\ 15 & -\ 3\sqrt{15} & +\ 6\sqrt{10} & \\
 & & -\ 5\sqrt{15} & +\ 15 & -\ 10\sqrt{6} \\
\hline
 & 30 & & &
\end{array}$$

10. $(3 + \sqrt{6} + \sqrt{15})(2 + \sqrt{6} - \sqrt{10})$.

$$\begin{array}{rrrrr}
3 & +\ \sqrt{6} & +\ \sqrt{15} & & \\
2 & +\ \sqrt{6} & -\ \sqrt{10} & & \\
\cline{1-3}
6 & +\ 2\sqrt{6} & +\ 2\sqrt{15} & & \\
 & +\ 3\sqrt{6} & +\ 6 & +\ 3\sqrt{10} & \\
 & & -\ 3\sqrt{10} & -\ 2\sqrt{15} & -\ 5\sqrt{6} \\
\hline
12 & & & &
\end{array}$$

11. $(2\sqrt{7} - 8\sqrt{28} - 5\sqrt{63})(2\sqrt{7} - 8\sqrt{28} - 5\sqrt{63})$.

$$\begin{aligned}
2\sqrt{7} - 8\sqrt{28} - 5\sqrt{63} &= 2\sqrt{7} - 16\sqrt{7} - 15\sqrt{7} \\
&= -29\sqrt{7}.
\end{aligned}$$

$$(-29\sqrt{7})(-29\sqrt{7}) = 841 \times 7 = 5887.$$

12. $(2\sqrt{2} - 3\sqrt{3} + 4\sqrt{5})(3\sqrt{5} + \sqrt{2} - \sqrt{3})$.

$$\begin{array}{rrrrr}
2\sqrt{2} & -\ 3\sqrt{3} & +\ 4\sqrt{5} & & \\
\sqrt{2} & -\ \sqrt{3} & +\ 3\sqrt{5} & & \\
\cline{1-3}
4 & -\ 3\sqrt{6} & +\ 4\sqrt{10} & & \\
 & -\ 2\sqrt{6} & +\ 9 & -\ 4\sqrt{15} & \\
 & & +\ 6\sqrt{10} & -\ 9\sqrt{15} & +\ 60 \\
\hline
73 & -\ 5\sqrt{6} & +\ 10\sqrt{10} & -\ 13\sqrt{15} &
\end{array}$$

13. $(3\sqrt{3} - 4\sqrt{8} + 5\sqrt{5})(5\sqrt{2} + \sqrt{3} + \sqrt{5})$.

$$3\sqrt{3} - 4\sqrt{8} + 5\sqrt{5} = 3\sqrt{3} - 8\sqrt{2} + 5\sqrt{5}.$$

$$\begin{array}{rrrrr}
-\ 8\sqrt{2} & +\ 3\sqrt{3} & +\ 5\sqrt{5} & & \\
5\sqrt{2} & +\ \sqrt{3} & +\ \sqrt{5} & & \\
\cline{1-3}
-\ 80 & +\ 15\sqrt{6} & +\ 25\sqrt{10} & & \\
 & -\ 8\sqrt{6} & +\ 9 & +\ 5\sqrt{15} & \\
 & & -\ 8\sqrt{10} & +\ 3\sqrt{15} & +\ 25 \\
\hline
-\ 46 & +\ 7\sqrt{6} & +\ 17\sqrt{10} & +\ 8\sqrt{15} &
\end{array}$$

14. $(\sqrt[3]{6}-2\sqrt[3]{2}+4\sqrt[3]{4})(5\sqrt[3]{4}+3\sqrt[3]{6}-2\sqrt[3]{2})$.

$$\begin{array}{rrrrrr}
-2\sqrt[3]{2} & +\ 4\sqrt[3]{4} & +\ \sqrt[3]{6} & & & \\
-2\sqrt[3]{2} & +\ 5\sqrt[3]{4} & +\ 3\sqrt[3]{6} & & & \\
\hline
4\sqrt[3]{4} & -16 & -\ 2\sqrt[3]{12} & & & \\
 & -20 & & +40\sqrt[3]{2} & +10\sqrt[3]{3} & \\
 & & -\ 6\sqrt[3]{12} & & +24\sqrt[3]{3} & +\ 3\sqrt[3]{36} \\
\hline
4\sqrt[3]{4} & -36 & -8\sqrt[3]{12} & +40\sqrt[3]{2} & +34\sqrt[3]{3} & +3\sqrt[3]{36}
\end{array}$$

15. $(2\sqrt{6}+5\sqrt{3}-7\sqrt{2})(\sqrt{6}-2\sqrt{3}+4\sqrt{2})$.

$$\begin{array}{rrrrr}
2\sqrt{6} & +\ 5\sqrt{3} & -\ 7\sqrt{2} & & \\
\sqrt{6} & -\ 2\sqrt{3} & +\ 4\sqrt{2} & & \\
\hline
12 & +15\sqrt{2} & -14\sqrt{3} & & \\
 & -12\sqrt{2} & -\ 30 & +14\sqrt{6} & \\
 & & +16\sqrt{3} & +20\sqrt{6} & -56 \\
\hline
 & 3\sqrt{2} & +\ 2\sqrt{3} & +34\sqrt{6} & -74
\end{array}$$

16. $(\sqrt[3]{9}-7\sqrt[3]{72}+6\sqrt[3]{1125})(3\sqrt[3]{2}+8\sqrt[3]{3}-4\sqrt[3]{\tfrac{1}{9}})$.

$$\sqrt[3]{9}-7\sqrt[3]{72}+6\sqrt[3]{1125}=\sqrt[3]{9}-14\sqrt[3]{9}+30\sqrt[3]{9}=17\sqrt[3]{9}.$$

$$\begin{aligned}3\sqrt[3]{2}+8\sqrt[3]{3}-4\sqrt[3]{\tfrac{1}{9}}&=3\sqrt[3]{2}+8\sqrt[3]{3}-\tfrac{4}{3}\sqrt[3]{3}\\&=3\sqrt[3]{2}+\tfrac{20}{3}\sqrt[3]{3}.\end{aligned}$$

$$\begin{array}{ll}
3\sqrt[3]{2} & +\ \tfrac{20}{3}\sqrt[3]{3} \\
17\sqrt[3]{9} & \\
\hline
51\sqrt[3]{18} & +\ 340
\end{array}$$

17. $(5\sqrt{112}+\sqrt{176}-\sqrt{4375})(3\sqrt{396}+\sqrt{175}-2\sqrt{539})$.

$5\sqrt{112}+\sqrt{176}-\sqrt{4375}=20\sqrt{7}+4\sqrt{11}-25\sqrt{7}=4\sqrt{11}-5\sqrt{7}.$

$3\sqrt{396}+\sqrt{175}-2\sqrt{539}=18\sqrt{11}+5\sqrt{7}-14\sqrt{11}=4\sqrt{11}+5\sqrt{7}.$

$(4\sqrt{11}-5\sqrt{7})(4\sqrt{11}+5\sqrt{7})=176-175=1.$

18. $(\sqrt{x}+\sqrt{y})(x^2+xy+y^2)(\sqrt{x}-\sqrt{y})=(x-y)(x^2+xy+y^2)=x^3-y^3.$

19. $(a+b+\sqrt[3]{a^2b}+\sqrt[3]{ab^2})(\sqrt[3]{a}-\sqrt[3]{b})$.

$$\begin{array}{llllll}
a & +\sqrt[3]{a^2b} & +\sqrt[3]{ab^2} & +\ b & & \\
\sqrt[3]{a} & -\sqrt[3]{b} & & & & \\
\hline
a\sqrt[3]{a} & +\ a\sqrt[3]{b} & +\sqrt[3]{a^2b^2} & +\ b\sqrt[3]{a} & & \\
 & -\ a\sqrt[3]{b} & -\sqrt[3]{a^2b^2} & -\ b\sqrt[3]{a} & -\ b\sqrt[3]{b} & \\
\hline
a\sqrt[3]{a} & & & & -\ b\sqrt[3]{b} &
\end{array}$$

Exercise 101. Page 247

1. Divide $\sqrt[3]{56}$ by $\sqrt[3]{7}$.

$$\frac{\sqrt[3]{56}}{\sqrt[3]{7}}=\sqrt[3]{8}=2.$$

2. Divide $\sqrt[3]{1080}$ by $\sqrt[3]{5}$.

$$\frac{\sqrt[3]{1080}}{\sqrt[3]{5}}=\sqrt[3]{216}=6.$$

3. Divide $\sqrt[4]{243}$ by $\sqrt[4]{3}$.

$$\frac{\sqrt[4]{243}}{\sqrt[4]{3}}=\sqrt[4]{81}=3.$$

4. Divide $\sqrt{\frac{2}{9}}$ by $\sqrt{\frac{1}{2}}$.

$$\frac{\sqrt{\frac{2}{9}}}{\sqrt{\frac{1}{2}}}=\sqrt{\tfrac{2}{9}\times 2}=\sqrt{\tfrac{4}{9}}=\tfrac{2}{3}.$$

5. Divide $\sqrt{a^5}$ by $\sqrt{a^8}$.

$$\frac{\sqrt{a^5}}{\sqrt{a^8}}=\frac{a^2\sqrt{a}}{a\sqrt{a}}=a.$$

6. Divide $\sqrt[3]{96\,a^8}$ by $\sqrt[3]{6\,a^5}$.

$$\frac{\sqrt[3]{96a^8}}{\sqrt[3]{6a^5}}=\frac{2a^2\sqrt[3]{12a^2}}{a\sqrt[3]{6\,a^2}}=2\,a\sqrt[3]{2}.$$

7. Divide $\sqrt[4]{125\,a^7}$ by $\sqrt[4]{\frac{1}{5}\,a^2}$.

$$\frac{\sqrt[4]{125a^7}}{\sqrt[4]{\frac{1}{5}\,a^2}}=\sqrt[4]{625a^5}=5a\sqrt[4]{a}.$$

8. Divide $\sqrt[4]{\frac{2}{5}\,m^3}$ by $\sqrt[4]{\frac{5}{8}\,m}$.

$$\frac{\sqrt[4]{\frac{2}{5}\,m^3}}{\sqrt[4]{\frac{5}{8}\,m}}=\sqrt[4]{\frac{16}{25}m^2}=2\sqrt[4]{\frac{m^2}{25}}$$

$$=2\sqrt{\frac{m}{5}}=\frac{2}{5}\sqrt{5\,m}.$$

9. Divide 6 by $\sqrt{72}$.

$$\frac{6}{\sqrt{72}}=\frac{6}{6\sqrt{2}}=\frac{1}{\sqrt{2}}=\frac{1}{2}\sqrt{2}.$$

10. Divide $\sqrt{a^3}$ by $\sqrt[9]{a^8}$.

$$\frac{\sqrt{a^3}}{\sqrt[9]{a^8}}=\frac{\sqrt[18]{a^{27}}}{\sqrt[18]{a^{16}}}=\sqrt[18]{a^{11}}$$

11. Divide ax^5 by $\sqrt{a^2x}$.

$$\frac{ax^5}{\sqrt{a^2x}}=\frac{ax^5}{a\sqrt{x}}=\frac{x^5}{\sqrt{x}}=x^4\sqrt{x}.$$

12. Divide $\sqrt[4]{8\,a^3b^2}$ by $\sqrt{2\,a^2b^3}$.

$$\frac{\sqrt[4]{8\,a^3b^2}}{\sqrt{2\,a^2b^3}}=\frac{\sqrt[4]{8\,a^3b^2}}{\sqrt[4]{4\,a^4b^6}}=\sqrt[4]{\frac{2}{ab^4}}=\frac{1}{b}\sqrt[4]{\frac{2}{a}}=\frac{1}{ab}\sqrt[4]{2\,a^3}.$$

13. Divide $15\sqrt{105}-36\sqrt{10}+30\sqrt{3}$ by $3\sqrt{15}$.

$$\frac{15\sqrt{105}-36\sqrt{10}+30\sqrt{3}}{3\sqrt{15}}=5\sqrt{7}-12\sqrt{\tfrac{2}{3}}+10\sqrt{\tfrac{1}{5}}$$

$$=5\sqrt{7}-4\sqrt{6}+2\sqrt{5},$$

14. Divide $56\sqrt{30} - 84\sqrt{10} + 100\sqrt{14}$ by $4\sqrt{35}$.

$$\begin{aligned}\frac{56\sqrt{30} - 84\sqrt{10} + 100\sqrt{14}}{4\sqrt{35}} &= 14\sqrt{\tfrac{6}{7}} - 21\sqrt{\tfrac{2}{7}} + 25\sqrt{\tfrac{2}{5}}\\ &= 2\sqrt{42} - 3\sqrt{14} + 5\sqrt{10}.\end{aligned}$$

15. Divide $10\sqrt{14} - 5\sqrt{63} + 4\sqrt{28}$ by $2\sqrt{7}$.

$$\begin{aligned}\frac{10\sqrt{14} - 5\sqrt{63} + 4\sqrt{28}}{2\sqrt{7}} &= 5\sqrt{2} - \tfrac{5}{2}\sqrt{9} + 2\sqrt{4}\\ &= 5\sqrt{2} - \tfrac{15}{2} + 4\\ &= 5\sqrt{2} - 3\tfrac{1}{2}.\end{aligned}$$

16. Divide $18\sqrt{22} + 9\sqrt{66} - 3\sqrt{44}$ by $3\sqrt{11}$.

$$\begin{aligned}\frac{18\sqrt{22} + 9\sqrt{66} - 3\sqrt{44}}{3\sqrt{11}} &= 6\sqrt{2} + 3\sqrt{6} - \sqrt{4}\\ &= 6\sqrt{2} + 3\sqrt{6} - 2.\end{aligned}$$

17. Divide $9\sqrt[3]{9} - 6\sqrt[3]{36} + 4\sqrt[3]{72}$ by $3\sqrt[3]{3}$.

$$9\sqrt[3]{9} - 6\sqrt[3]{36} + 4\sqrt[3]{72} = 9\sqrt[3]{9} - 6\sqrt[3]{36} + 8\sqrt[3]{9} = 17\sqrt[3]{9} - 6\sqrt[3]{36}.$$

$$\frac{17\sqrt[3]{9} - 6\sqrt[3]{36}}{3\sqrt[3]{3}} = \frac{17}{3}\sqrt[3]{3} - 2\sqrt[3]{12}.$$

18. Divide $16\sqrt[3]{25} - 8\sqrt[3]{75} + 4\sqrt[3]{225}$ by $4\sqrt[3]{5}$.

$$\frac{16\sqrt[3]{25} - 8\sqrt[3]{75} + 4\sqrt[3]{225}}{4\sqrt[3]{5}} = 4\sqrt[3]{5} - 2\sqrt[3]{15} + \sqrt[3]{45}.$$

Exercise 102. Page 249

1. Divide $\sqrt{2}$ by $\sqrt{3} - \sqrt{2}$.

$$\frac{\sqrt{2}}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{2}(\sqrt{3} + \sqrt{2})}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})} = \frac{\sqrt{6} + 2}{1} = \sqrt{6} + 2.$$

2. Divide 12 by $4 - \sqrt{7}$.

$$\frac{12}{4 - \sqrt{7}} = \frac{12(4 + \sqrt{7})}{(4 - \sqrt{7})(4 + \sqrt{7})} = \frac{48 + 12\sqrt{7}}{9} = \frac{16}{3} + \frac{4}{3}\sqrt{7}.$$

3. Divide $7-\sqrt{5}$ by $3+\sqrt{5}$.

$$\frac{7-\sqrt{5}}{3+\sqrt{5}}=\frac{(7-\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}=\frac{26-10\sqrt{5}}{4}=\frac{13}{2}-\frac{5}{2}\sqrt{5}.$$

4. Divide $\sqrt{3}+\sqrt{2}$ by $\sqrt{3}-\sqrt{2}$.

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{(\sqrt{3}+\sqrt{2})(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\frac{5+2\sqrt{6}}{1}=5+2\sqrt{6}$$

5. Divide $9-5\sqrt{3}$ by $7-3\sqrt{3}$.

$$\frac{9-5\sqrt{3}}{7-3\sqrt{3}}=\frac{(9-5\sqrt{3})(7+3\sqrt{3})}{(7-3\sqrt{3})(7+3\sqrt{3})}=\frac{18-8\sqrt{3}}{22}=\frac{9-4\sqrt{3}}{11}.$$

6. Divide $a+b\sqrt{x}$ by $c+d\sqrt{x}$.

$$\frac{a+b\sqrt{x}}{c+d\sqrt{x}}=\frac{(a+b\sqrt{x})(c-d\sqrt{x})}{(c+d\sqrt{x})(c-d\sqrt{x})}=\frac{ac-bdx+(bc-ad)\sqrt{x}}{c^2-d^2x}.$$

7. Divide $\sqrt{x}-\sqrt{y}$ by $\sqrt{x}+\sqrt{y}$.

$$\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}=\frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}-\sqrt{y})}{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}=\frac{x+y-2\sqrt{xy}}{x-y}.$$

8. Divide $3+\sqrt{6}$ by $\sqrt{3}+\sqrt{2}$.

$$\frac{3+\sqrt{6}}{\sqrt{3}+\sqrt{2}}=\frac{(3+\sqrt{6})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}=\frac{\sqrt{3}}{1}=\sqrt{3}.$$

9. Divide $5\sqrt{3}-3\sqrt{5}$ by $\sqrt{5}-\sqrt{3}$.

$$\frac{5\sqrt{3}-3\sqrt{5}}{\sqrt{5}-\sqrt{3}}=\frac{(5\sqrt{3}-3\sqrt{5})(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})}=\frac{2\sqrt{15}}{2}=\sqrt{15}.$$

10. Divide $7\sqrt{5}+5\sqrt{7}$ by $\sqrt{7}+\sqrt{5}$.

$$\frac{7\sqrt{5}+5\sqrt{7}}{\sqrt{7}+\sqrt{5}}=\frac{(7\sqrt{5}+5\sqrt{7})(\sqrt{7}-\sqrt{5})}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})}=\frac{2\sqrt{35}}{2}=\sqrt{35}.$$

11. Divide $2\sqrt{3}+\sqrt{6}$ by $\sqrt{3}+\sqrt{6}$.

$$\frac{2\sqrt{3}+\sqrt{6}}{\sqrt{3}+\sqrt{6}}=\frac{(2\sqrt{3}+\sqrt{6})(\sqrt{3}-\sqrt{6})}{(\sqrt{3}+\sqrt{6})(\sqrt{3}-\sqrt{6})}=\frac{-3\sqrt{2}}{-3}=\sqrt{2}.$$

12. Divide $3 + 2\sqrt{x}$ by $5 + 3\sqrt{x}$.

$$\frac{3 + 2\sqrt{x}}{5 + 3\sqrt{x}} = \frac{(3 + 2\sqrt{x})(5 - 3\sqrt{x})}{(5 + 3\sqrt{x})(5 - 3\sqrt{x})} = \frac{15 - 6x + \sqrt{x}}{25 - 9x}.$$

13. Divide $2\sqrt{6}$ by $\sqrt{2} + \sqrt{3} + \sqrt{5}$.

$$\begin{aligned}\frac{2\sqrt{6}}{\sqrt{2} + \sqrt{3} + \sqrt{5}} &= \frac{2\sqrt{6}(\sqrt{2} + \sqrt{3} - \sqrt{5})}{(\sqrt{2} + \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5})}\\ &= \frac{2\sqrt{6}(\sqrt{2} + \sqrt{3} - \sqrt{5})}{2\sqrt{6}} = \sqrt{2} + \sqrt{3} - \sqrt{5}.\end{aligned}$$

14. Divide $2\sqrt{15}$ by $\sqrt{3} + \sqrt{5} + 2\sqrt{2}$.

$$\begin{aligned}\frac{2\sqrt{15}}{\sqrt{3} + \sqrt{5} + 2\sqrt{2}} &= \frac{2\sqrt{15}(\sqrt{3} + \sqrt{5} - 2\sqrt{2})}{(\sqrt{3} + \sqrt{5} + 2\sqrt{2})(\sqrt{3} + \sqrt{5} - 2\sqrt{2})}\\ &= \frac{2\sqrt{15}(\sqrt{3} + \sqrt{5} - 2\sqrt{2})}{2\sqrt{15}}\\ &= \sqrt{3} + \sqrt{5} - 2\sqrt{2}.\end{aligned}$$

15. Divide $a\sqrt{x} - b\sqrt{y}$ by $c\sqrt{x} - d\sqrt{y}$.

$$\begin{aligned}\frac{a\sqrt{x} - b\sqrt{y}}{c\sqrt{x} - d\sqrt{y}} &= \frac{(a\sqrt{x} - b\sqrt{y})(c\sqrt{x} + d\sqrt{y})}{(c\sqrt{x} - d\sqrt{y})(c\sqrt{x} + d\sqrt{y})}\\ &= \frac{acx - bdy + (ad - bc)\sqrt{xy}}{c^2x - d^2y}.\end{aligned}$$

16. Divide $25\sqrt{14} - 4\sqrt{2}$ by $8\sqrt{2} + 2\sqrt{7}$.

$$\begin{aligned}\frac{25\sqrt{14} - 4\sqrt{2}}{8\sqrt{2} + 2\sqrt{7}} &= \frac{(25\sqrt{14} - 4\sqrt{2})(8\sqrt{2} - 2\sqrt{7})}{(8\sqrt{2} + 2\sqrt{7})(8\sqrt{2} - 2\sqrt{7})}\\ &= \frac{400\sqrt{7} - 350\sqrt{2} + 8\sqrt{14} - 64}{100}\\ &= 4\sqrt{7} - \tfrac{7}{2}\sqrt{2} + \tfrac{2}{25}\sqrt{14} - \tfrac{16}{25}.\end{aligned}$$

17. Divide $1+3\sqrt{2}-2\sqrt{3}$ by $\sqrt{6}+\sqrt{3}+\sqrt{2}$.

$$\begin{aligned}\frac{1+3\sqrt{2}-2\sqrt{3}}{\sqrt{6}+\sqrt{3}+\sqrt{2}} &= \frac{(1+3\sqrt{2}-2\sqrt{3})(\sqrt{6}+\sqrt{3}-\sqrt{2})}{(\sqrt{6}+\sqrt{3}+\sqrt{2})(\sqrt{6}+\sqrt{3}-\sqrt{2})}\\ &= \frac{6\sqrt{6}+7\sqrt{3}-7\sqrt{2}-12}{7+6\sqrt{2}}\\ &= \frac{(6\sqrt{6}+7\sqrt{3}-7\sqrt{2}-12)(7-6\sqrt{2})}{(7+6\sqrt{2})(7-6\sqrt{2})}\\ &= \frac{23\sqrt{2}-23\sqrt{3}}{-23}\\ &= \sqrt{3}-\sqrt{2}.\end{aligned}$$

18. Divide $3-\sqrt{5}-\sqrt{2}$ by $3+\sqrt{5}+\sqrt{2}$.

$$\begin{aligned}\frac{3-\sqrt{5}-\sqrt{2}}{3+\sqrt{5}+\sqrt{2}} &= \frac{(3-\sqrt{5}-\sqrt{2})(3+\sqrt{5}-\sqrt{2})}{(3+\sqrt{5}+\sqrt{2})(3+\sqrt{5}-\sqrt{2})}\\ &= \frac{6-6\sqrt{2}}{12+6\sqrt{5}} = \frac{1-\sqrt{2}}{2+\sqrt{5}}\\ &= \frac{(1-\sqrt{2})(2-\sqrt{5})}{(2+\sqrt{5})(2-\sqrt{5})}\\ &= \frac{2-2\sqrt{2}-\sqrt{5}+\sqrt{10}}{-1}\\ &= 2\sqrt{2}+\sqrt{5}-\sqrt{10}-2.\end{aligned}$$

19. Divide $60\sqrt{2}+12\sqrt{3}$ by $5\sqrt{6}+3\sqrt{2}-2\sqrt{3}$.

$$\begin{aligned}\frac{60\sqrt{2}+12\sqrt{3}}{5\sqrt{6}+3\sqrt{2}-2\sqrt{3}} &= \frac{12(5\sqrt{2}+\sqrt{3})(5\sqrt{6}+3\sqrt{2}+2\sqrt{3})}{(5\sqrt{6}+3\sqrt{2}-2\sqrt{3})(5\sqrt{6}+3\sqrt{2}+2\sqrt{3})}\\ &= \frac{12(50\sqrt{3}+15\sqrt{2}+13\sqrt{6}+36)}{156+60\sqrt{3}}\\ &= \frac{50\sqrt{3}+15\sqrt{2}+13\sqrt{6}+36}{13+5\sqrt{3}}\\ &= \frac{(50\sqrt{3}+15\sqrt{2}+13\sqrt{6}+36)(13-5\sqrt{3})}{(13+5\sqrt{3})(13-5\sqrt{3})}\\ &= \frac{470\sqrt{3}+94\sqrt{6}-282}{94}\\ &= 5\sqrt{3}+\sqrt{6}-3.\end{aligned}$$

20. Divide $7 - 2\sqrt{3} + 3\sqrt{2}$ by $3 + 3\sqrt{3} - 2\sqrt{2}$.

$$\frac{7 - 2\sqrt{3} + 3\sqrt{2}}{3 + 3\sqrt{3} - 2\sqrt{2}} = \frac{(7 - 2\sqrt{3} + 3\sqrt{2})(3 + 3\sqrt{3} + 2\sqrt{2})}{(3 + 3\sqrt{3} - 2\sqrt{2})(3 + 3\sqrt{3} + 2\sqrt{2})}$$

$$= \frac{15 + 15\sqrt{3} + 23\sqrt{2} + 5\sqrt{6}}{28 + 18\sqrt{3}}$$

$$= \frac{(15 + 15\sqrt{3} + 23\sqrt{2} + 5\sqrt{6})(28 - 18\sqrt{3})}{(28 + 18\sqrt{3})(28 - 18\sqrt{3})}$$

$$= \frac{-390 + 150\sqrt{3} + 374\sqrt{2} - 274\sqrt{6}}{-188}$$

$$= \frac{195 - 75\sqrt{3} - 187\sqrt{2} + 137\sqrt{6}}{94}.$$

21. Divide $2 + 3\sqrt{6} - 4\sqrt{2}$ by $2 - \sqrt{6} + 2\sqrt{2}$.

$$\frac{2 + 3\sqrt{6} - 4\sqrt{2}}{2 - \sqrt{6} + 2\sqrt{2}} = \frac{(2 + 3\sqrt{6} - 4\sqrt{2})(2 + \sqrt{6} + 2\sqrt{2})}{(2 - \sqrt{6} + 2\sqrt{2})(2 + \sqrt{6} + 2\sqrt{2})}$$

$$= \frac{6 + 8\sqrt{6} - 4\sqrt{2} + 4\sqrt{3}}{6 + 8\sqrt{2}}$$

$$= \frac{3 + 4\sqrt{6} - 2\sqrt{2} + 2\sqrt{3}}{3 + 4\sqrt{2}}$$

$$= \frac{(3 + 4\sqrt{6} - 2\sqrt{2} + 2\sqrt{3})(3 - 4\sqrt{2})}{(3 + 4\sqrt{2})(3 - 4\sqrt{2})}$$

$$= \frac{25 + 4\sqrt{6} - 18\sqrt{2} - 26\sqrt{3}}{-23}$$

$$= \frac{18\sqrt{2} + 26\sqrt{3} - 4\sqrt{6} - 25}{23}.$$

22. Divide $2\sqrt{3} - 3\sqrt{6}$ by $2 + \sqrt{2} + \sqrt{3} + \sqrt{6}$.

$$\frac{2\sqrt{3} - 3\sqrt{6}}{2 + \sqrt{2} + \sqrt{3} + \sqrt{6}} = \frac{2\sqrt{3} - 3\sqrt{6}}{(\sqrt{2} + 1)(\sqrt{2} + \sqrt{3})}$$

$$= \frac{(2\sqrt{3} - 3\sqrt{6})(\sqrt{2} - 1)(\sqrt{2} - \sqrt{3})}{(\sqrt{2} + 1)(\sqrt{2} - 1)(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})}$$

$$= \frac{-15\sqrt{2} + 10\sqrt{3} - 8\sqrt{6} + 24}{-1}$$

$$= 15\sqrt{2} - 10\sqrt{3} + 8\sqrt{6} - 24.$$

Exercise 103. Page 250

Perform the operations indicated :

1. $(\sqrt[4]{a})^5 = (a^{\frac{1}{4}})^5 = a^{\frac{5}{4}} = \sqrt[4]{a^5} = a\sqrt[4]{a}.$

2. $(\sqrt[8]{x^2y^3})^7 = (x^{\frac{2}{8}}y^{\frac{3}{8}})^7 = x^{\frac{14}{8}}y^{\frac{21}{8}} = \sqrt[8]{x^{14}y^{21}} = xy^2\sqrt[8]{x^6y^5}.$

3. $(\sqrt[3]{2a^4})^2 = (2^{\frac{1}{3}}a^{\frac{4}{3}})^2 = 2^{\frac{2}{3}}a^{\frac{8}{3}} = \sqrt[3]{2^2a^8} = a^2\sqrt[3]{4a^2}.$

4. $(\sqrt[3]{ax^4})^6 = (a^{\frac{1}{3}}x^{\frac{4}{3}})^6 = a^2x^8.$

5. $(-\sqrt[3]{xy^2})^3 = (-x^{\frac{1}{3}}y^{\frac{2}{3}})^3 = -xy^2.$

6. $(-\sqrt[3]{4})^6 = (-2^{\frac{2}{3}})^6 = (-2)^4 = 16.$

7. $(-\sqrt[4]{127})^4 = (-127^{\frac{1}{4}})^4 = (-127)^1 = -127.$

8. $(-\sqrt[3]{9})^4 = (-3^{\frac{2}{3}})^4 = (-3)^{\frac{8}{3}} = \sqrt[3]{(-3)^8} = \sqrt[3]{3^8} = 3^2\sqrt[3]{3^2} = 9\sqrt[3]{9}.$

9. $(\sqrt[8]{a})^4 = (a^{\frac{1}{8}})^4 = a^{\frac{1}{2}} = \sqrt{a}.$

10. $\sqrt[3]{\sqrt{b}} = (b^{\frac{1}{2}})^{\frac{1}{3}} = b^{\frac{1}{6}} = \sqrt[6]{b}.$

11. $\sqrt[4]{\sqrt[3]{a}} = (a^{\frac{1}{3}})^{\frac{1}{4}} = a^{\frac{1}{12}} = \sqrt[12]{a}.$

12. $\sqrt[6]{\sqrt[3]{a^6}} = [(a^6)^{\frac{1}{3}}]^{\frac{1}{6}} = a^{\frac{1}{3}} = \sqrt[3]{a}.$

13. $\sqrt[5]{\sqrt[6]{2a}} = [(2a)^{\frac{1}{6}}]^{\frac{1}{5}} = (2a)^{\frac{1}{30}} = 2^{\frac{1}{30}}a^{\frac{1}{30}} = \sqrt[30]{2a}.$

14. $\sqrt[4]{\sqrt{3a^2}} = [(3a^2)^{\frac{1}{2}}]^{\frac{1}{4}} = (3a^2)^{\frac{1}{8}} = 3^{\frac{1}{8}}a^{\frac{2}{8}} = \sqrt[8]{3a^2}.$

15. $\sqrt[5]{\sqrt[3]{a^{10}}} = [(a^{10})^{\frac{1}{3}}]^{\frac{1}{5}} = (a^{\frac{10}{3}})^{\frac{1}{5}} = a^{\frac{2}{3}} = \sqrt[3]{a^2}.$

16. $\sqrt[4]{\sqrt[3]{a^5}} = [(a^5)^{\frac{1}{3}}]^{\frac{1}{4}} = (a^{\frac{5}{3}})^{\frac{1}{4}} = a^{\frac{5}{12}} = \sqrt[12]{a^5}.$

17. $\sqrt[5]{\sqrt[3]{(m+n)^{30}}} = [\{(m+n)^{30}\}^{\frac{1}{3}}]^{\frac{1}{5}} = [(m+n)^{10}]^{\frac{1}{5}} = (m+n)^2.$

18. $\sqrt{\sqrt[5]{(1-x^2)^{20}}} = [\{(1-x^2)^{20}\}^{\frac{1}{5}}]^{\frac{1}{2}} = [(1-x^2)^4]^{\frac{1}{2}} = (1-x^2)^2.$

19. $\sqrt[6]{\sqrt[3]{512a^{18}}} = [(512a^{18})^{\frac{1}{3}}]^{\frac{1}{6}} = [(2^9a^{18})^{\frac{1}{3}}]^{\frac{1}{6}} = (2^3a^6)^{\frac{1}{6}} = 2^{\frac{1}{2}}a = a\sqrt{2}.$

20. $\sqrt[13]{6\sqrt[5]{a^{65}}} = [6(a^{65})^{\frac{1}{5}}]^{\frac{1}{13}} = (6a^{13})^{\frac{1}{13}} = 6^{\frac{1}{13}}a = a\sqrt[13]{6}.$

21. $\sqrt[11]{\sqrt[3]{a^{33}b^{22}}} = [(a^{33}b^{22})^{\frac{1}{3}}]^{\frac{1}{11}} = (a^{11}b^{\frac{22}{3}})^{\frac{1}{11}} = ab^{\frac{2}{3}} = a\sqrt[3]{b^2}.$

22. $\sqrt[5]{a\sqrt[3]{b^2}} = [a(b^2)^{\frac{1}{3}}]^{\frac{1}{5}} = (ab^{\frac{2}{3}})^{\frac{1}{5}} = a^{\frac{1}{5}}b^{\frac{2}{15}} = a^{\frac{3}{15}}b^{\frac{2}{15}} = \sqrt[15]{a^3b^2}.$

23. $\sqrt[5]{a\sqrt[8]{b}} = (ab^{\frac{1}{8}})^{\frac{1}{5}} = a^{\frac{1}{5}}b^{\frac{1}{40}} = a^{\frac{8}{40}}b^{\frac{1}{40}} = \sqrt[40]{a^8b}.$

24. $a\sqrt[4]{a^{-3}\sqrt[4]{a^{-3}}} = a[a^{-3}(a^{-3})^{\frac{1}{4}}]^{\frac{1}{4}} = a(a^{-3}\times a^{-\frac{3}{4}})^{\frac{1}{4}} = a(a^{-\frac{15}{4}})^{\frac{1}{4}}$
$= a\times a^{-\frac{15}{16}} = a^{\frac{1}{16}} = \sqrt[16]{a}.$

Exercise 104. Page 252

1. Find the square root of $17 + 12\sqrt{2}$.

$17 + 12\sqrt{2} = 17 + 2\sqrt{6^2 \times 2} = 17 + 2\sqrt{2^3 \times 3^2}$.

Two numbers whose sum is 17 and product $2^3 \times 3^2$ are 9 and 8.

$\therefore\ 17 + 12\sqrt{2} = 9 + 2\sqrt{9 \times 8} + 8 = (\sqrt{9} + \sqrt{8})^2 = (3 + 2\sqrt{2})^2$.

Therefore, the square root of $17 + 12\sqrt{2}$ is $3 + 2\sqrt{2}$.

2. Find the square root of $52 - 30\sqrt{3}$.

$52 - 30\sqrt{3} = 52 - 2\sqrt{15^2 \times 3} = 52 - 2\sqrt{3^3 \times 5^2}$.

Two numbers whose sum is 52 and product $3^3 \times 5^2$ are 25 and 27.

$\therefore\ 52 - 30\sqrt{3} = 25 - 2\sqrt{25 \times 27} + 27 = (\sqrt{25} - \sqrt{27})^2 = (5 - 3\sqrt{3})^2$.

Therefore, the square root of $52 - 30\sqrt{3}$ is $5 - 3\sqrt{3}$.

3. Find the square root of $102 - 28\sqrt{2}$.

$102 - 28\sqrt{2} = 102 - 2\sqrt{14^2 \times 2} = 102 - 2\sqrt{2^3 \times 7^2}$.

Two numbers whose sum is 102 and product $2^3 \times 7^2$ are 4 and 98.

$\therefore\ 102 - 28\sqrt{2} = 4 - 2\sqrt{4 \times 98} + 98 = (\sqrt{4} - \sqrt{98})^2 = (2 - 7\sqrt{2})^2$.

Therefore, the square root of $102 - 28\sqrt{2}$ is $2 - 7\sqrt{2}$.

4. Find the square root of $30 + 12\sqrt{6}$.

$30 + 12\sqrt{6} = 30 + 2\sqrt{6^2 \times 6} = 30 + 2\sqrt{2^3 \times 3^3}$.

Two numbers whose sum is 30 and product $2^3 \times 3^3$ are 12 and 18.

$\therefore\ 30 + 12\sqrt{6} = 12 + 2\sqrt{12 \times 18} + 18 = (\sqrt{12} + \sqrt{18})^2 = (2\sqrt{3} + 3\sqrt{2})^2$.

Therefore, the square root of $30 + 12\sqrt{6}$ is $2\sqrt{3} + 3\sqrt{2}$.

5. Find the square root of $220 - 30\sqrt{35}$.

$220 - 30\sqrt{35} = 220 - 2\sqrt{15^2 \times 35} = 220 - 2\sqrt{3^2 \times 5^3 \times 7}$.

Two numbers whose sum is 220 and product $3^2 \times 5^3 \times 7$ are 45 and 175.

$$\begin{aligned}\therefore\ 220 - 30\sqrt{35} &= 45 - 2\sqrt{45 \times 175} + 175 = (\sqrt{45} - \sqrt{175})^2\\ &= (3\sqrt{5} - 5\sqrt{7})^2.\end{aligned}$$

Therefore, the square root of $220 - 30\sqrt{35}$ is $3\sqrt{5} - 5\sqrt{7}$.

6. Find the square root of $23 - 4\sqrt{15}$.

$23 - 4\sqrt{15} = 23 - 2\sqrt{2^2 \times 15} = 23 - 2\sqrt{2^2 \times 3 \times 5}$.

Two numbers whose sum is 23 and product $2^2 \times 3 \times 5$ are 20 and 3.

$\therefore\ 23 - 4\sqrt{15} = 20 - 2\sqrt{20 \times 3} + 3 = (\sqrt{20} - \sqrt{3})^2 = (2\sqrt{5} - \sqrt{3})^2$.

Therefore, the square root of $23 - 4\sqrt{15}$ is $2\sqrt{5} - \sqrt{3}$.

7. Find the square root of $57 - 12\sqrt{15}$.

$57 - 12\sqrt{15} = 57 - 2\sqrt{6^2 \times 15} = 57 - 2\sqrt{2^2 \times 3^3 \times 5}$.

Two numbers whose sum is 57 and product $2^2 \times 3^3 \times 5$ are 12 and 45.

$$\therefore\ 57 - 12\sqrt{15} = 12 - 2\sqrt{12 \times 45} + 45 = (\sqrt{12} - \sqrt{45})^2$$
$$= (2\sqrt{3} - 3\sqrt{5})^2.$$

Therefore, the square root of $57 - 12\sqrt{15}$ is $2\sqrt{3} - 3\sqrt{5}$.

8. Find the square root of $95 - 24\sqrt{14}$.

$95 - 24\sqrt{14} = 95 - 2\sqrt{12^2 \times 14} = 95 - 2\sqrt{2^5 \times 3^2 \times 7}$.

Two numbers whose sum is 95 and product $2^5 \times 3^2 \times 7$ are 63 and 32.

$$\therefore\ 95 - 24\sqrt{14} = 63 - 2\sqrt{63 \times 32} + 32 = (\sqrt{63} - \sqrt{32})^2$$
$$= (3\sqrt{7} - 4\sqrt{2})^2.$$

Therefore, the square root of $95 - 24\sqrt{14} = 3\sqrt{7} - 4\sqrt{2}$.

9. Find the square root of $314 - 28\sqrt{30}$.

$314 - 28\sqrt{30} = 314 - 2\sqrt{14^2 \times 30} = 314 - 2\sqrt{2^3 \times 3 \times 5 \times 7^2}$.

Two numbers whose sum is 314 and product $2^3 \times 3 \times 5 \times 7^2$ are 20 and 294.

$$\therefore\ 314 - 28\sqrt{30} = 20 - 2\sqrt{20 \times 294} + 294 = (\sqrt{20} + \sqrt{294})^2$$
$$= (2\sqrt{5} - 7\sqrt{6})^2.$$

Therefore, the square root of $314 - 28\sqrt{30}$ is $2\sqrt{5} - 7\sqrt{6}$.

10. Find the square root of $62 - 24\sqrt{3}$.

$62 - 24\sqrt{3} = 62 - 2\sqrt{12^2 \times 3} = 62 - 2\sqrt{2^4 \times 3^3}$.

Two numbers whose sum is 62 and product $2^4 \times 3^3$ are 54 and 8.

$\therefore\ 62 - 24\sqrt{3} = 54 - 2\sqrt{54 \times 8} + 8 = (\sqrt{54} - \sqrt{8})^2 = (3\sqrt{6} - 2\sqrt{2})^2$.

Therefore, the square root of $62 - 24\sqrt{3}$ is $3\sqrt{6} - 2\sqrt{2}$.

11. Find the square root of $68 + 16\sqrt{15}$.

$68 + 16\sqrt{15} = 68 + 2\sqrt{8^2 \times 15} = 68 + 2\sqrt{2^6 \times 3 \times 5}.$

Two numbers whose sum is 68 and product $2^6 \times 3 \times 5$ are 20 and 48.

$\therefore\ 68 + 16\sqrt{15} = 20 + 2\sqrt{20 \times 48} + 48 = (\sqrt{20} + \sqrt{48})^2$
$= (2\sqrt{5} + 4\sqrt{3})^2.$

Therefore, the square root of $68 + 16\sqrt{15}$ is $2\sqrt{5} + 4\sqrt{3}$.

12. Find the square root of $139 + 24\sqrt{21}$.

$139 + 24\sqrt{21} = 139 + 2\sqrt{12^2 \times 21} = 139 + 2\sqrt{2^4 \times 3^3 \times 7}.$

Two numbers whose sum is 139 and product $2^4 \times 3^3 \times 7$ are 27 and 112.

$\therefore\ 139 + 24\sqrt{21} = 27 + 2\sqrt{27 \times 112} + 112 = (\sqrt{27} + \sqrt{112})^2$
$= (3\sqrt{3} + 4\sqrt{7})^2.$

Therefore, the square root of $139 + 24\sqrt{21}$ is $3\sqrt{3} + 4\sqrt{7}$.

13. Find the square root of $66 - 36\sqrt{2}$.

$66 - 36\sqrt{2} = 66 - 2\sqrt{18^2 \times 2} = 66 - 2\sqrt{2^3 \times 3^4}.$

Two numbers whose sum is 66 and product $2^3 \times 3^4$ are 12 and 54.

$\therefore\ 66 - 36\sqrt{2} = 12 - 2\sqrt{12 \times 54} + 54 = (\sqrt{12} - \sqrt{54})^2$
$= (2\sqrt{3} - 3\sqrt{6})^2.$

Therefore, the square root of $66 - 36\sqrt{2}$ is $2\sqrt{3} - 3\sqrt{6}$.

14. Find the square root of $69 - 28\sqrt{5}$.

$69 - 28\sqrt{5} = 69 - 2\sqrt{14^2 \times 5} = 69 - 2\sqrt{2^2 \times 5 \times 7^2}.$

Two numbers whose sum is 69 and product $2^2 \times 5 \times 7^2$ are 49 and 20.

$\therefore\ 69 - 28\sqrt{5} = 49 - 2\sqrt{49 \times 20} + 20 = (\sqrt{49} - \sqrt{20})^2 = (7 - 2\sqrt{5})^2.$

Therefore, the square root of $69 - 28\sqrt{5}$ is $7 - 2\sqrt{5}$.

15. Find the square root of $207 + 40\sqrt{14}$.

$207 + 40\sqrt{14} = 207 + 2\sqrt{20^2 \times 14} = 207 + 2\sqrt{2^5 \times 5^2 \times 7}.$

Two numbers whose sum is 207 and product $2^5 \times 5^2 \times 7$ are 175 and 32.

$\therefore\ 207 + 40\sqrt{14} = 175 + 2\sqrt{175 \times 32} + 32 = (\sqrt{175} + \sqrt{32})^2$
$= (5\sqrt{7} + 4\sqrt{2})^2.$

Therefore, the square root of $207 + 40\sqrt{14}$ is $5\sqrt{7} + 4\sqrt{2}$.

16. Find the square root of $180 + 80\sqrt{2}$.

$$180 + 80\sqrt{2} = 180 + 2\sqrt{40^2 \times 2} = 180 + 2\sqrt{2^7 \times 5^2}.$$

Two numbers whose sum is 180 and product $2^7 \times 5^2$ are 20 and 160.

$$\therefore\ 180 + 80\sqrt{2} = 20 + 2\sqrt{20 \times 160} + 160 = (\sqrt{20} + \sqrt{160})^2$$
$$= (2\sqrt{5} + 4\sqrt{10})^2.$$

Therefore, the square root of $180 + 80\sqrt{2}$ is $2\sqrt{5} + 4\sqrt{10}$.

17. Find the square root of $735 + 300\sqrt{6}$.

$$735 + 300\sqrt{6} = 735 + 2\sqrt{150^2 \times 6} = 735 + 2\sqrt{2^3 \times 3^3 \times 5^4}.$$

Two numbers whose sum is 735 and product $2^3 \times 3^3 \times 5^4$ are 360 and 375.

$$\therefore\ 735 + 300\sqrt{6} = 360 + 2\sqrt{360 \times 375} + 375 = (\sqrt{360} + \sqrt{375})^2$$
$$= (6\sqrt{10} + 5\sqrt{15}).$$

Therefore, the square root of $735 + 300\sqrt{6}$ is $6\sqrt{10} + 5\sqrt{15}$.

18. Find the square root of $162 + 108\sqrt{2}$.

$$162 + 108\sqrt{2} = 162 + 2\sqrt{54^2 \times 2} = 162 + 2\sqrt{2^3 \times 3^6}.$$

Two numbers whose sum is 162 and product $2^3 \times 3^6$ are 54 and 108.

$$\therefore\ 162 + 108\sqrt{2} = 54 + 2\sqrt{54 \times 108} + 108 = (\sqrt{54} + \sqrt{108})^2$$
$$= (3\sqrt{6} + 6\sqrt{3})^2.$$

Therefore, the square root of $162 + 108\sqrt{2}$ is $3\sqrt{6} + 6\sqrt{3}$.

Exercise 105. Page 253

1. Solve $9 + 4\sqrt{x} = 11$.

Transpose and combine, $4\sqrt{x} = 2$.

Divide by 2, $2\sqrt{x} = 1$.

Square, $4x = 1$.

$\therefore\ x = \frac{1}{4}$.

2. Solve $6 - 3\sqrt{x} = 4$.

Transpose and combine, $3\sqrt{x} = 2$.

Square, $9x = 4$.

$\therefore\ x = \frac{4}{9}$.

3. Solve $\sqrt[5]{80x - 43} = -3.$

Raise to the fifth power, $80x - 43 = -243.$

Transpose and combine, $80x = -200.$

$\therefore x = -2\frac{1}{2}.$

4. Solve $3\sqrt{16x - 9} = 6\sqrt{4x} - 9.$

Divide by 3, $\sqrt{16x - 9} = 2\sqrt{4x} - 3.$

Square, $16x - 9 = 16x - 12\sqrt{4x} + 9.$

Transpose and combine, $12\sqrt{4x} = 18.$

Divide by 6, $2\sqrt{4x} = 3.$

Square, $16x = 9.$

$\therefore x = \frac{9}{16}.$

5. Solve $\sqrt{22 - x} - 2 = \sqrt{10 - x}.$

Square, $22 - x - 4\sqrt{22 - x} + 4 = 10 - x.$

Transpose and combine, $4\sqrt{22 - x} = 16.$

Divide by 4, $\sqrt{22 - x} = 4.$

Square, $22 - x = 16.$

$\therefore x = 6.$

6. Solve $\sqrt{12 + x} = 6 - \sqrt{x}.$

Square, $12 + x = 36 - 12\sqrt{x} + x.$

Transpose and combine, $12\sqrt{x} = 24.$

Divide by 12, $\sqrt{x} = 2.$

$\therefore x = 4.$

7. Solve $7\sqrt{3x} - 1 = 5\sqrt{3x} + 5.$

Transpose and combine, $2\sqrt{3x} = 6.$

Divide by 2, $\sqrt{3x} = 3.$

Square, $3x = 9.$

$\therefore x = 3.$

8. Solve $a\sqrt{x} - b = c\sqrt{x} - d.$

Transpose and combine, $(a - c)\sqrt{x} = b - d.$

Divide by $a - c$, $\sqrt{x} = \frac{b - d}{a - c}.$

Square, $x = \frac{(b - d)^2}{(a - c)^2}.$

9. Solve $\sqrt{3x} - \sqrt{2x} = 1.$

Square, $3x - 2\sqrt{6}x + 2x = 1.$

Combine, $(5 - 2\sqrt{6})x = 1.$

$$\therefore x = \frac{1}{5 - 2\sqrt{6}} = 5 + 2\sqrt{6}.$$

10. Solve $\sqrt{3x} - \sqrt{x} = 2.$

Square, $3x - 2\sqrt{3}x + x = 4.$

Combine, $(4 - 2\sqrt{3})x = 4.$

Divide by 2, $(2 - \sqrt{3})x = 2.$

$$\therefore x = \frac{2}{2 - \sqrt{3}} = 4 + 2\sqrt{3}.$$

11. Solve $\sqrt{ax} + \sqrt{bx} = c.$

Factor, $\sqrt{x}(\sqrt{a} + \sqrt{b}) = c.$

$$\therefore \sqrt{x} = \frac{c}{\sqrt{a} + \sqrt{b}}.$$

Square,

$$x = \frac{c^2}{a + 2\sqrt{ab} + b}$$

$$= \frac{c^2(a + b - 2\sqrt{ab})}{a^2 - 2ab + b^2}.$$

12. Solve $\sqrt{x} + \sqrt{2x} = 1.$

Factor, $\sqrt{x}(1 + \sqrt{2}) = 1.$

$$\therefore \sqrt{x} = \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1.$$

Square, $x = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}.$

13. Solve $5\sqrt{x} + \sqrt{3x} = 22.$

Factor, $\sqrt{x}(5 + \sqrt{3}) = 22.$

$$\therefore \sqrt{x} = \frac{22}{5 + \sqrt{3}} = 5 - \sqrt{3}.$$

Square, $x = (5 - \sqrt{3})^2 = 28 - 10\sqrt{3}.$

14. Solve $2\sqrt{x} - \sqrt{2x} = 2 + \sqrt{2}.$

Factor, $\sqrt{x}(2 - \sqrt{2}) = 2 + \sqrt{2}.$

$$\therefore \sqrt{x} = \frac{2 + \sqrt{2}}{2 - \sqrt{2}} = 3 + 2\sqrt{2}.$$

Square, $x = (3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2}.$

15. **Solve** $b(1+\sqrt{x})=a(1-\sqrt{x}).$

Simplify, $b+b\sqrt{x}=a-a\sqrt{x}.$

Transpose and combine, $(a+b)\sqrt{x}=a-b.$

$\therefore\ \sqrt{x}=\frac{a-b}{a+b}.$

Square, $x=\frac{(a-b)^2}{(a+b)^2}.$

16. **Solve** $7+\sqrt{x^2-11x+4}=x.$

Transpose, $\sqrt{x^2-11x+4}=x-7.$

Square, $x^2-11x+4=x^2-14x+49.$

Transpose and combine, $3x=45.$

$\therefore\ x=15.$

17. Solve $\frac{2}{3}(7\sqrt{x}+5)-5=\frac{3}{2}(3\sqrt{x}-1).$

Multiply by 6, $28\sqrt{x}+20-30=27\sqrt{x}-9.$

Transpose and combine, $\sqrt{x}=1.$

Square, $x=1.$

18. Solve $\frac{1}{2}(16-\sqrt{x})-\frac{1}{3}(10-\sqrt{x})=\sqrt{x}.$

Multiply by 6, $48-3\sqrt{x}-20+2\sqrt{x}=6\sqrt{x}.$

Transpose and combine, $7\sqrt{x}=28.$

$\therefore\ \sqrt{x}=4.$

Square, $x=16.$

19. Solve $\sqrt{x+60}=2\sqrt{x+5}+\sqrt{x}.$

Square, $x+60=4x+20+4\sqrt{x^2+5x}+x.$

Transpose and combine, $4\sqrt{x^2+5x}=-4x+40.$

Divide by 4, $\sqrt{x^2+5x}=-x+10.$

Square, $x^2+5x=x^2-20x+100.$

Transpose and combine, $25x=100.$

$\therefore\ x=4.$

20. Solve $\sqrt{9x+7}+\sqrt{4x+1}=\sqrt{25x+14}.$

Square, $9x+7+2\sqrt{36x^2+37x+7}+4x+1=25x+14.$

Transpose and combine, $2\sqrt{36x^2+37x+7}=12x+6.$

Divide by 2, $\sqrt{36x^2+37x+7}=6x+3.$

Square, $36x^2+37x+7=36x^2+36x+9.$

Transpose and combine, $x=2.$

21. Solve $\sqrt{4x+9}-\sqrt{x-1}=\sqrt{x+6}$.

Square, $4x+9-2\sqrt{4x^2+5x-9}+x-1=x+6$.

Transpose and combine, $2\sqrt{4x^2+5x-9}=4x+2$.

Divide by 2, $\sqrt{4x^2+5x-9}=2x+1$.

Square, $4x^2+5x-9=4x^2+4x+1$.

Transpose and combine, $x=10$.

22. Solve $2\sqrt{x+5}+3\sqrt{x-7}=\sqrt{25x-79}$.

Square, $4x+20+12\sqrt{x^2-2x-35}+9x-63=25x-79$.

Transpose and combine, $12\sqrt{x^2-2x-35}=12x-36$.

Divide by 12, $\sqrt{x^2-2x-35}=x-3$.

Square, $x^2-2x-35=x^2-6x+9$.

Transpose and combine, $4x=44$.

$\therefore x=11$.

23. Solve $3\sqrt{x+3}-2\sqrt{x-12}=5\sqrt{x-9}$.

Square, $9x+27-12\sqrt{x^2-9x-36}+4x-48=25x-225$.

Transpose and combine, $12\sqrt{x^2-9x-36}=204-12x$.

Divide by 12, $\sqrt{x^2-9x-36}=17-x$.

Square, $x^2-9x-36=289-34x+x^2$.

Transpose and combine, $25x=325$.

$\therefore x=13$.

24. Solve $\sqrt{x-9}+\sqrt{x+12}=\sqrt{x-4}+\sqrt{x+3}$.

Square,

$$x-9+2\sqrt{x^2+3x-108}+x+12=x-4+2\sqrt{x^2-x-12}+x+3.$$

Transpose and combine, $2\sqrt{x^2+3x-108}=2\sqrt{x^2-x-12}-4$.

Divide by 2, $\sqrt{x^2+3x-108}=\sqrt{x^2-x-12}-2$.

Square, $x^2+3x-108=x^2-x-12-4\sqrt{x^2-x-12}+4$.

Transpose and combine, $4\sqrt{x^2-x-12}=100-4x$.

Divide by 4, $\sqrt{x^2-x-12}=25-x$.

Square, $x^2-x-12=625-50x+x^2$.

Transpose and combine, $49x=637$.

$\therefore x=13$.

25. Solve $\sqrt{x-7}+\sqrt{x-2}-\sqrt{x-10}=\sqrt{x+5}.$

Transpose, $\sqrt{x-7}+\sqrt{x-2}=\sqrt{x+5}+\sqrt{x-10}.$

Square,

$$x-7+2\sqrt{x^2-9x+14}+x-2=x+5+2\sqrt{x^2-5x-50}+x-10.$$

Transpose and combine, $2\sqrt{x^2-9x+14}-4=2\sqrt{x^2-5x-50}.$

Divide by 2, $\sqrt{x^2-9x+14}-2=\sqrt{x^2-5x-50}.$

Square, $x^2-9x+14-4\sqrt{x^2-9x+14}+4=x^2-5x-50.$

Transpose and combine, $68-4x=4\sqrt{x^2-9x+14}.$

Divide by 4, $17-x=\sqrt{x^2-9x+14}.$

Square, $289-34x+x^2=x^2-9x+14.$

Transpose and combine, $25x=275.$

$\therefore x=11.$

26. Solve $\sqrt{x+15}+\sqrt{x-24}-\sqrt{x-13}=\sqrt{x}.$

Transpose, $\sqrt{x+15}+\sqrt{x-24}=\sqrt{x}+\sqrt{x-13}.$

Square, $x+15+2\sqrt{x^2-9x-360}+x-24=x+2\sqrt{x^2-13x}+x-13.$

Transpose and combine, $2\sqrt{x^2-9x-360}=2\sqrt{x^2-13x}-4.$

Divide by 2, $\sqrt{x^2-9x-360}=\sqrt{x^2-13x}-2.$

Square, $x^2-9x-360=x^2-13x-4\sqrt{x^2-13x}+4.$

Transpose and combine, $4\sqrt{x^2-13x}=364-4x.$

Divide by 4, $\sqrt{x^2-13x}=91-x.$

Square, $x^2-13x=8281-182x+x^2.$

Transpose and combine, $169x=8281.$

$\therefore x=49.$

Exercise 106. Page 260

Reduce to the typical form $a\sqrt{-1}$:

1. $\sqrt{-9}=\sqrt{9\times(-1)}=3\sqrt{-1}.$
2. $\sqrt{-25}=\sqrt{25\times(-1)}=5\sqrt{-1}.$
3. $\sqrt{-36}=\sqrt{36\times(-1)}=6\sqrt{-1}.$
4. $2\sqrt{-49}=2\sqrt{49\times(-1)}=2\times 7\sqrt{-1}=14\sqrt{-1}.$
5. $3\sqrt{-625a^2}=3\sqrt{625a^2(-1)}=3\times 25a\sqrt{-1}=75a\sqrt{-1}.$
6. $2\sqrt{-81x^2y^2}=2\sqrt{81x^2y^2(-1)}=2\times 9xy\sqrt{-1}=18xy\sqrt{-1}.$
7. $\sqrt{-x^2-y^2}=\sqrt{(x^2+y^2)(-1)}=\sqrt{x^2+y^2}\sqrt{-1}.$
8. $\sqrt{-\frac{1}{4}}=\sqrt{\frac{1}{4}\times(-1)}=\frac{1}{2}\sqrt{-1}.$

9. $ax\sqrt{-a^3x^4} = ax\sqrt{a^3x^4(-1)} = ax \times ax^2\sqrt{a}\sqrt{-1} = a^2x^3\sqrt{a}\sqrt{-1}.$

10. $a^2\sqrt{-729a^6} = a^2\sqrt{729a^6(-1)} = a^2 \times 27a^3\sqrt{-1} = 27a^5\sqrt{-1}.$

11. $\sqrt{-a^6b^{-8}} = \sqrt{a^6b^{-8}(-1)} = a^3b^{-4}\sqrt{-1} = \dfrac{a^3}{b^4}\sqrt{-1}.$

12. $\sqrt{-a^{-4}b^{-8}c^2} = \sqrt{a^{-4}b^{-8}c^2(-1)} = a^{-2}b^{-4}c\sqrt{-1} = \dfrac{c}{a^2b^4}\sqrt{-1}.$

Find the sum of:

13. $\sqrt{-25} + \sqrt{-36} + \sqrt{-49} + \sqrt{-64} + \sqrt{-81}$
$$\begin{aligned} &= 5\sqrt{-1} + 6\sqrt{-1} + 7\sqrt{-1} + 8\sqrt{-1} + 9\sqrt{-1} \\ &= (5 + 6 + 7 + 8 + 9)\sqrt{-1} \\ &= 35\sqrt{-1}. \end{aligned}$$

14. $2\sqrt{-a^4} + 3\sqrt{-a^6} - 4\sqrt{-16a^8} + 5\sqrt{-25a^4}$
$$\begin{aligned} &= 2a^2\sqrt{-1} + 3a^3\sqrt{-1} - 4 \times 4a^4\sqrt{-1} + 5 \times 5a^2\sqrt{-1} \\ &= 2a^2\sqrt{-1} + 3a^3\sqrt{-1} - 16a^4\sqrt{-1} + 25a^2\sqrt{-1} \\ &= (2a^2 + 25a^2 + 3a^3 - 16a^4)\sqrt{-1} \\ &= (27a^2 + 3a^3 - 16a^4)\sqrt{-1}. \end{aligned}$$

15. $(3 + 7\sqrt{-1}) + (2 - 6\sqrt{-1}) - (3 + 4\sqrt{-1})$
$$\begin{aligned} &= 3 + 7\sqrt{-1} + 2 - 6\sqrt{-1} - 3 - 4\sqrt{-1} \\ &= 3 + 2 - 3 + (7 - 6 - 4)\sqrt{-1} \\ &= 2 - 3\sqrt{-1}. \end{aligned}$$

16. $(a + b\sqrt{-1}) + (c - d\sqrt{-1}) - (a - d\sqrt{-1})$
$$\begin{aligned} &= a + b\sqrt{-1} + c - d\sqrt{-1} - a + d\sqrt{-1} \\ &= a + c - a + (b - d + d)\sqrt{-1} \\ &= c + b\sqrt{-1}. \end{aligned}$$

17. Find the product of

$(3 + 5\sqrt{-1})(7 + 4\sqrt{-1}).$

$$\begin{array}{rrr} 3 + & 5\sqrt{-1} & \\ 7 + & 4\sqrt{-1} & \\ \hline 21 + & 35\sqrt{-1} & \\ + & 12\sqrt{-1} & - 20 \\ \hline 1 + & 47\sqrt{-1} & \end{array}$$

18. Find the product of

$(5 - 2\sqrt{-7})(6 - 2\sqrt{-7}).$

$$\begin{array}{rrr} 5 - & 2\sqrt{-7} & \\ 6 - & 2\sqrt{-7} & \\ \hline 30 - & 12\sqrt{-7} & \\ - & 10\sqrt{-7} & - 28 \\ \hline 2 - & 22\sqrt{-7} & \end{array}$$

19. Find the product of

$(2\sqrt{7}+3\sqrt{-8})(3\sqrt{7}-10\sqrt{-2})$.

$2\sqrt{7}+3\sqrt{-8}=2\sqrt{7}+6\sqrt{-2}$.

$$\begin{array}{rrr}
2\sqrt{7} & +\ 6\sqrt{-2} & \\
3\sqrt{7} & -10\sqrt{-2} & \\
\hline
42 & +18\sqrt{-14} & \\
 & -20\sqrt{-14} & +120 \\
\hline
162 & -\ 2\sqrt{-14} &
\end{array}$$

20. Find the product of

$(3\sqrt{3}+2\sqrt{-2})(3\sqrt{3}-2\sqrt{-2})$.

$$\begin{array}{rrr}
3\sqrt{3} & +2\sqrt{-2} & \\
3\sqrt{3} & -2\sqrt{-2} & \\
\hline
27 & +6\sqrt{-6} & \\
 & -6\sqrt{-6} & +8 \\
\hline
35 & &
\end{array}$$

21. Find the product of

$(4+7\sqrt{-1})(2+3\sqrt{-1})$.

$$\begin{array}{rrr}
4\ + & 7\sqrt{-1} & \\
2\ + & 3\sqrt{-1} & \\
\hline
8\ + & 14\sqrt{-1} & \\
 & +12\sqrt{-1} & -21 \\
\hline
 & 26\sqrt{-1} & -13
\end{array}$$

22. Find the product of

$(a+b\sqrt{-1})(a-c\sqrt{-1})$.

$$\begin{array}{lll}
a\ + & b\sqrt{-1} & \\
a\ - & c\sqrt{-1} & \\
\hline
a^2 + & ab\sqrt{-1} & \\
 & -ac\sqrt{-1} & +bc \\
\hline
\multicolumn{3}{l}{a^2+(ab-ac)\sqrt{-1}+bc}
\end{array}$$

23. Find the product of $(a\sqrt{-1}-b)(c\sqrt{-1}-d)$.

$$\begin{array}{rrr}
 & a\sqrt{-1}-b & \\
 & c\sqrt{-1}-d & \\
\hline
 & -ac & -bc\sqrt{-1} \\
 & & -ad\sqrt{-1}+bd \\
\hline
bd-ac & & -(ad+bc)\sqrt{-1}
\end{array}$$

24. Find the product of $(2\sqrt{5}+5\sqrt{-2})(2\sqrt{5}-3\sqrt{-5})$.

$$\begin{array}{rlll}
2\sqrt{5}\ + & 5\sqrt{-2} & & \\
2\sqrt{5}\ - & 3\sqrt{-5} & & \\
\hline
20 & +10\sqrt{-10} & & \\
 & & -30\sqrt{-1} & +15\sqrt{10} \\
\hline
20 & +10\sqrt{-10} & -30\sqrt{-1} & +15\sqrt{10}
\end{array}$$

Perform the divisions indicated:

25. $\dfrac{4}{1+\sqrt{-3}}=\dfrac{4(1-\sqrt{-3})}{(1+\sqrt{-3})(1-\sqrt{-3})}=\dfrac{4(1-\sqrt{-3})}{4}=1-\sqrt{-3}.$

26. $$\frac{21}{4+3\sqrt{-6}}=\frac{21(4-3\sqrt{-6})}{(4+3\sqrt{-6})(4-3\sqrt{-6})}=\frac{21(4-3\sqrt{-6})}{70}$$
$$=\frac{3(4-3\sqrt{-6})}{10}.$$

27. $$\frac{5}{\sqrt{2}-\sqrt{-3}}=\frac{5(\sqrt{2}+\sqrt{-3})}{(\sqrt{2}-\sqrt{-3})(\sqrt{2}+\sqrt{-3})}=\frac{5(\sqrt{2}+\sqrt{-3})}{5}$$
$$=\sqrt{2}+\sqrt{-3}.$$

28. $$\frac{1+\sqrt{-3}}{1-\sqrt{-3}}=\frac{(1+\sqrt{-3})(1+\sqrt{-3})}{(1-\sqrt{-3})(1+\sqrt{-3})}=\frac{2\sqrt{-3}-2}{4}=\frac{\sqrt{-3}-1}{2}$$

29. $$\frac{1+\sqrt{-1}}{1-\sqrt{-1}}=\frac{(1+\sqrt{-1})(1+\sqrt{-1})}{(1-\sqrt{-1})(1+\sqrt{-1})}=\frac{2\sqrt{-1}}{2}=\sqrt{-1}.$$

30. $$\frac{5-29\sqrt{-5}}{7-3\sqrt{-5}}=\frac{(5-29\sqrt{-5})(7+3\sqrt{-5})}{(7-3\sqrt{-5})(7+3\sqrt{-5})}=\frac{470-188\sqrt{-5}}{94}$$
$$=5-2\sqrt{-5}.$$

31. $$\frac{3}{\sqrt{2}+\sqrt{-1}}=\frac{3(\sqrt{2}-\sqrt{-1})}{(\sqrt{2}+\sqrt{-1})(\sqrt{2}-\sqrt{-1})}=\frac{3(\sqrt{2}-\sqrt{-1})}{3}$$
$$=\sqrt{2}-\sqrt{-1}.$$

32. $$\frac{\sqrt{3}+\sqrt{-2}}{\sqrt{3}-\sqrt{-2}}=\frac{(\sqrt{3}+\sqrt{-2})(\sqrt{3}+\sqrt{-2})}{(\sqrt{3}-\sqrt{-2})(\sqrt{3}+\sqrt{-2})}=\frac{1+2\sqrt{-6}}{5}.$$

33. $$\frac{\sqrt{-a}+\sqrt{-b}}{\sqrt{-a}-\sqrt{-b}}=\frac{(\sqrt{-a}+\sqrt{-b})(\sqrt{-a}+\sqrt{-b})}{(\sqrt{-a}-\sqrt{-b})(\sqrt{-a}+\sqrt{-b})}$$
$$=\frac{-a-2\sqrt{ab}-b}{-a+b}=\frac{a+2\sqrt{ab}+b}{a-b}.$$

34. $$\frac{a-b\sqrt{-1}}{a\sqrt{-1}+b}=\frac{(a-b\sqrt{-1})(a\sqrt{-1}-b)}{(a\sqrt{-1}+b)(a\sqrt{-1}-b)}=\frac{a^2\sqrt{-1}+b^2\sqrt{-1}}{-a^2-b^2}$$
$$=\frac{\sqrt{-1}}{-1}=-\sqrt{-1}.$$

35. $$\frac{x+i\sqrt{1-x^2}}{x-i\sqrt{1-x^2}}=\frac{(x+i\sqrt{1-x^2})(x+i\sqrt{1-x^2})}{(x-i\sqrt{1-x^2})(x+i\sqrt{1-x^2})}$$
$$=\frac{2x^2-1+2xi\sqrt{1-x^2}}{1}=2x^2-1+2x\sqrt{x^2-1}.$$

36. $\dfrac{\sqrt{a-b}+\sqrt{b-a}}{\sqrt{a-b}-\sqrt{b-a}}=\dfrac{(\sqrt{a-b}+\sqrt{b-a})(\sqrt{a-b}+\sqrt{b-a})}{(\sqrt{a-b}-\sqrt{b-a})(\sqrt{a-b}+\sqrt{b-a})}$

$$=\frac{2(a-b)\sqrt{-1}}{2a-2b}=\sqrt{-1}.$$

37. Extract the square root of $5+12\sqrt{-1}$.

$$5+12\sqrt{-1}=5+2\sqrt{6^2\times(-1)}=5+2\sqrt{2^2\times 3^2\times(-1)}.$$

Two numbers whose sum is 5 and product $2^2\times 3^2\times(-1)$ are 9 and -4.

$$\therefore\ 5+12\sqrt{-1}=9+2\sqrt{9\times(-4)}+(-4)=(\sqrt{9}+\sqrt{-4})^2$$
$$=(3+2\sqrt{-1})^2.$$

Therefore, the square root of $5+12\sqrt{-1}$ is $3+2\sqrt{-1}$.

38. Extract the square root of $-21+20\sqrt{-1}$.

$$-21+20\sqrt{-1}=-21+2\sqrt{10^2\times(-1)}=-21+2\sqrt{2^2\times 5^2\times(-1)}.$$

Two numbers whose sum is -21 and product $2^2\times 5^2\times(-1)$ are 4 and -25.

$$\therefore\ -21+20\sqrt{-1}=4+2\sqrt{4\times(-25)}+(-25)=(\sqrt{4}+\sqrt{-25})^2$$
$$=(2+5\sqrt{-1})^2.$$

Therefore, the square root of $-21+20\sqrt{-1}$ is $2+5\sqrt{-1}$.

39. Extract the square root of $16-30\sqrt{-1}$.

$$16-30\sqrt{-1}=16-2\sqrt{15^2\times(-1)}=16-2\sqrt{3^2\times 5^2\times(-1)}.$$

Two numbers whose sum is 16 and product $3^2\times 5^2\times(-1)$ are 25 and -9.

$$\therefore\ 16-30\sqrt{-1}=25-2\sqrt{25\times(-9)}+(-9)=(\sqrt{25}-\sqrt{-9})^2$$
$$=(5-3\sqrt{-1})^2.$$

Therefore, the square root of $16-30\sqrt{-1}$ is $5-3\sqrt{-1}$.

40. Extract the square root of $-40-42\sqrt{-1}$.

$$-40-42\sqrt{-1}=-40-2\sqrt{21^2\times(-1)}=-40-2\sqrt{3^2\times 7^2\times(-1)}.$$

Two numbers whose sum is -40 and product $3^2\times 7^2\times(-1)$ are 9 and -49.

$$\therefore\ -40-42\sqrt{-1}=9-2\sqrt{9\times(-49)}+(-49)=(\sqrt{9}-\sqrt{-49})^2$$
$$=(3-7\sqrt{-1})^2.$$

Therefore, the square root of $-40-42\sqrt{-1}$ is $3-7\sqrt{-1}$.

Exercise 107. Page 264

1. Solve $x^2 - 36 = 0$.
Transpose, $x^2 = 36$.
Extract the root, $x = \pm 6$.

2. Solve $x^2 - 144 = 0$.
Transpose, $x^2 = 144$.
Extract the root, $x = \pm 12$.

3. Solve $4x^2 - 144 = 0$.
Divide by 4, $x^2 - 36 = 0$.
Transpose, $x^2 = 36$.
Extract the root, $x = \pm 6$.

4. Solve $x^2 - a^2 = 0$.
Transpose, $x^2 = a^2$.
Extract the root, $x = \pm a$.

5. Solve $x^2 - a = 0$.
Transpose, $x^2 = a$.
Extract the root, $x = \pm \sqrt{a}$.

6. Solve $4x^2 - a^2 = 0$.
Transpose, $4x^2 = a^2$.
Extract the root, $2x = \pm a$.
$\therefore x = \pm \frac{1}{2} a$.

7. Solve $4x^2 - a = 0$.
Transpose, $4x^2 = a$.
Extract the root, $2x = \pm \sqrt{a}$.
$\therefore x = \pm \frac{1}{2}\sqrt{a}$.

8. Solve $a^2x^2 - b^2 = 0$.
Transpose, $a^2x^2 = b^2$.
Extract the root, $ax = \pm b$.
$\therefore x = \pm \frac{b}{a}$.

9. Solve $a^2x^2 - b = 0$.
Transpose, $a^2x^2 = b$
Extract the root,
$ax = \pm \sqrt{b}$.
$\therefore x = \pm \frac{\sqrt{b}}{a}$.

10. Solve $x^2 - 5 = 0$.
Transpose, $x^2 = 5$.
Extract the root, $x = \pm \sqrt{5}$.

11. Solve $4x^2 - 15 = 0$.
Transpose, $4x^2 = 15$.
Extract the root, $2x = \pm \sqrt{15}$.
$\therefore x = \pm \frac{1}{2}\sqrt{15}$.

12. Solve $x^2 + 5 = 0$.
Transpose, $x^2 = -5$.
Extract the root, $x = \pm \sqrt{-5}$.

13. Solve $4x^2 + 15 = 0$.
Transpose, $4x^2 = -15$.
Extract the root,
$2x = \pm \sqrt{-15}$.
$\therefore x = \pm \frac{1}{2}\sqrt{-15}$.

14. Solve $5x^2 + 3x^2 = 72$.
Combine, $8x^2 = 72$.
Divide by 8, $x^2 = 9$.
Extract the root, $x = \pm 3$.

15. Solve $6x^2 + 5x^2 = 176$.
Combine, $11x^2 = 176$.
Divide by 11, $x^2 = 16$.
Extract the root, $x = \pm 4$.

16. Solve $7x^2 - 2x^2 = 125$.
Combine, $5x^2 = 125$.
Divide by 5, $x^2 = 25$.
Extract the root, $x = \pm 5$.

17. Solve $9x^2 - 4x^2 = 80.$
Combine, $5x^2 = 80.$
Divide by 5, $x^2 = 16.$
Extract the root, $x = \pm 4.$

18. Solve $12x^2 - 5x^2 = 343.$
Combine, $7x^2 = 343.$
Divide by 7, $x^2 = 49.$
Extract the root, $x = \pm 7.$

19. Solve $3x^2 + 12 = 7x^2 - 88.$
Transpose and combine, $-4x^2 = -100.$
Divide by -4, $x^2 = 25.$
Extract the root, $x = \pm 5.$

20. Solve $9x^2 - 53 = 6x^2 + 94.$
Transpose and combine, $3x^2 = 147.$
Divide by 3, $x^2 = 49.$
Extract the root, $\therefore x = \pm 7.$

21. Solve $13x^2 - 19 = 7x^2 + 5.$
Transpose and combine, $6x^2 = 24.$
Divide by 6, $x^2 = 4.$
Extract the root, $x = \pm 2.$

22. Solve $17x^2 - 7 = 418.$
Transpose and combine, $17x^2 = 425.$
Divide by 17, $x^2 = 25.$
Extract the root, $x = \pm 5.$

23. Solve $4x^2 + 7 = 6x^2 - 3.$
Transpose and combine, $-2x^2 = -10.$
Divide by -2, $x^2 = 5.$
Extract the root, $x = \pm\sqrt{5}.$

24. Solve $6x^2 - 12 = 3x^2 - 27.$
Transpose and combine, $3x^2 = -15.$
Divide by 3, $x^2 = -5.$
Extract the root, $x = \pm\sqrt{-5}.$

25. Solve $(x + \frac{1}{2})(x - \frac{1}{2}) = \frac{5}{16}$.

Expand, $x^2 - \frac{1}{4} = \frac{5}{16}$.
Transpose and combine, $x^2 = \frac{9}{16}$.
Extract the root, $x = \pm \frac{3}{4}$.

26. Solve $(3x - 7)(3x + 7) = 32$.

Expand, $9x^2 - 49 = 32$.
Transpose and combine, $9x^2 = 81$.
Divide by 9, $x^2 = 9$.
Extract the root, $x = \pm 3$.

27. Solve $mx^2 = a^2 - nx^2$.

Transpose, $mx^2 + nx^2 = a^2$.
Factor, $(m + n)x^2 = a^2$.
Divide by $m + n$, $x^2 = \frac{a^2}{m + n}$.
Extract the root, $x = \pm \frac{a}{m + n}\sqrt{m + n}$.

28. Solve $ax^2 - b = cx^2 + d$.

Transpose, $ax^2 - cx^2 = b + d$.
Factor, $(a - c)x^2 = b + d$.
Divide by $a - c$, $x^2 = \frac{b + d}{a - c}$.
Extract the root, $x = \pm\sqrt{\frac{b + d}{a - c}}$.

29. Solve $(a + x)(x - b) = (a - x)(x + b)$.

Expand, $ax + x^2 - ab - bx = ax - x^2 + ab - bx$.
Transpose and combine, $2x^2 = 2ab$.
Divide by 2, $x^2 = ab$.
Extract the root, $x = \pm\sqrt{ab}$.

30. Solve $(ax + b)(c + dx) = (cx + d)(a + bx)$.

Expand, $acx + adx^2 + bc + bdx = acx + bcx^2 + ad + bdx$.
Transpose and combine, $adx^2 - bcx^2 = ad - bc$.
Divide by $ad - bc$, $x^2 = 1$.
Extract the root, $x = \pm 1$.

31. Solve $(a-x)(b-x)=(1-ax)(1-bx).$
Expand, $ab-ax-bx+x^2=1-bx-ax+abx^2.$
Transpose and combine, $x^2-abx^2=1-ab.$
Divide by $1-ab$, $x^2=1.$
Extract the root, $x=\pm 1.$

32. Solve $ax^2+a=bx^2-b.$
Transpose, $ax^2-bx^2=-a-b.$
Factor, $(a-b)x^2=-a-b.$
Divide by $a-b$, $x^2=\frac{a+b}{b-a}.$
Extract the root, $x=\pm\sqrt{\frac{b+a}{b-a}}.$

33. Solve $(x+3)(x-3)=66-2x^2.$
Expand, $x^2-9=66-2x^2.$
Transpose and combine, $3x^2=75.$
Divide by 3, $x^2=25.$
Extract the root, $x=\pm 5.$

34. Solve $(x+2)(x+3)=2x^2+5x.$
Expand, $x^2+5x+6=2x^2+5x.$
Transpose and combine, $x^2=6.$
Extract the root, $x=\pm\sqrt{6}.$

35. Solve $x^2-5x+9=3x^2-5x-9.$
Transpose and combine, $-2x^2=-18.$
Divide by -2, $x^2=9.$
Extract the root, $x=\pm 3.$

36. Solve $(a+x)(b-x)=(x-a)(b+x).$
Expand, $ab-ax+bx-x^2=bx+x^2-ab-ax.$
Transpose and combine, $-2x^2=-2ab.$
Divide by -2, $x^2=ab.$
Extract the root, $x=\pm\sqrt{ab}.$

37. Solve $\frac{25+x}{9+x}=\frac{13+x}{47-x}.$
Clear of fractions, $1175+22x-x^2=117+22x+x^2.$
Transpose and combine, $-2x^2=-1058.$
Divide by -2, $x^2=529.$
Extract the root, $x=\pm 23.$

38. Solve $$\frac{35+3x}{1+x}=\frac{x-55}{3x-53}.$$

Clear of fractions, $9x^2-54x-1855=x^2-54x-55.$
Transpose and combine, $8x^2=1800.$
Divide by 8, $x^2=225.$
Extract the root, $x=\pm 15.$

39. Solve $$\frac{x-2}{3x+14}=\frac{3(8-x)}{28-x}.$$

Clear of fractions, $30x-x^2-56=336+30x-9x^2.$
Transpose and combine, $8x^2=392.$
Divide by 8, $x^2=49.$
Extract the root, $x=\pm 7.$

40. Solve $$\frac{x+5a+b}{x-3a+b}=\frac{x-a+b}{a-x+3b}.$$

Clear of fractions, $-x^2-4ax+2bx+5a^2+16ab+3b^2$
$=x^2-4ax+2bx+3a^2-4ab+b^2.$
Transpose and combine, $-2x^2=-2a^2-20ab-2b^2.$
Divide by -2, $x^2=a^2+10ab+b^2.$
Extract the root, $x=\pm\sqrt{a^2+10ab+b^2}.$

41. Solve $$\frac{3a-2b+3x}{a-2b+x}=\frac{x-7a+8b}{3x-5a+4b}.$$

Clear of fractions, $9x^2-6ax+6bx-15a^2+22ab-8b^2$
$=x^2-6ax+6bx-7a^2+22ab-16b^2.$
Transpose and combine, $8x^2=8a^2-8b^2.$
Divide by 8, $x^2=a^2-b^2.$
Extract the root, $x=\pm\sqrt{a^2-b^2}.$

42. Solve $$\frac{x+a-b}{x-a+b}=\frac{a(x+a+5b)}{b(x+5a+b)}.$$

Clear of fractions,
$$bx^2+6abx+5a^2b-4ab^2-b^3=ax^2+6abx-a^3-4a^2b+5ab^2.$$
Transpose and combine, $ax^2-bx^2=a^3+9a^2b-9ab^2-b^3.$
Divide by $a-b$, $x^2=a^2+10ab+b^2.$
Extract the root, $x=\pm\sqrt{a^2+10ab+b^2}.$

43. Solve $(x+a)(x-a)+(x+b)(x-b)-(x+c)(x-c)=0.$

Expand, $x^2-a^2+x^2-b^2-x^2+c^2=0.$

Transpose and combine, $x^2=a^2+b^2-c^2.$

Extract the root, $x=\pm\sqrt{a^2+b^2-c^2}.$

44. Solve $(a+bx)(b-ax)+(b+cx)(c-bx)+(c+ax)(a-cx)=0.$

Expand,

$$ab-a^2x+b^2x-abx^2+bc-b^2x+c^2x-bcx^2+ac-c^2x+a^2x-acx^2=0.$$

Transpose and combine,

$$-abx^2-bcx^2-acx^2=-ab-bc-ac.$$

Divide by $-ab-bc-ac$, $x^2=1.$

Extract the root, $x=\pm 1.$

45. Solve $$\frac{x^2-x+1}{x+1}+6=\frac{x^2+x+1}{x-1}.$$

Clear of fractions,

$$x^3-2x^2+2x-1+6x^2-6=x^3+2x^2+2x+1.$$

Transpose and combine, $2x^2=8.$

Divide by 2, $x^2=4.$

Extract the root, $x=\pm 2.$

46. Solve $$\frac{7a-b+x}{7b-a+x}=\frac{a(a+5b+x)}{b(5a+b+x)}.$$

Clear of fractions, $35a^2b+2ab^2-b^3+12abx+bx^2$

$$=-a^3+2a^2b+35ab^2+12abx+ax^2.$$

Transpose and combine, $bx^2-ax^2=b^3+33ab^2-33a^2b-a^3.$

Divide by $b-a$, $x^2=a^2+34ab+b^2.$

Extract the root, $x=\pm\sqrt{a^2+34ab+b^2}.$

47. Solve $$\frac{17a+b-x}{a+17b-x}=\frac{a^2(a+17b+x)}{b^2(17a+b+x)}.$$

Clear of fractions,

$$289a^2b^2+34ab^3+b^4-b^2x^2=a^4+34a^3b+289a^2b^2-a^2x^2.$$

Transpose and combine, $a^2x^2-b^2x^2=a^4+34a^3b-34ab^3-b^4.$

Divide by a^2-b^2, $x^2=a^2+34ab+b^2.$

Extract the root, $x=\pm\sqrt{a^2+34ab+b^2}.$

48. Solve $(2x+3)(3x+4)(4x+5)-(2x-3)(3x-4)(4x-5)=184.$

Expand, $24x^3+98x^2+133x+60-24x^3+98x^2-133x+60=184.$

Transpose and combine, $196x^2=64.$

Divide by 4, $49x^2=16.$

Extract the root, $7x=\pm 4.$

$\therefore\ x=\pm\frac{4}{7}.$

49. Solve $(a+5b+x)(5a+b+x)=3(a+b+x)^2.$

Expand,

$$5a^2+26ab+5b^2+6ax+6bx+x^2=3a^2+3b^2+3x^2+6ab+6ax+6bx.$$

Transpose and combine, $-2x^2=-2a^2-20ab-2b^2.$

Divide by -2, $x^2=a^2+10ab+b^2.$

Extract the root, $x=\pm\sqrt{a^2+10ab+b^2}.$

50. Solve $\frac{x^2-2bx+2ax-b^2}{x^3-a^3}+\frac{x+2b}{x^2+ax+a^2}=\frac{1}{x-a}.$

Clear of fractions,

$$x^2-2bx+2ax-b^2+x^2-ax+2bx-2ab=x^2+ax+a^2.$$

Transpose and combine, $x^2=a^2+2ab+b^2.$

Extract the root, $x=\pm(a+b).$

51. Solve $\frac{3a}{x-5a}+\frac{x+4a}{x+3a}=\frac{7a^2+2ax-x^2}{x^2-2ax-15a^2}.$

Clear of fractions,

$$3ax+9a^2+x^2-ax-20a^2=7a^2+2ax-x^2.$$

Transpose and combine, $2x^2=18a^2.$

Divide by 2, $x^2=9a^2.$

Extract the root, $x=\pm 3a.$

52. Solve $(a+x)(b-x)+(1+ax)(1-bx)=(a+b)(1+x^2).$

Expand,

$$ab-ax+bx-x^2+1-bx+ax-abx^2=a+b+ax^2+bx^2.$$

Transpose and combine,

$$-ax^2-bx^2-abx^2-x^2=a+b-ab-1.$$

Divide by $-(a+b+ab+1)$, $x^2=\frac{ab+1-a-b}{a+b+ab+1}.$

$\therefore\ x^2=\frac{(1-a)(1-b)}{(1+a)(1+b)}.$

Extract the root, $x=\pm\sqrt{\frac{(1-a)(1-b)}{(1+a)(1+b)}}.$

Exercise 108. Page 267

1. Solve $x^2 + 2x = 63.$

Complete the square, $x^2 + 2x + 1 = 64.$
Extract the root, $x + 1 = \pm 8.$
$\therefore x = 7, \text{ or } -9.$

2. Solve $x^2 - 8x + 15 = 0.$

Transpose, $x^2 - 8x = -15.$
Complete the square, $x^2 - 8x + 16 = 1.$
Extract the root, $x - 4 = \pm 1.$
$\therefore x = 5, \text{ or } 3.$

3. Solve $x^2 - 6x + 4 = 0.$

Transpose, $x^2 - 6x = -4.$
Complete the square, $x^2 - 6x + 9 = 5.$
Extract the root, $x - 3 = \pm\sqrt{5}.$
$\therefore x = 3 \pm \sqrt{5}.$

4. Solve $x^2 - 10x + 32 = 0.$

Transpose, $x^2 - 10x = -32.$
Complete the square, $x^2 - 10x + 25 = -7.$
Extract the root, $x - 5 = \pm\sqrt{-7}.$
$\therefore x = 5 \pm \sqrt{-7}.$

5. Solve $x^2 + 2x - 1 = 0.$

Transpose, $x^2 + 2x = 1.$
Complete the square, $x^2 + 2x + 1 = 2.$
Extract the root, $x + 1 = \pm\sqrt{2}.$
$\therefore x = -1 \pm \sqrt{2}.$

6. Solve $x^2 + 6x - 91 = 0.$

Transpose, $x^2 + 6x = 91.$
Complete the square, $x^2 + 6x + 9 = 100.$
Extract the root, $x + 3 = \pm 10.$
$\therefore x = 7, \text{ or } -13.$

7. Solve $x^2 - 7x - 30 = 0.$

Transpose, $x^2 - 7x = 30.$
Complete the square, $x^2 - 7x + \frac{49}{4} = \frac{169}{4}.$
Extract the root, $x - \frac{7}{2} = \pm \frac{13}{2}.$
$\therefore x = 10, \text{ or } -3.$

8. Solve $x^2 - 12x + 35 = 0.$

Transpose, $x^2 - 12x = -35.$
Complete the square, $x^2 - 12x + 36 = 1.$
Extract the root, $x - 6 = \pm 1.$
$\therefore x = 7$, or $5.$

9. Solve $x^2 + x - 56 = 0.$

Transpose, $x^2 + x = 56.$
Complete the square, $x^2 + x + \frac{1}{4} = \frac{225}{4}.$
Extract the root, $x + \frac{1}{2} = \pm \frac{15}{2}.$
$\therefore x = 7$, or $-8.$

10. Solve $x^2 + 2x - 15 = 0.$

Transpose, $x^2 + 2x = 15.$
Complete the square, $x^2 + 2x + 1 = 16.$
Extract the root, $x + 1 = \pm 4.$
$\therefore x = 3$, or $-5.$

11. Solve $x^2 - 4x - 77 = 0.$

Transpose, $x^2 - 4x = 77.$
Complete the square, $x^2 - 4x + 4 = 81.$
Extract the root, $x - 2 = \pm 9.$
$\therefore x = 11$, or $-7.$

12. Solve $x^2 - 14x + 45 = 0.$

Transpose, $x^2 - 14x = -45.$
Complete the square, $x^2 - 14x + 49 = 4.$
Extract the root, $x - 7 = \pm 2.$
$\therefore x = 9$, or $5.$

13. Solve $x^2 - 3x - 54 = 0.$

Transpose, $x^2 - 3x = 54.$
Complete the square, $x^2 - 3x + \frac{9}{4} = \frac{225}{4}.$
Extract the root, $x - \frac{3}{2} = \pm \frac{15}{2}.$
$\therefore x = 9$, or $-6.$

14. Solve $x^2 + 6x + 9 = 0.$

Extract the root, $x + 3 = 0.$
$\therefore x = -3.$

15. Solve $x^2 + 4x - 60 = 0.$

Transpose, $x^2 + 4x = 60.$
Complete the square, $x^2 + 4x + 4 = 64.$
Extract the root, $x + 2 = \pm 8.$
$\therefore x = 6, \text{ or } -10.$

16. Solve $12x^2 + 8x - 15 = 0.$

Transpose, $12x^2 + 8x = 15.$
Divide by 12, $x^2 + \frac{2}{3}x = \frac{5}{4}.$
Complete the square, $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{49}{36}.$
Extract the root, $x + \frac{1}{3} = \pm \frac{7}{6}.$
$\therefore x = \frac{5}{6}, \text{ or } -\frac{3}{2}.$

17. Solve $2x^2 + 6x + 15 = 0.$

Transpose, $2x^2 + 6x = -15.$
Divide by 2, $x^2 + 3x = -\frac{15}{2}.$
Complete the square, $x^2 + 3x + \frac{9}{4} = -\frac{21}{4}.$
Extract the root, $x + \frac{3}{2} = \pm \frac{1}{2}\sqrt{-21}.$
$\therefore x = -\frac{3}{2} \pm \frac{1}{2}\sqrt{-21}.$

18. Solve $3x^2 + 5x - 22 = 0.$

Transpose, $3x^2 + 5x = 22.$
Divide by 3, $x^2 + \frac{5}{3}x = \frac{22}{3}.$
Complete the square, $x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{289}{36}.$
Extract the root, $x + \frac{5}{6} = \pm \frac{17}{6}.$
$\therefore x = 2, \text{ or } -\frac{11}{3}.$

19. Solve $\dfrac{5x-7}{9} + \dfrac{14}{2x-3} = x - 1.$

Clear of fractions, $10x^2 - 29x + 21 + 126 = 18x^2 - 45x + 27.$
Transpose and combine, $8x^2 - 16x = 120.$
Divide by 8, $x^2 - 2x = 15.$
Complete the square, $x^2 - 2x + 1 = 16.$
Extract the root, $x - 1 = \pm 4.$
$\therefore x = 5, \text{ or } -3.$

20. Solve $\dfrac{6x+4}{5}-\dfrac{15-2x}{x-3}=\dfrac{7(x-1)}{5}.$

Transpose, $\dfrac{6x+4}{5}-\dfrac{7x-7}{5}=\dfrac{15-2x}{x-3}.$

Combine, $\dfrac{11-x}{5}=\dfrac{15-2x}{x-3}.$

Clear of fractions, $-33+14x-x^2=75-10x.$

Transpose and combine, $x^2-24x=-108.$

Complete the square, $x^2-24x+144=36.$

Extract the root, $x-12=\pm 6.$

$\therefore x=18, \text{ or } 6.$

21. Solve $\dfrac{5x-1}{9}+\dfrac{3x-1}{5}=\dfrac{2}{x}+x-1.$

Clear of fractions, $25x^2-5x+27x^2-9x=90+45x^2-45x.$

Transpose and combine, $7x^2+31x=90.$

Divide by 7, $x^2+\frac{31}{7}x=\frac{90}{7}.$

Complete the square, $x^2+\frac{31}{7}x+\frac{961}{196}=\frac{3481}{196}.$

Extract the root, $x+\frac{31}{14}=\pm\frac{59}{14}.$

$\therefore x=2, \text{ or } -\frac{45}{7}.$

22. Solve $\dfrac{16-x}{4}-\dfrac{2(x-11)}{x-6}=\dfrac{x-4}{12}.$

Clear of fractions,

$$-3x^2+66x-288-24x+264=x^2-10x+24.$$

Transpose and combine, $-4x^2+52x=48.$

Divide by -4, $x^2-13x=-12.$

Complete the square, $x^2-13x+\frac{169}{4}=\frac{121}{4}.$

Extract the root, $x-\frac{13}{2}=\pm\frac{11}{2}.$

$\therefore x=12, \text{ or } 1.$

23. Solve $2x^2-7x+3=0.$

Transpose, $2x^2-7x=-3.$

Divide by 2, $x^2-\frac{7}{2}x=-\frac{3}{2}.$

Complete the square, $x^2-\frac{7}{2}x+\frac{49}{16}=\frac{25}{16}.$

Extract the root, $x-\frac{7}{4}=\pm\frac{5}{4}.$

$\therefore x=3, \text{ or } \frac{1}{2}.$

24. Solve $2x^2-5x-3=0.$

Transpose, $2x^2-5x=3.$
Divide by 2, $x^2-\frac{5}{2}x=\frac{3}{2}.$
Complete the square, $x^2-\frac{5}{2}x+\frac{25}{16}=\frac{49}{16}.$
Extract the root, $x-\frac{5}{4}=\pm\frac{7}{4}.$
$\therefore x=3, \text{ or } -\frac{1}{2}.$

25. Solve $3x^2-17x+10=0.$

Transpose, $3x^2-17x=-10.$
Divide by 3, $x^2-\frac{17}{3}x=-\frac{10}{3}.$
Complete the square, $x^2-\frac{17}{3}x+\frac{289}{36}=\frac{169}{36}.$
Extract the root, $x-\frac{17}{6}=\pm\frac{13}{6}.$
$\therefore x=5, \text{ or } \frac{2}{3}.$

26. Solve $6x^2-5x-6=0.$

Transpose, $6x^2-5x=6.$
Divide by 6, $x^2-\frac{5}{6}x=1.$
Complete the square, $x^2-\frac{5}{6}x+\frac{25}{144}=\frac{169}{144}.$
Extract the root, $x-\frac{5}{12}=\pm\frac{13}{12}.$
$\therefore x=\frac{3}{2}, \text{ or } -\frac{2}{3}.$

27. Solve $4x^2-3x-2=0.$

Transpose, $4x^2-3x=2.$
Divide by 4, $x^2-\frac{3}{4}x=\frac{1}{2}.$
Complete the square, $x^2-\frac{3}{4}x+\frac{9}{64}=\frac{41}{64}.$
Extract the root, $x-\frac{3}{8}=\pm\frac{1}{8}\sqrt{41}.$
$\therefore x=\frac{3}{8}\pm\frac{1}{8}\sqrt{41}.$

28. Solve $3x^2+4x+5=0.$

Transpose, $3x^2+4x=-5.$
Divide by 3, $x^2+\frac{4}{3}x=-\frac{5}{3}.$
Complete the square, $x^2+\frac{4}{3}x+\frac{4}{9}=-\frac{11}{9}.$
Extract the root, $x+\frac{2}{3}=\pm\frac{1}{3}\sqrt{-11}.$
$\therefore x=-\frac{2}{3}\pm\frac{1}{3}\sqrt{-11}.$

29. Solve $\frac{x-5}{x+3}+\frac{x-8}{x-3}=\frac{3(1-x)}{x^2-9}.$

Clear of fractions,

$$x^2-8x+15+x^2-5x-24=3-3x.$$

Transpose and combine, $2x^2-10x=12.$
Divide by 2, $x^2-5x=6.$

Complete the square, $x^2 - 5x + \frac{25}{4} = \frac{49}{4}.$

Extract the root, $x - \frac{5}{2} = \pm \frac{7}{2}.$

$\therefore x = 6, \text{ or } -1.$

30. Solve $$\frac{3x+4}{2x-1} + \frac{5x+12}{2x+1} = \frac{4x+3}{4x^2-1} + 7.$$

Clear of fractions,

$$6x^2 + 11x + 4 + 10x^2 + 19x - 12 = 4x + 3 + 28x^2 - 7.$$

Transpose and combine, $12x^2 - 26x = -4.$

Divide by 12, $x^2 - \frac{13}{6}x = -\frac{1}{3}.$

Complete the square, $x^2 - \frac{13}{6}x + \frac{169}{144} = \frac{121}{144}.$

Extract the root, $x - \frac{13}{12} = \pm \frac{11}{12}.$

$\therefore x = 2, \text{ or } \frac{1}{6}.$

31. Solve $$\frac{3x-3}{x+1} + \frac{3x-1}{x+2} - \frac{3x-11}{x-1} = 3.$$

Clear of fractions,

$$3x^3-9x+6+3x^3-x^2-3x+1-3x^3+2x^2+27x+22=3x^3+6x^2-3x-6.$$

Transpose and combine, $5x^2 - 18x = 35.$

Divide by 5, $x^2 - \frac{18}{5}x = 7.$

Complete the square, $x^2 - \frac{18}{5}x + \frac{81}{25} = \frac{256}{25}.$

Extract the root, $x - \frac{9}{5} = \pm \frac{16}{5}.$

$\therefore x = 5, \text{ or } -\frac{7}{5}.$

Exercise 109. Page 269

1. Solve $25x^2 - 30x - 27 = 0.$

Transpose, $25x^2 - 30x = 27.$

Complete the square, $25x^2 - 30x + 9 = 36.$

Extract the root, $5x - 3 = \pm 6.$

$5x = 9, \text{ or } -3.$

$\therefore x = \frac{9}{5}, \text{ or } -\frac{3}{5}.$

2. Solve $4x^2 - 4x - 15 = 0.$

Transpose, $4x^2 - 4x = 15.$

Complete the square, $4x^2 - 4x + 1 = 16.$

Extract the root, $2x - 1 = \pm 4.$

$2x = 5, \text{ or } -3.$

$\therefore x = \frac{5}{2}, \text{ or } -\frac{3}{2}.$

3. Solve $4x^2+24x+35=0.$

Transpose, $4x^2+24x=-35.$

Complete the square, $4x^2+24x+36=1.$

Extract the root, $2x+6=\pm 1.$

$2x=-5, \text{ or } -7.$

$\therefore x=-\frac{5}{2}, \text{ or } -\frac{7}{2}.$

4. Solve $4x^2-24x+35=0.$

Transpose, $4x^2-24x=-35.$

Complete the square, $4x^2-24x+36=1.$

Extract the root, $2x-6=\pm 1.$

$2x=7, \text{ or } 5.$

$\therefore x=\frac{7}{2}, \text{ or } \frac{5}{2}.$

5. Solve $9x^2+15x-6=0.$

Transpose, $9x^2+15x=6.$

Complete the square, $9x^2+15x+\frac{25}{4}=\frac{49}{4}.$

Extract the root, $3x+\frac{5}{2}=\pm\frac{7}{2}.$

$3x=1, \text{ or } -6.$

$\therefore x=\frac{1}{3}, \text{ or } -2.$

6. Solve $16x^2-32x+15=0.$

Transpose, $16x^2-32x=-15.$

Complete the square, $16x^2-32x+16=1.$

Extract the root, $4x-4=\pm 1.$

$4x=5, \text{ or } 3.$

$\therefore x=\frac{5}{4}, \text{ or } \frac{3}{4}.$

7. Solve $16x^2+8x-35=0.$

Transpose, $16x^2+8x=35.$

Complete the square, $16x^2+8x+1=36.$

Extract the root, $4x+1=\pm 6.$

$4x=5, \text{ or } -7.$

$\therefore x=\frac{5}{4}, \text{ or } -\frac{7}{4}.$

8. Solve $16x^2-8x-35=0.$

Transpose, $16x^2-8x=35.$

Complete the square, $16x^2-8x+1=36.$

Extract the root, $4x-1=\pm 6.$

$4x=7, \text{ or } -5.$

$\therefore x=\frac{7}{4}, \text{ or } -\frac{5}{4}.$

9. Solve $4x^2 + 24x + 27 = 0.$

Transpose, $4x^2 + 24x = -27.$

Complete the square, $4x^2 + 24x + 36 = 9.$

Extract the root, $2x + 6 = \pm 3.$

$2x = -3, \text{ or } -9.$

$\therefore x = -\frac{3}{2}, \text{ or } -\frac{9}{2}.$

10. Solve $3x^2 - 22x + 35 = 0.$

Transpose, $3x^2 - 22x = -35.$

Multiply by 3, $9x^2 - 66x = -105.$

Complete the square, $9x^2 - 66x + 121 = 16.$

Extract the root, $3x - 11 = \pm 4.$

$3x = 15, \text{ or } 7.$

$\therefore x = 5, \text{ or } \frac{7}{3}.$

11. Solve $3x^2 - 7x - 16 = 0.$

Transpose, $3x^2 - 7x = 16.$

Multiply by 3, $9x^2 - 21x = 48.$

Complete the square, $9x^2 - 21x + \frac{49}{4} = \frac{241}{4}.$

Extract the root, $3x - \frac{7}{2} = \pm \frac{1}{2}\sqrt{241}.$

$3x = \frac{7}{2} \pm \frac{1}{2}\sqrt{241}.$

$\therefore x = \frac{7}{6} \pm \frac{1}{6}\sqrt{241}.$

12. Solve $2x^2 - 8 = 3x + 12.$

Transpose and combine, $2x^2 - 3x = 20.$

Multiply by 2, $4x^2 - 6x = 40.$

Complete the square, $4x^2 - 6x + \frac{9}{4} = \frac{169}{4}.$

Extract the root, $2x - \frac{3}{2} = \pm \frac{13}{2}.$

$2x = 8, \text{ or } -5.$

$\therefore x = 4, \text{ or } -\frac{5}{2}.$

13. Solve $2x^2 - x - 15 = 0.$

Transpose, $2x^2 - x = 15.$

Multiply by 2, $4x^2 - 2x = 30.$

Complete the square, $4x^2 - 2x + \frac{1}{4} = \frac{121}{4}.$

Extract the root, $2x - \frac{1}{2} = \pm \frac{11}{2}.$

$2x = 6, \text{ or } -5.$

$\therefore x = 3, \text{ or } -\frac{5}{2}.$

14. Solve $6x^2-17x+7=0.$

Transpose, $6x^2-17x=-7.$

Multiply by 6, $36x^2-102x=-42.$

Complete the square, $36x^2-102x+\frac{289}{4}=\frac{121}{4}.$

Extract the root, $6x-\frac{17}{2}=\pm\frac{11}{2}.$

$6x=14,\text{ or }3.$

$\therefore x=\frac{7}{3},\text{ or }\frac{1}{2}.$

15. Solve $3x^2+14x+15=0.$

Transpose, $3x^2+14x=-15.$

Multiply by 3, $9x^2+42x=-45.$

Complete the square, $9x^2+42x+49=4.$

Extract the root, $3x+7=\pm 2.$

$3x=-5,\text{ or }-9.$

$\therefore x=-\frac{5}{3},\text{ or }-3.$

16. Solve $4x^2-5x-9=0.$

Transpose, $4x^2-5x=9.$

Complete the square, $4x^2-5x+\frac{25}{16}=\frac{169}{16}.$

Extract the root, $2x-\frac{5}{4}=\pm\frac{13}{4}.$

$2x=\frac{9}{2},\text{ or }-2.$

$\therefore x=\frac{9}{4},\text{ or }-1.$

17. Solve $5x^2+6x-27=0.$

Transpose, $5x^2+6x=27.$

Multiply by 5, $25x^2+30x=135.$

Complete the square, $25x^2+30x+9=144.$

Extract the root, $5x+3=\pm 12.$

$5x=9,\text{ or }-15.$

$\therefore x=\frac{9}{5},\text{ or }-3.$

18. Solve $3x^2-22x+21=0.$

Transpose, $3x^2-22x=-21.$

Multiply by 3, $9x^2-66x=-63.$

Complete the square, $9x^2-66x+121=58.$

Extract the root, $3x-11=\pm\sqrt{58}.$

$3x=11\pm\sqrt{58}.$

$\therefore x=\frac{11}{3}\pm\frac{1}{3}\sqrt{58}.$

19. Solve $(2x-5)^2-(x-6)^2=80.$

Expand, $4x^2-20x+25-x^2+12x-36=80.$

Transpose and combine, $3x^2-8x=91.$

Multiply by 3, $9x^2-24x=273.$

Complete the square, $9x^2-24x+16=289.$

Extract the root, $3x-4=\pm 17.$

$3x=21, \text{ or } -13.$

$\therefore x=7, \text{ or } -\frac{13}{3}.$

20. Solve $(1-3x)(x-6)=5(x+2).$

Expand, $-3x^2+19x-6=5x+10.$

Transpose and combine, $3x^2-14x=-16.$

Multiply by 3, $9x^2-42x=-48.$

Complete the square, $9x^2-42x+49=1.$

Extract the root, $3x-7=\pm 1.$

$3x=8, \text{ or } 6.$

$\therefore x=\frac{8}{3}, \text{ or } 2.$

21. Solve $4x^2+24x-35=10.$

Transpose and combine, $4x^2+24x=45.$

Complete the square, $4x^2+24x+36=81.$

Extract the root, $2x+6=\pm 9.$

$2x=3, \text{ or } -15.$

$\therefore x=\frac{3}{2}, \text{ or } -\frac{15}{2}.$

22. Solve $(3x-5)^2-(2x+1)^2+32=0.$

Expand, $9x^2-30x+25-4x^2-4x-1+32=0.$

Transpose and combine, $5x^2-34x=-56.$

Multiply by 5, $25x^2-170x=-280.$

Complete the square, $25x^2-170x+289=9.$

Extract the root, $5x-17=\pm 3.$

$5x=20, \text{ or } 14.$

$\therefore x=4, \text{ or } \frac{14}{5}.$

23. Solve $(2x-3)^2-5(2x-5)^2-4=0.$

Expand, $4x^2-12x+9-20x^2+100x-125-4=0.$

Transpose and combine, $16x^2-88x=-120.$

Complete the square, $16x^2-88x+121=1.$

Extract the root, $4x-11=\pm 1.$

$4x=12, \text{ or } 10.$

$\therefore x=3, \text{ or } \frac{5}{2}.$

24. Solve $(5x+2)^2-5(3x-4)^2+331=0.$

Expand, $25x^2+20x+4-45x^2+120x-80+331=0.$

Transpose and combine, $-20x^2+140x=-255.$

Divide by -5, $4x^2-28x=51.$

Complete the square, $4x^2-28x+49=100.$

Extract the root, $2x-7=\pm 10.$

$2x=17, \text{ or } -3.$

$\therefore\ x=\frac{17}{2}, \text{ or } -\frac{3}{2}.$

25. Solve $\dfrac{15}{3x-1}-\dfrac{3}{4}=\dfrac{220}{9x^2-1}-\dfrac{11}{3x+1}.$

Clear of fractions, $180x+60-27x^2+3=880-132x+44.$

Transpose and combine, $-27x^2+312x=861.$

Multiply by -3, $81x^2-936x=-2583.$

Complete the square, $81x^2-936x+2704=121.$

Extract the root, $9x-52=\pm 11.$

$9x=63, \text{ or } 41.$

$\therefore\ x=7, \text{ or } \frac{41}{9}.$

26. Solve $\dfrac{2x-1}{x-2}+\dfrac{3x+1}{x-3}-\dfrac{11(x+1)}{(x-2)(x-3)}=0.$

Clear of fractions, $2x^2-7x+3+3x^2-5x-2-11x-11=0.$

Transpose and combine, $5x^2-23x=10.$

Multiply by 5, $25x^2-115x=50.$

Complete the square, $25x^2-115x+\frac{529}{4}=\frac{729}{4}.$

Extract the root, $5x-\frac{23}{2}=\pm\frac{27}{2}.$

$5x=25, \text{ or } -2.$

$\therefore\ x=5, \text{ or } -\frac{2}{5}.$

27. Solve $\dfrac{1}{x-3}+\dfrac{7}{x+3}=\dfrac{4}{x^2-9}+\dfrac{16-3x}{x-3}.$

Clear of fractions, $x+3+7x-21=4+48+7x-3x^2.$

Transpose and combine, $3x^2+x=70.$

Multiply by 3, $9x^2+3x=210.$

Complete the square, $9x^2+3x+\frac{1}{4}=\frac{841}{4}.$

Extract the root, $3x+\frac{1}{2}=\pm\frac{29}{2}.$

$3x=14, \text{ or } -15.$

$\therefore\ x=\frac{14}{3}, \text{ or } -5.$

28. Solve $\frac{2x+1}{x+3}+\frac{x-1}{x^2-9}=\frac{x+3}{3-x}+\frac{9x+20}{2(3+x)}.$

Clear of fractions,

$$4x^2-10x-6+2x-2=-2x^2-12x-18+9x^2-7x-60.$$

Transpose and combine, $3x^2-11x=70.$

Multiply by 3, $9x^2-33x=210.$

Complete the square, $9x^2-33x+\frac{121}{4}=\frac{961}{4}.$

Extract the root, $3x-\frac{11}{2}=\pm\frac{31}{2}.$

$3x=21, \text{ or } -10.$

$\therefore x=7, \text{ or } -\frac{10}{3}.$

29. Solve $\frac{x+7}{2x+3}-\frac{x+1}{2x-3}=\frac{4x-5}{4x^2-9}-\frac{x-3}{2x-3}.$

Clear of fractions,

$$2x^2+11x-21-2x^2-5x-3=4x-5-2x^2+3x+9.$$

Transpose and combine, $2x^2-x=28.$

Multiply by 2, $4x^2-2x=56.$

Complete the square, $4x^2-2x+\frac{1}{4}=\frac{225}{4}.$

Extract the root, $2x-\frac{1}{2}=\pm\frac{15}{2}.$

$2x=8, \text{ or } -7.$

$\therefore x=4, \text{ or } -\frac{7}{2}.$

Exercise 110. Page 271

1. Solve $6x^2-19x-20=0.$

Transpose, $6x^2-19x=20.$

Complete the square, $144x^2-(\ )+361=841.$

Extract the root, $12x-19=\pm 29.$

$12x=48, \text{ or } -10.$

$\therefore x=4, \text{ or } -\frac{5}{6}.$

2. Solve $6x^2+17x+12=0.$

Transpose, $6x^2+17x=-12.$

Complete the square, $144x^2+(\ )+289=1.$

Extract the root $12x+17=\pm 1.$

$12x=-16, \text{ or } -18.$

$\therefore x=-\frac{4}{3}, \text{ or } -\frac{3}{2}.$

3. Solve $6x^2 - 11x - 10 = 0.$

Transpose, $6x^2 - 11x = 10.$

Complete the square, $144x^2 - (\ ) + 121 = 361.$

Extract the root, $12x - 11 = \pm 19.$

$12x = 30, \text{ or } -8.$

$\therefore\ x = \frac{5}{2}, \text{ or } -\frac{2}{3}.$

4. Solve $8x^2 + 2x - 15 = 0.$

Transpose, $8x^2 + 2x = 15.$

Complete the square, $64x^2 + (\ ) + 1 = 121.$

Extract the root, $8x + 1 = \pm 11.$

$8x = 10, \text{ or } -12.$

$\therefore\ x = \frac{5}{4}, \text{ or } -\frac{3}{2}.$

5. Solve $6x^2 - 17x - 14 = 0.$

Transpose, $6x^2 - 17x = 14.$

Complete the square, $144x^2 - (\ ) + 289 = 625.$

Extract the root, $12x - 17 = \pm 25.$

$12x = 42, \text{ or } -8.$

$\therefore\ x = \frac{7}{2}, \text{ or } -\frac{2}{3}.$

6. Solve $6x^2 + 13x + 6 = 0.$

Transpose, $6x^2 + 13x = -6.$

Complete the square, $144x^2 + (\ ) + 169 = 25.$

Extract the root, $12x + 13 = \pm 5.$

$12x = -8, \text{ or } -18.$

$\therefore\ x = -\frac{2}{3}, \text{ or } -\frac{3}{2}.$

7. Solve $10x^2 + 19x - 15 = 0.$

Transpose, $10x^2 + 19x = 15.$

Complete the square, $400x^2 + (\ ) + 361 = 961.$

Extract the root, $20x + 19 = \pm 31.$

$20x = 12, \text{ or } -50.$

$\therefore\ x = \frac{3}{5}, \text{ or } -\frac{5}{2}.$

8. Solve $3x^2 - 7x + 4 = 0.$

Transpose, $3x^2 - 7x = -4.$

Complete the square, $36x^2 - (\ ) + 49 = 1.$

Extract the root, $6x - 7 = \pm 1.$

$6x = 8, \text{ or } 6.$

$\therefore\ x = \frac{4}{3}, \text{ or } 1.$

9. Solve $4x^2 - 11x - 45 = 0.$

Transpose, $4x^2 - 11x = 45.$
Complete the square, $64x^2 - (\ ) + 121 = 841.$
Extract the root, $8x - 11 = \pm 29.$
$8x = 40, \text{ or } -18.$
$\therefore x = 5, \text{ or } -\frac{9}{4}.$

10. Solve $3x^2 + 14x + 15 = 0.$

Transpose, $3x^2 + 14x = -15.$
Complete the square, $9x^2 + (\ ) + 49 = 4.$
Extract the root, $3x + 7 = \pm 2.$
$3x = -5, \text{ or } -9.$
$\therefore x = -\frac{5}{3}, \text{ or } -3.$

11. Solve $2x^2 - 14x - 36 = 0.$

Transpose, $2x^2 - 14x = 36.$
Complete the square, $4x^2 - (\ ) + 49 = 121.$
Extract the root, $2x - 7 = \pm 11.$
$2x = 18, \text{ or } -4.$
$\therefore x = 9, \text{ or } -2.$

12. Solve $3x^2 - 5x + 7 = 0.$

Transpose, $3x^2 - 5x = -7.$
Complete the square, $36x^2 - (\ ) + 25 = -59.$
Extract the root, $6x - 5 = \pm\sqrt{-59}.$
$6x = 5 \pm \sqrt{-59}.$
$\therefore x = \frac{5}{6} \pm \frac{1}{6}\sqrt{-59}.$

13. Solve $5x^2 - 2x + 14 = 0.$

Transpose, $5x^2 - 2x = -14.$
Complete the square, $25x^2 - (\ ) + 1 = -69.$
Extract the root, $5x - 1 = \pm\sqrt{-69}.$
$5x = 1 \pm \sqrt{-69}.$
$\therefore x = \frac{1}{5} \pm \frac{1}{5}\sqrt{-69}.$

14. Solve $3x^2 + 8x - 21 = 0.$

Transpose, $3x^2 + 8x = 21.$
Complete the square, $9x^2 + (\ ) + 16 = 79.$
Extract the root, $3x + 4 = \pm\sqrt{79}.$
$3x = -4 \pm \sqrt{79}.$
$\therefore x = -\frac{4}{3} \pm \frac{1}{3}\sqrt{79}.$

15. Solve $6x^2 - 5x + 42 = 0.$

Transpose, $6x^2 - 5x = -42.$

Complete the square, $144x^2 - (\ ) + 25 = -983.$

Extract the root, $12x - 5 = \pm\sqrt{-983}.$

$$12x = 5 \pm \sqrt{-983}.$$

$$\therefore x = \tfrac{5}{12} \pm \tfrac{1}{12}\sqrt{-983}.$$

16. Solve $5x^2 + 12x - 17 = 0.$

Transpose, $5x^2 + 12x = 17.$

Complete the square, $25x^2 + (\ ) + 36 = 121.$

Extract the root, $5x + 6 = \pm 11.$

$$5x = 5, \text{ or } -17.$$

$$\therefore x = 1, \text{ or } -\tfrac{17}{5}.$$

17. Solve $3x^2 + 7x - 76 = 0.$

Transpose, $3x^2 + 7x = 76.$

Complete the square, $36x^2 + (\ ) + 49 = 961.$

Extract the root, $6x + 7 = \pm 31.$

$$6x = 24, \text{ or } -38.$$

$$\therefore x = 4, \text{ or } -\tfrac{19}{3}.$$

18. Solve $5x^2 - 8x - 189 = 0.$

Transpose, $5x^2 - 8x = 189.$

Complete the square, $25x^2 - (\ ) + 16 = 961.$

Extract the root, $5x - 4 = \pm 31.$

$$5x = 35, \text{ or } -27.$$

$$\therefore x = 7, \text{ or } -\tfrac{27}{5}.$$

19. Solve $\dfrac{2x-3}{x-2} + \dfrac{x+1}{x-1} = \dfrac{3x+11}{x+1}.$

Clear of fractions,

$$2x^3 - 3x^2 - 2x + 3 + x^3 - 3x - 2 = 3x^3 + 2x^2 - 27x + 22.$$

Transpose and combine, $5x^2 - 22x = -21.$

Complete the square, $25x^2 - (\ ) + 121 = 16.$

Extract the root, $5x - 11 = \pm 4.$

$$5x = 15, \text{ or } 7.$$

$$\therefore x = 3, \text{ or } \tfrac{7}{5}.$$

20. Solve $$\frac{5+x}{3-x}-\frac{8-3x}{x}=\frac{2x}{x-2}.$$

Clear of fractions,

$$x^3+3x^2-10x-3x^3+23x^2-58x+48=-2x^3+6x^2.$$

Transpose and combine, $20x^2-68x=-48.$

Divide by 4, $5x^2-17x=-12.$

Complete the square, $100x^2-(\ )+289=49.$

Extract the root, $10x-17=\pm 7.$

$10x=24, \text{ or } 10.$

$\therefore\ x=\frac{12}{5}, \text{ or } 1.$

21. Solve $$\frac{1}{x+6}+\frac{1}{x+30}+\frac{1}{3x}=\frac{1}{x}.$$

Clear of fractions,

$$3x^2+90x+3x^2+18x+x^2+36x+180=3x^2+108x+540.$$

Transpose and combine, $4x^2+36x=360.$

Complete the square, $4x^2+36x+81=441.$

Extract the root, $2x+9=\pm 21.$

$2x=12, \text{ or } -30.$

$\therefore\ x=6, \text{ or } -15.$

22. Solve $$\frac{2}{x-2}+\frac{3}{x-3}=\frac{8}{x-4}.$$

Clear of fractions,

$$2x^2-14x+24+3x^2-18x+24=8x^2-40x+48.$$

Transpose and combine, $3x^2-8x=0.$

Complete the square, $9x^2-(\ )+16=16.$

Extract the root, $3x-4=\pm 4.$

$3x=8, \text{ or } 0.$

$\therefore\ x=\frac{8}{3}, \text{ or } 0.$

23. Solve $$\frac{3}{x+2}+\frac{2}{x+4}-\frac{1}{x+6}=\frac{4}{x}.$$

Clear of fractions,

$$3x^3+30x^2+72x+2x^3+16x^2+24x-x^3-6x^2-8x$$
$$=4x^3+48x^2+176x+192.$$

Transpose and combine, $-8x^2-88x=192.$

Divide by -8, $x^2+11x=-24.$

Complete the square, $4x^2+(\ )+121=25.$

Extract the root, $2x+11=\pm 5.$

$2x=-6, \text{ or } -16.$

$\therefore\ x=-3, \text{ or } -8.$

24. Solve $$\frac{3x}{2}-\frac{3x-20}{18-2x}=2+\frac{3x^2-80}{2(x-1)}.$$

Clear of fractions,

$$\begin{aligned}-3x^3+30x^2-27x-3x^2+23x-20\\ =-4x^2+40x-36-3x^3+27x^2+80x-720.\end{aligned}$$

Transpose and combine, $4x^2-124x=-736.$

Complete the square, $4x^2-(\;)+961=225.$

Extract the root, $2x-31=\pm 15.$

$$2x=46, \text{ or } 16.$$

$$\therefore x=23, \text{ or } 8.$$

25. Solve $$\frac{x}{2x-1}+\frac{24}{4x^2-1}=\frac{2(x-4)}{2x+1}-\frac{1}{9}.$$

Clear of fractions,

$$18x^2+9x+216=36x^2-162x+72-4x^2+1.$$

Transpose and combine, $14x^2-171x=143.$

Complete the square,

$$4\times 14^2x^2-(\;)+29241=37249.$$

Extract the root, $28x-171=\pm 193.$

$$28x=364, \text{ or } -22.$$

$$\therefore x=13, \text{ or } -\tfrac{11}{14}.$$

Exercise 111. Page 272

1. Resolve $6x^2-7x+2$ into two factors.

$$\begin{aligned}&6x^2-7x+2\\ &=6(x^2-\tfrac{7}{6}x+\tfrac{1}{3})\\ &=6[(x^2-\tfrac{7}{6}x+\tfrac{49}{144})-\tfrac{49}{144}+\tfrac{1}{3}]\\ &=6[(x-\tfrac{7}{12})^2-\tfrac{1}{144}]\\ &=6(x-\tfrac{7}{12}+\tfrac{1}{12})(x-\tfrac{7}{12}-\tfrac{1}{12})\\ &=6(x-\tfrac{1}{2})(x-\tfrac{2}{3})\\ &=(2x-1)(3x-2).\end{aligned}$$

Another Method

$$\begin{aligned}&6x^2-7x+2\\ &=6(x^2-\tfrac{7}{6}x+\tfrac{2}{6})\\ &=6(x^2-\tfrac{7}{6}x+\tfrac{12}{36})\\ &=6(x-\tfrac{3}{6})(x-\tfrac{4}{6})\\ &=6(x-\tfrac{1}{2})(x-\tfrac{2}{3})\\ &=(2x-1)(3x-2).\end{aligned}$$

2. Resolve $12x^2-7x-10$ into two factors.

$$\begin{aligned}&12x^2-7x-10\\ &=12(x^2-\tfrac{7}{12}x-\tfrac{5}{6})\\ &=12[(x^2-\tfrac{7}{12}x+\tfrac{49}{576})-\tfrac{49}{576}-\tfrac{5}{6}]\\ &=12[(x-\tfrac{7}{24})^2-\tfrac{529}{576}]\\ &=12(x-\tfrac{7}{24}+\tfrac{23}{24})(x-\tfrac{7}{24}-\tfrac{23}{24})\\ &=12(x+\tfrac{2}{3})(x-\tfrac{5}{4})\\ &=(3x+2)(4x-5).\end{aligned}$$

Another Method

$$\begin{aligned}&12x^2-7x-10\\ &=12(x^2-\tfrac{7}{12}x-\tfrac{10}{12})\\ &=12(x^2-\tfrac{7}{12}x-\tfrac{120}{144})\\ &=12(x+\tfrac{8}{12})(x-\tfrac{15}{12})\\ &=12(x+\tfrac{2}{3})(x-\tfrac{5}{4})\\ &=(3x+2)(4x-5).\end{aligned}$$

3. Resolve $21x^2 - 13x + 2$ into two factors.

$$\begin{aligned}
&21x^2 - 13x + 2\\
&= 21(x^2 - \tfrac{13}{21}x + \tfrac{2}{21})\\
&= 21[(x^2 - \tfrac{13}{21}x + \tfrac{169}{1764}) - \tfrac{169}{1764} + \tfrac{2}{21}]\\
&= 21[(x - \tfrac{13}{42})^2 - \tfrac{1}{1764}]\\
&= 21(x - \tfrac{13}{42} + \tfrac{1}{42})(x - \tfrac{13}{42} - \tfrac{1}{42})\\
&= 21(x - \tfrac{2}{7})(x - \tfrac{1}{3})\\
&= (7x - 2)(3x - 1).
\end{aligned}$$

Another Method

$$\begin{aligned}
&21x^2 - 13x + 2\\
&= 21(x^2 - \tfrac{13}{21}x + \tfrac{2}{21})\\
&= 21(x^2 - \tfrac{13}{21}x + \tfrac{42}{441})\\
&= 21(x - \tfrac{6}{21})(x - \tfrac{7}{21})\\
&= 21(x - \tfrac{2}{7})(x - \tfrac{1}{3})\\
&= (7x - 2)(3x - 1).
\end{aligned}$$

4. Resolve $3x^2 + 8x + 4$ into two factors.

$$\begin{aligned}
&3x^2 + 8x + 4\\
&= 3(x^2 + \tfrac{8}{3}x + \tfrac{4}{3})\\
&= 3[(x^2 + \tfrac{8}{3}x + \tfrac{16}{9}) - \tfrac{16}{9} + \tfrac{4}{3}]\\
&= 3[(x + \tfrac{4}{3})^2 - \tfrac{4}{9}]\\
&= 3(x + \tfrac{4}{3} + \tfrac{2}{3})(x + \tfrac{4}{3} - \tfrac{2}{3})\\
&= 3(x + 2)(x + \tfrac{2}{3})\\
&= (x + 2)(3x + 2).
\end{aligned}$$

Another Method

$$\begin{aligned}
&3x^2 + 8x + 4\\
&= 3(x^2 + \tfrac{8}{3}x + \tfrac{4}{3})\\
&= 3(x^2 + \tfrac{8}{3}x + \tfrac{12}{9})\\
&= 3(x + \tfrac{6}{3})(x + \tfrac{2}{3})\\
&= 3(x + 2)(x + \tfrac{2}{3})\\
&= (x + 2)(3x + 2).
\end{aligned}$$

5. Resolve $6x^2 - 11x + 3$ into two factors.

$$\begin{aligned}
&6x^2 - 11x + 3\\
&= 6(x^2 - \tfrac{11}{6}x + \tfrac{1}{2})\\
&= 6[(x^2 - \tfrac{11}{6}x + \tfrac{121}{144}) - \tfrac{121}{144} + \tfrac{1}{2}]\\
&= 6[(x - \tfrac{11}{12})^2 - \tfrac{49}{144}]\\
&= 6(x - \tfrac{11}{12} + \tfrac{7}{12})(x - \tfrac{11}{12} - \tfrac{7}{12})\\
&= 6(x - \tfrac{1}{3})(x - \tfrac{3}{2})\\
&= (3x - 1)(2x - 3).
\end{aligned}$$

Another Method

$$\begin{aligned}
&6x^2 - 11x + 3\\
&= 6(x^2 - \tfrac{11}{6}x + \tfrac{3}{6})\\
&= 6(x^2 - \tfrac{11}{6}x + \tfrac{18}{36})\\
&= 6(x - \tfrac{2}{6})(x - \tfrac{9}{6})\\
&= 6(x - \tfrac{1}{3})(x - \tfrac{3}{2})\\
&= (3x - 1)(2x - 3).
\end{aligned}$$

6. Resolve $3x^2 + x - 10$ into two factors.

$$\begin{aligned}
&3x^2 + x - 10\\
&= 3(x^2 + \tfrac{1}{3}x - \tfrac{10}{3})\\
&= 3[(x^2 + \tfrac{1}{3}x + \tfrac{1}{36}) - \tfrac{1}{36} - \tfrac{10}{3}]\\
&= 3[(x + \tfrac{1}{6})^2 - \tfrac{121}{36}]\\
&= 3(x + \tfrac{1}{6} + \tfrac{11}{6})(x + \tfrac{1}{6} - \tfrac{11}{6})\\
&= 3(x + 2)(x - \tfrac{5}{3})\\
&= (x + 2)(3x - 5).
\end{aligned}$$

Another Method

$$\begin{aligned}
&3x^2 + x - 10\\
&= 3(x^2 + \tfrac{1}{3}x - \tfrac{10}{3})\\
&= 3(x^2 + \tfrac{1}{3}x - \tfrac{30}{9})\\
&= 3(x + \tfrac{6}{3})(x - \tfrac{5}{3})\\
&= 3(x + 2)(x - \tfrac{5}{3})\\
&= (x + 2)(3x - 5).
\end{aligned}$$

7. Resolve $8x^2 + 22x + 5$ into two factors.

$$
\begin{aligned}
&8x^2 + 22x + 5\\
&\quad 8\left(x^2 + \tfrac{11}{4}x + \tfrac{5}{8}\right)\\
&\quad 8\left[\left(x^2 + \tfrac{11}{4}x + \tfrac{121}{64}\right) - \tfrac{121}{64} + \tfrac{5}{8}\right]\\
&\quad 8\left[\left(x + \tfrac{11}{8}\right)^2 - \tfrac{81}{64}\right]\\
&\quad 8\left(x + \tfrac{11}{8} + \tfrac{9}{8}\right)\left(x + \tfrac{11}{8} - \tfrac{9}{8}\right)\\
&\quad 8\left(x + \tfrac{5}{2}\right)\left(x + \tfrac{1}{4}\right)\\
&= (2x + 5)(4x + 1).
\end{aligned}
$$

Another Method

$$
\begin{aligned}
&8x^2 + 22x + 5\\
&\quad = 8\left(x^2 + \tfrac{22}{8}x + \tfrac{5}{8}\right)\\
&\quad = 8\left(x^2 + \tfrac{11}{4}x + \tfrac{10}{16}\right)\\
&\quad = 8\left(x + \tfrac{10}{4}\right)\left(x + \tfrac{1}{4}\right)\\
&\quad = 8\left(x + \tfrac{5}{2}\right)\left(x + \tfrac{1}{4}\right)\\
&\quad = (2x + 5)(4x + 1).
\end{aligned}
$$

8. Resolve $9x^2 - 3x - 2$ into two factors.

$$
\begin{aligned}
&9x^2 - 3x - 2\\
&\quad 9\left(x^2 - \tfrac{1}{3}x - \tfrac{2}{9}\right)\\
&\quad 9\left[\left(x^2 - \tfrac{1}{3}x + \tfrac{1}{36}\right) - \tfrac{1}{36} - \tfrac{2}{9}\right]\\
&= 9\left[\left(x - \tfrac{1}{6}\right)^2 - \tfrac{1}{4}\right]\\
&\quad 9\left(x - \tfrac{1}{6} + \tfrac{1}{2}\right)\left(x - \tfrac{1}{6} - \tfrac{1}{2}\right)\\
&\quad 9\left(x + \tfrac{1}{3}\right)\left(x - \tfrac{2}{3}\right)\\
&= (3x + 1)(3x - 2).
\end{aligned}
$$

Another Method

$$
\begin{aligned}
&9x^2 - 3x - 2\\
&\quad = 9\left(x^2 - \tfrac{3}{9}x - \tfrac{2}{9}\right)\\
&\quad = 9\left(x^2 - \tfrac{1}{3}x - \tfrac{2}{9}\right)\\
&\quad = 9\left(x + \tfrac{1}{3}\right)\left(x - \tfrac{2}{3}\right)\\
&\quad = (3x + 1)(3x - 2).
\end{aligned}
$$

9. Resolve $6x^2 + 13x - 5$ into two factors.

$$
\begin{aligned}
&6x^2 + 13x - 5\\
&\quad 6\left(x^2 + \tfrac{13}{6}x - \tfrac{5}{6}\right)\\
&\quad 6\left[\left(x^2 + \tfrac{13}{6}x + \tfrac{169}{144}\right) - \tfrac{169}{144} - \tfrac{5}{6}\right]\\
&\quad 6\left[\left(x + \tfrac{13}{12}\right)^2 - \tfrac{289}{144}\right]\\
&\quad 6\left(x + \tfrac{13}{12} + \tfrac{17}{12}\right)\left(x + \tfrac{13}{12} - \tfrac{17}{12}\right)\\
&\quad 6\left(x + \tfrac{5}{2}\right)\left(x - \tfrac{1}{3}\right)\\
&\quad (2x + 5)(3x - 1).
\end{aligned}
$$

Another Method

$$
\begin{aligned}
&6x^2 + 13x - 5\\
&\quad = 6\left(x^2 + \tfrac{13}{6}x - \tfrac{5}{6}\right)\\
&\quad = 6\left(x^2 + \tfrac{13}{6}x - \tfrac{30}{36}\right)\\
&\quad = 6\left(x + \tfrac{15}{6}\right)\left(x - \tfrac{2}{6}\right)\\
&\quad = 6\left(x + \tfrac{5}{2}\right)\left(x - \tfrac{1}{3}\right)\\
&\quad = (2x + 5)(3x - 1).
\end{aligned}
$$

10. Resolve $6x^2 + x - 15$ into two factors.

$$
\begin{aligned}
&6x^2 + x - 15\\
&= 6\left(x^2 + \tfrac{1}{6}x - \tfrac{5}{2}\right)\\
&= 6\left[\left(x^2 + \tfrac{1}{6}x + \tfrac{1}{144}\right) - \tfrac{1}{144} - \tfrac{5}{2}\right]\\
&\quad 6\left[\left(x + \tfrac{1}{12}\right)^2 - \tfrac{361}{144}\right]\\
&= 6\left(x + \tfrac{1}{12} + \tfrac{19}{12}\right)\left(x + \tfrac{1}{12} - \tfrac{19}{12}\right)\\
&\quad 6\left(x + \tfrac{5}{3}\right)\left(x - \tfrac{3}{2}\right)\\
&= (3x + 5)(2x - 3).
\end{aligned}
$$

Another Method

$$
\begin{aligned}
&6x^2 + x - 15\\
&\quad = 6\left(x^2 + \tfrac{1}{6}x - \tfrac{15}{6}\right)\\
&\quad = 6\left(x^2 + \tfrac{1}{6}x - \tfrac{90}{36}\right)\\
&\quad = 6\left(x + \tfrac{10}{6}\right)\left(x - \tfrac{9}{6}\right)\\
&\quad = 6\left(x + \tfrac{5}{3}\right)\left(x - \tfrac{3}{2}\right)\\
&\quad = (3x + 5)(2x - 3).
\end{aligned}
$$

11. Resolve $9x^2 - 12x - 1$ into two factors.

$$\begin{aligned}9x^2 - 12x - 1 &= 9(x^2 - \tfrac{4}{3}x - \tfrac{1}{9})\\ &= 9[(x^2 - \tfrac{4}{3}x + \tfrac{4}{9}) - \tfrac{4}{9} - \tfrac{1}{9}]\\ &= 9[(x - \tfrac{2}{3})^2 - \tfrac{5}{9}]\\ &= 9(x - \tfrac{2}{3} + \tfrac{1}{3}\sqrt{5})(x - \tfrac{2}{3} - \tfrac{1}{3}\sqrt{5})\\ &= (3x - 2 + \sqrt{5})(3x - 2 - \sqrt{5}).\end{aligned}$$

12. Resolve $2x^2 + 10x - 9$ into two factors.

$$\begin{aligned}2x^2 + 10x - 9 &= 2(x^2 + 5x - \tfrac{9}{2})\\ &= 2[(x^2 + 5x + \tfrac{25}{4}) - \tfrac{25}{4} - \tfrac{9}{2}]\\ &= 2[(x + \tfrac{5}{2})^2 - \tfrac{43}{4}]\\ &= 2(x + \tfrac{5}{2} + \tfrac{1}{2}\sqrt{43})(x + \tfrac{5}{2} - \tfrac{1}{2}\sqrt{43}).\end{aligned}$$

13. Resolve $3x^2 - 7x - 5$ into two factors.

$$\begin{aligned}3x^2 - 7x - 5 &= 3(x^2 - \tfrac{7}{3}x - \tfrac{5}{3})\\ &= 3[(x^2 - \tfrac{7}{3}x + \tfrac{49}{36}) - \tfrac{49}{36} - \tfrac{5}{3}]\\ &= 3[(x - \tfrac{7}{6})^2 - \tfrac{109}{36}]\\ &= 3(x - \tfrac{7}{6} + \tfrac{1}{6}\sqrt{109})(x - \tfrac{7}{6} - \tfrac{1}{6}\sqrt{109}).\end{aligned}$$

14. Resolve $6x^2 - 5x + 5$ into two factors.

$$\begin{aligned}6x^2 - 5x + 5 &= 6(x^2 - \tfrac{5}{6}x + \tfrac{5}{6})\\ &= 6[(x^2 - \tfrac{5}{6}x + \tfrac{25}{144}) - \tfrac{25}{144} + \tfrac{5}{6}]\\ &= 6[(x - \tfrac{5}{12})^2 - (-\tfrac{95}{144})]\\ &= 6(x - \tfrac{5}{12} + \tfrac{1}{12}\sqrt{-95})(x - \tfrac{5}{12} - \tfrac{1}{12}\sqrt{-95}).\end{aligned}$$

Exercise 112. Page 275

Find all the roots of:

1. $x^2 + 11x + 30 = 0.$

$(x + 5)(x + 6) = 0.$

$\therefore x = -5, \text{ or } -6.$

2. $x^2 - 2x - 24 = 0.$

$(x - 6)(x + 4) = 0.$

$\therefore x = 6, \text{ or } -4.$

3. $x^2 - 10x + 21 = 0.$

$(x - 3)(x - 7) = 0.$

$\therefore x = 3, \text{ or } 7.$

4. $x^2 - 2x - 3 = 0.$

$(x - 3)(x + 1) = 0.$

$\therefore x = 3, \text{ or } -1.$

5. $36x^2 - 35x + 6 = 0.$

$(4x - 3)(9x - 2) = 0.$

$4x - 3 = 0,$

or $9x - 2 = 0.$

$4x = 3,$

or $9x = 2.$

$\therefore x = \frac{3}{4}, \text{ or } \frac{2}{9}.$

6. $(7x-8)(2x-3)=0.$

$$7x-8=0, \text{ or } 2x-3=0.$$
$$7x=8, \text{ or } 2x=3.$$
$$\therefore x=\tfrac{8}{7}, \text{ or } \tfrac{3}{2}.$$

7. $(3x+2)(5x+9)=0.$

$$3x+2=0, \text{ or } 5x+9=0.$$
$$3x=-2, \text{ or } 5x=-9.$$
$$\therefore x=-\tfrac{2}{3}, \text{ or } -\tfrac{9}{5}.$$

8. $(10-x)(6+5x)=0.$

$$10-x=0, \text{ or } 6+5x=0.$$
$$-x=-10, \text{ or } 5x=-6.$$
$$\therefore x=10, \text{ or } -\tfrac{6}{5}.$$

9. $x^2-16x=0.$

$$x(x-16)=0.$$
$$\therefore x=0, \text{ or } 16.$$

10. $x^2+23x=0.$

$$x(x+23)=0.$$
$$\therefore x=0, \text{ or } -23.$$

11. $4x^2+28x+49=0.$

$$(2x+7)(2x+7)=0.$$
$$2x+7=0, \text{ or } 2x+7=0.$$
$$2x=-7, \text{ or } 2x=-7.$$
$$\therefore x=-\tfrac{7}{2}, \text{ or } -\tfrac{7}{2}.$$

12. $x^3-1=0.$

$$(x-1)(x^2+x+1)=0.$$
$$\therefore x=1, \text{ or } x^2+x+1=0.$$
$$x^2+x=-1.$$
$$4x^2+(\ )+1=-3.$$
$$2x+1=\pm\sqrt{-3}.$$
$$2x=-1\pm\sqrt{-3}.$$
$$\therefore x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$
$$\therefore x=1, \text{ or } -\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$

13. $x^3+1=0.$

$$(x+1)(x^2-x+1)=0.$$
$$\therefore x=-1, \text{ or } x^2-x+1=0.$$
$$x^2-x=-1.$$
$$4x^2-(\ )+1=-3.$$
$$2x-1=\pm\sqrt{-3}.$$
$$2x=1\pm\sqrt{-3}.$$
$$\therefore x=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$
$$\therefore x=-1, \text{ or } \tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$

14. $x^3-27=0.$

$$(x-3)(x^2+3x+9)=0.$$
$$\therefore x=3, \text{ or } x^2+3x+9=0.$$
$$x^2+3x=-9.$$
$$4x^2+(\ )+9=-27.$$

$$\begin{aligned} 2x+3&=\pm 3\sqrt{-3}.\\ 2x&=-3\pm 3\sqrt{-3}.\\ \therefore\ x&=-\tfrac{3}{2}\pm\tfrac{3}{2}\sqrt{-3}.\\ \therefore\ x&=3,\text{ or }-\tfrac{3}{2}\pm\tfrac{3}{2}\sqrt{-3}.\end{aligned}$$

15. $x^4-16=0.$

$$\begin{aligned} (x^2-4)(x^2+4)&=0.\\ (x-2)(x+2)(x^2+4)&=0.\\ \therefore\ x&=2,\text{ or }-2,\text{ or }x^2+4=0.\\ x^2&=-4.\\ \therefore\ x&=\pm 2\sqrt{-1}.\\ \therefore\ x&=\pm 2,\text{ or }\pm 2\sqrt{-1}.\end{aligned}$$

16. $8x^3-27=0.$

$$\begin{aligned} (2x-3)(4x^2+6x+9)&=0.\\ \therefore\ 2x-3&=0,\text{ or }4x^2+6x+9=0.\end{aligned}$$

(i)
$$\begin{aligned} 2x-3&=0.\\ 2x&=3.\\ \therefore\ x&=\tfrac{3}{2}.\end{aligned}$$

(ii)
$$\begin{aligned} 4x^2+6x+9&=0.\\ 4x^2+6x&=-9.\\ 16x^2+24x+9&=-27.\\ 4x+3&=\pm 3\sqrt{-3}.\\ 4x&=-3\pm 3\sqrt{-3}.\\ \therefore\ x&=-\tfrac{3}{4}\pm\tfrac{3}{4}\sqrt{-3}.\\ \therefore\ x&=\tfrac{3}{2},\text{ or }-\tfrac{3}{4}\pm\tfrac{3}{4}\sqrt{-3}.\end{aligned}$$

17. $8x^3+27=0.$

$$\begin{aligned} (2x+3)(4x^2-6x+9)&=0.\\ \therefore\ 2x+3&=0,\text{ or }4x^2-6x+9=0.\end{aligned}$$

(i)
$$\begin{aligned} 2x+3&=0.\\ 2x&=-3.\\ \therefore\ x&=-\tfrac{3}{2}.\end{aligned}$$

(ii)
$$\begin{aligned} 4x^2-6x+9&=0.\\ 4x^2-6x&=-9.\\ 16x^2-24x+9&=-27.\\ 4x-3&=\pm 3\sqrt{-3}.\\ 4x&=3\pm 3\sqrt{-3}.\\ \therefore\ x&=\tfrac{3}{4}\pm\tfrac{3}{4}\sqrt{-3}.\\ \therefore\ x&=-\tfrac{3}{2},\text{ or }\tfrac{3}{4}\pm\tfrac{3}{4}\sqrt{-3}.\end{aligned}$$

18. $216 + 125x^3 = 0.$

$$(6 + 5x)(36 - 30x + 25x^2) = 0.$$

$$\therefore\ 6 + 5x = 0, \text{ or } 36 - 30x + 25x^2 = 0.$$

(i)
$$\begin{aligned} 6 + 5x &= 0. \\ 5x &= -6. \\ \therefore\ x &= -\tfrac{6}{5}. \end{aligned}$$

(ii)
$$\begin{aligned} 36 - 30x + 25x^2 &= 0. \\ 25x^2 - 30x &= -36. \\ 25x^2 - 30x + 9 &= -27. \\ 5x - 3 &= \pm 3\sqrt{-3}. \\ 5x &= 3 \pm 3\sqrt{-3}. \\ \therefore\ x &= \tfrac{3}{5} \pm \tfrac{3}{5}\sqrt{-3}. \\ \therefore\ x &= -\tfrac{6}{5}, \text{ or } \tfrac{3}{5} \pm \tfrac{3}{5}\sqrt{-3}. \end{aligned}$$

19. $x^4 + 4x^2 + 16 = 0.$

$$\begin{aligned} x^4 + 4x^2 + 16 &= x^4 + 8x^2 + 16 - 4x^2 \\ &= (x^2 + 4)^2 - (2x)^2 \\ &= (x^2 + 2x + 4)(x^2 - 2x + 4). \end{aligned}$$

$$\therefore\ (x^2 + 2x + 4)(x^2 - 2x + 4) = 0.$$

(i)
$$\begin{aligned} x^2 + 2x + 4 &= 0. \\ x^2 + 2x &= -4. \\ x^2 + 2x + 1 &= -3. \\ x + 1 &= \pm\sqrt{-3}. \\ \therefore\ x &= -1 \pm \sqrt{-3}. \end{aligned}$$

(ii)
$$\begin{aligned} x^2 - 2x + 4 &= 0. \\ x^2 - 2x &= -4. \\ x^2 - 2x + 1 &= -3. \\ x - 1 &= \pm\sqrt{-3}. \\ \therefore\ x &= 1 \pm \sqrt{-3}. \\ \therefore\ x &= 1 \pm \sqrt{-3}, \text{ or } -1 \pm \sqrt{-3}. \end{aligned}$$

20. $x^4 + 64 = 0.$

$$\begin{aligned} x^4 + 64 &= x^4 + 16x^2 + 64 - 16x^2 \\ &= (x^2 + 8)^2 - (4x)^2 \\ &= (x^2 + 4x + 8)(x^2 - 4x + 8). \end{aligned}$$

$$\therefore\ (x^2 + 4x + 8)(x^2 - 4x + 8) = 0.$$

(i)
$$\begin{aligned} x^2 + 4x + 8 &= 0. \\ x^2 + 4x &= -8. \\ x^2 + 4x + 4 &= -4. \\ x + 2 &= \pm 2\sqrt{-1}. \\ \therefore\ x &= -2 \pm 2\sqrt{-1}. \end{aligned}$$

(ii)

$$\begin{aligned}
x^2 - 4x + 8 &= 0.\\
x^2 - 4x &= -8.\\
x^2 - 4x + 4 &= -4.\\
x - 2 &= \pm 2\sqrt{-1}.\\
\therefore\ x &= 2 \pm 2\sqrt{-1}.\\
\therefore\ x &= 2 \pm 2\sqrt{-1}, \text{ or } -2 \pm 2\sqrt{-1}.
\end{aligned}$$

21. $4x^4 - 29x^2 + 25 = 0.$

$$\begin{aligned}
4x^4 - 29x^2 + 25 &= 4x^4 - 20x^2 + 25 - 9x^2\\
&= (2x^2 - 5)^2 - (3x)^2\\
&= (2x^2 + 3x - 5)(2x^2 - 3x - 5).\\
2x^2 + 3x - 5 &= (x - 1)(2x + 5).\\
2x^2 - 3x - 5 &= (x + 1)(2x - 5).\\
\therefore\ 4x^4 - 29x^2 + 25 &= (x - 1)(x + 1)(2x - 5)(2x + 5).\\
\therefore\ x &= 1,\ -1,\ \tfrac{5}{2}, \text{ or } -\tfrac{5}{2}.\\
\therefore\ x &= \pm 1, \text{ or } \pm \tfrac{5}{2}.
\end{aligned}$$

22. $35x^2 - 83x + 36 = 0.$

$$\begin{aligned}
35x^2 - 83x + 36 &= 35\left(x^2 - \tfrac{83}{35}x + \tfrac{36}{35}\right)\\
&= 35\left[\left(x^2 - \tfrac{83}{35}x + \tfrac{6889}{4900}\right) - \tfrac{6889}{4900} + \tfrac{36}{35}\right]\\
&= 35\left[\left(x - \tfrac{83}{70}\right)^2 - \tfrac{1849}{4900}\right]\\
&= 35\left(x - \tfrac{83}{70} + \tfrac{43}{70}\right)\left(x - \tfrac{83}{70} - \tfrac{43}{70}\right)\\
&= 35\left(x - \tfrac{4}{7}\right)\left(x - \tfrac{9}{5}\right)\\
&= (7x - 4)(5x - 9).\\
\therefore\ (7x - 4)(5x - 9) &= 0.\\
\therefore\ 7x - 4 &= 0, \text{ or } 5x - 9 = 0.\\
\therefore\ x &= \tfrac{4}{7}, \text{ or } \tfrac{9}{5}.
\end{aligned}$$

23. $42x^2 - 59x + 20 = 0.$

$$\begin{aligned}
42x^2 - 59x + 20 &= 42\left(x^2 - \tfrac{59}{42}x + \tfrac{10}{21}\right)\\
&= 42\left[\left(x^2 - \tfrac{59}{42}x + \tfrac{3481}{7056}\right) - \tfrac{3481}{7056} + \tfrac{10}{21}\right]\\
&= 42\left[\left(x - \tfrac{59}{84}\right)^2 - \tfrac{121}{7056}\right]\\
&= 42\left(x - \tfrac{59}{84} + \tfrac{11}{84}\right)\left(x - \tfrac{59}{84} - \tfrac{11}{84}\right)\\
&= 42\left(x - \tfrac{4}{7}\right)\left(x - \tfrac{5}{6}\right)\\
&= (7x - 4)(6x - 5).\\
\therefore\ (7x - 4)(6x - 5) &= 0.\\
\therefore\ 7x - 4 &= 0, \text{ or } 6x - 5 = 0.\\
\therefore\ x &= \tfrac{4}{7}, \text{ or } \tfrac{5}{6}.
\end{aligned}$$

[illegible]

25. $x^3 - [illegible]x^2 - [illegible]x = 0.$

$x(x^2 - [illegible]x - [illegible]) = 0.$

$x(x - 4)(x - 17) = 0.$

$\therefore x = 0, x - 4 = 0,$ or $x - 17 = 0.$

$\therefore x = 0, 4,$ or $17.$

26. $y^5 + 5y^4 - 36y^3 = 0.$

$y^3(y^2 + 5y - 36) = 0.$

$y^3(y - 4)(y + 9) = 0.$

$\therefore y^3 = 0, y - 4 = 0,$ or $y + 9 = 0.$

$\therefore y = 0, 0, 0, 4,$ or $-9.$

27. $x^3 + x^2 - 22x - 40 = 0.$

$(x + 2)(x^2 - x - 20) = 0.$ § 147

$(x + 2)(x - 5)(x + 4) = 0.$

$\therefore x + 2 = 0, x - 5 = 0,$ or $x + 4 = 0.$

$\therefore x = -2, 5,$ or $-4.$

28. $9x^4 + 6x^3 + 3x^2 + 2x = 0.$

$$\begin{aligned} 9x^4 + 6x^3 + 3x^2 + 2x &= x(9x^3 + 6x^2 + 3x + 2) \\ &= x[3x^2(3x + 2) + (3x + 2)] \\ &= x(3x + 2)(3x^2 + 1). \end{aligned}$$

$\therefore\ x(3x + 2)(3x^2 + 1) = 0.$

$\therefore\ x = 0,\ 3x + 2 = 0,\ 3x^2 + 1 = 0.$

(i) $x = 0.$

(ii) $3x + 2 = 0.$

$\therefore\ x = -\frac{2}{3}.$

(iii) $3x^2 + 1 = 0.$

$3x^2 = -1.$

$x^2 = -\frac{1}{3}.$

$x = \pm\sqrt{-\frac{1}{3}} = \pm\frac{1}{3}\sqrt{-3}.$

$\therefore\ x = 0,\ -\frac{2}{3}$, or $\pm\frac{1}{3}\sqrt{-3}.$

29. $4x^3 - 12x^2 + 9x - 1 = 0.$

$4x^3 - 12x^2 + 9x - 1 = (x - 1)(4x^2 - 8x + 1).$

$\therefore\ (x - 1)(4x^2 - 8x + 1) = 0.$

$\therefore\ x - 1 = 0$, or $4x^2 - 8x + 1 = 0.$

(i) $x - 1 = 0.$

$\therefore\ x = 1.$

(ii) $4x^2 - 8x + 1 = 0.$

$4x^2 - 8x = -1.$

$4x^2 - 8x + 4 = 3.$

$2x - 2 = \pm\sqrt{3}.$

$2x = 2 \pm \sqrt{3}.$

$\therefore\ x = 1 \pm \frac{1}{2}\sqrt{3}.$

$\therefore\ x = 1$, or $1 \pm \frac{1}{2}\sqrt{3}.$

30. $6x^4 + 25x^3 + 5x^2 - 60x - 36 = 0.$

$6x^4 + 25x^3 + 5x^2 - 60x - 36$

$= (x + 2)(6x^3 + 13x^2 - 21x - 18)$ § 147

$= (x + 2)(x + 3)(6x^2 - 5x - 6).$ § 147

$$\begin{aligned} 6x^2 - 5x - 6 &= 6(x^2 - \tfrac{5}{6}x - \tfrac{6}{6}) \\ &= 6(x^2 - \tfrac{5}{6}x - \tfrac{36}{36}) \\ &= 6(x + \tfrac{4}{6})(x - \tfrac{9}{6}) \\ &= 6(x + \tfrac{2}{3})(x - \tfrac{3}{2}) \\ &= (3x + 2)(2x - 3). \end{aligned}$$

$\therefore\ (x + 2)(x + 3)(3x + 2)(2x - 3) = 0.$

$\therefore\ x + 2 = 0,\ x + 3 = 0,\ 3x + 2 = 0$, or $2x - 3 = 0.$

$\therefore\ x = -2,\ -3,\ -\frac{2}{3}$, or $\frac{3}{2}.$

31. $6x^4 + 13x^3 - 85x^2 - 104x + 240 = 0.$

$6x^4 + 13x^3 - 85x^2 - 104x + 240$
$= (x - 3)(6x^3 + 31x^2 + 8x - 80)$ § 147
$= (x - 3)(x + 4)(6x^2 + 7x - 20).$ § 147
$6x^2 + 7x - 20 = 6(x^2 + \frac{7}{6}x - \frac{20}{6})$
$= 6(x^2 + \frac{7}{6}x - \frac{120}{36})$
$= 6(x + \frac{15}{6})(x - \frac{8}{6})$
$= 6(x + \frac{5}{2})(x - \frac{4}{3})$
$= (2x + 5)(3x - 4).$
$\therefore (x - 3)(x + 4)(2x + 5)(3x - 4) = 0.$
$\therefore x - 3 = 0,\ x + 4 = 0,\ 2x + 5 = 0,$ or $3x - 4 = 0.$
$\therefore x = 3,\ -4,\ -\frac{5}{2},$ or $\frac{4}{3}.$

32. $x^4 - 17x^2 - 36x - 20 = 0.$

$x^4 - 17x^2 - 36x - 20$
$= (x + 1)(x^3 - x^2 - 16x - 20)$ § 147
$= (x + 1)(x + 2)(x^2 - 3x - 10)$ § 147
$= (x + 1)(x + 2)(x + 2)(x - 5).$
$\therefore (x + 1)(x + 2)(x + 2)(x - 5) = 0.$
$\therefore x = -1,\ -2,\ -2,$ or $5.$

33. $6x^4 + 49x^3 + 146x^2 + 189x + 90 = 0.$

$6x^4 + 49x^3 + 146x^2 + 189x + 90$
$= (x + 2)(6x^3 + 37x^2 + 72x + 45)$ § 147
$= (x + 2)(x + 3)(6x^2 + 19x + 15).$ § 147
$6x^2 + 19x + 15 = 6(x^2 + \frac{19}{6}x + \frac{15}{6})$
$= 6(x^2 + \frac{19}{6} + \frac{90}{36})$
$= 6(x + \frac{10}{6})(x + \frac{9}{6})$
$= 6(x + \frac{5}{3})(x + \frac{3}{2})$
$= (3x + 5)(2x + 3).$
$\therefore (x + 2)(x + 3)(3x + 5)(2x + 3) = 0.$
$\therefore x + 2 = 0,\ x + 3 = 0,\ 3x + 5 = 0,$ or $2x + 3 = 0.$
$\therefore x = -2,\ -3,\ -\frac{5}{3},$ or $-\frac{3}{2}.$

34. $2x^4 - 3x^3 - 4x^2 + 3x + 2 = 0.$

$2x^4 - 3x^3 - 4x^2 + 3x - 2$
$= (x - 1)(2x^3 - x^2 - 5x - 2)$ § 147
$= (x - 1)(x + 1)(2x^2 - 3x - 2)$ § 147
$= (x - 1)(x + 1)(x - 2)(2x + 1).$ § 147
$\therefore (x - 1)(x + 1)(x - 2)(2x + 1) = 0.$
$\therefore x - 1 = 0,\ x + 1 = 0,\ x - 2 = 0,$ or $2x + 1 = 0.$
$\therefore x = 1,\ -1,\ 2,$ or $-\frac{1}{2}.$

35. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0.$

$6x^4 - 35x^3 + 62x^2 - 35x + 6$

$= (x-2)(6x^3 - 23x^2 + 16x - 3)$ § 147

$= (x-2)(x-3)(6x^2 - 5x + 1)$ § 147

$= (x-2)(x-3)(2x-1)(3x-1).$

$\therefore (x-2)(x-3)(2x-1)(3x-1) = 0.$

$\therefore x-2=0,\ x-3=0,\ 2x-1=0, \text{ or } 3x-1=0.$

$\therefore x = 2,\ 3,\ \frac{1}{2}, \text{ or } \frac{1}{3}.$

36. $x^5 - 3x^4 - 17x^3 - x^2 + 24x - 4 = 0.$

$x^5 - 3x^4 - 17x^3 - x^2 + 24x - 4$

$= (x-1)(x^4 - 2x^3 - 19x^2 - 20x + 4)$ § 147

$= (x-1)(x+2)(x^3 - 4x^2 - 11x + 2)$ § 147

$= (x-1)(x+2)(x+2)(x^2 - 6x + 1).$ § 147

$\therefore (x-1)(x+2)(x+2)(x^2-6x+1) = 0.$

$\therefore x-1=0,\ x+2=0,\ x+2=0, \text{ or } x^2-6x+1=0.$

$\therefore x = 1,\ -2, \text{ or } -2.$

$$x^2 - 6x + 1 = 0.$$
$$x^2 - 6x = -1.$$
$$x^2 - 6x + 9 = 8.$$
$$x - 3 = \pm 2\sqrt{2}.$$
$$\therefore x = 3 \pm 2\sqrt{2}.$$
$$\therefore x = 1,\ -2,\ -2, \text{ or } 3 \pm 2\sqrt{2}.$$

37. $(x^2 - 5x + 7)^2 - (x-2)(x-3) - 1 = 0.$

$$(x^2 - 5x + 7)^2 - (x^2 - 5x + 6) - 1 = 0.$$
$$(x^2 - 5x + 7)^2 - (x^2 - 5x + 7) = 0.$$
$$(x^2 - 5x + 7)(x^2 - 5x + 7 - 1) = 0.$$
$$(x^2 - 5x + 6)(x^2 - 5x + 7) = 0.$$
$$(x-2)(x-3)(x^2 - 5x + 7) = 0.$$
$$\therefore x-2=0,\ x-3=0, \text{ or } x^2-5x+7=0.$$
$$\therefore x = 2, \text{ or } 3.$$
$$x^2 - 5x + 7 = 0.$$
$$x^2 - 5x = -7.$$
$$4x^2 - (\ ) + 25 = -3.$$
$$2x - 5 = \pm\sqrt{-3}.$$
$$2x = 5 \pm \sqrt{-3}.$$
$$\therefore x = \tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{-3}.$$
$$\therefore x = 2,\ 3, \text{ or } \tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{-3}.$$

7. Resolve $8x^2 + 22x + 5$ into two factors.

$$\begin{aligned}
&8x^2 + 22x + 5\\
&= 8(x^2 + \tfrac{11}{4}x + \tfrac{5}{8})\\
&= 8[(x^2 + \tfrac{11}{4}x + \tfrac{121}{64}) - \tfrac{121}{64} + \tfrac{5}{8}]\\
&= 8[(x + \tfrac{11}{8})^2 - \tfrac{81}{64}]\\
&= 8(x + \tfrac{11}{8} + \tfrac{9}{8})(x + \tfrac{11}{8} - \tfrac{9}{8})\\
&= 8(x + \tfrac{5}{2})(x + \tfrac{1}{4})\\
&= (2x + 5)(4x + 1).
\end{aligned}$$

Another Method

$$\begin{aligned}
&8x^2 + 22x + 5\\
&= 8(x^2 + \tfrac{22}{8}x + \tfrac{5}{8})\\
&= 8(x^2 + \tfrac{11}{4}x + \tfrac{10}{16})\\
&= 8(x + \tfrac{10}{4})(x + \tfrac{1}{4})\\
&= 8(x + \tfrac{5}{2})(x + \tfrac{1}{4})\\
&= (2x + 5)(4x + 1).
\end{aligned}$$

8. Resolve $9x^2 - 3x - 2$ into two factors.

$$\begin{aligned}
&9x^2 - 3x - 2\\
&= 9(x^2 - \tfrac{1}{3}x - \tfrac{2}{9})\\
&= 9[(x^2 - \tfrac{1}{3}x + \tfrac{1}{36}) - \tfrac{1}{36} - \tfrac{2}{9}]\\
&= 9[(x - \tfrac{1}{6})^2 - \tfrac{1}{4}]\\
&= 9(x - \tfrac{1}{6} + \tfrac{1}{2})(x - \tfrac{1}{6} - \tfrac{1}{2})\\
&= 9(x + \tfrac{1}{3})(x - \tfrac{2}{3})\\
&= (3x + 1)(3x - 2).
\end{aligned}$$

Another Method

$$\begin{aligned}
&9x^2 - 3x - 2\\
&= 9(x^2 - \tfrac{3}{9}x - \tfrac{2}{9})\\
&= 9(x^2 - \tfrac{1}{3}x - \tfrac{2}{9})\\
&= 9(x + \tfrac{1}{3})(x - \tfrac{2}{3})\\
&= (3x + 1)(3x - 2).
\end{aligned}$$

9. Resolve $6x^2 + 13x - 5$ into two factors.

$$\begin{aligned}
&6x^2 + 13x - 5\\
&= 6(x^2 + \tfrac{13}{6}x - \tfrac{5}{6})\\
&= 6[(x^2 + \tfrac{13}{6}x + \tfrac{169}{144}) - \tfrac{169}{144} - \tfrac{5}{6}]\\
&= 6[(x + \tfrac{13}{12})^2 - \tfrac{289}{144}]\\
&= 6(x + \tfrac{13}{12} + \tfrac{17}{12})(x + \tfrac{13}{12} - \tfrac{17}{12})\\
&= 6(x + \tfrac{5}{2})(x - \tfrac{1}{3})\\
&= (2x + 5)(3x - 1).
\end{aligned}$$

Another Method

$$\begin{aligned}
&6x^2 + 13x - 5\\
&= 6(x^2 + \tfrac{13}{6}x - \tfrac{5}{6})\\
&= 6(x^2 + \tfrac{13}{6}x - \tfrac{30}{36})\\
&= 6(x + \tfrac{15}{6})(x - \tfrac{2}{6})\\
&= 6(x + \tfrac{5}{2})(x - \tfrac{1}{3})\\
&= (2x + 5)(3x - 1).
\end{aligned}$$

10. Resolve $6x^2 + x - 15$ into two factors.

$$\begin{aligned}
&6x^2 + x - 15\\
&= 6(x^2 + \tfrac{1}{6}x - \tfrac{5}{2})\\
&= 6[(x^2 + \tfrac{1}{6}x + \tfrac{1}{144}) - \tfrac{1}{144} - \tfrac{5}{2}]\\
&= 6[(x + \tfrac{1}{12})^2 - \tfrac{361}{144}]\\
&= 6(x + \tfrac{1}{12} + \tfrac{19}{12})(x + \tfrac{1}{12} - \tfrac{19}{12})\\
&= 6(x + \tfrac{5}{3})(x - \tfrac{3}{2})\\
&= (3x + 5)(2x - 3).
\end{aligned}$$

Another Method

$$\begin{aligned}
&6x^2 + x - 15\\
&= 6(x^2 + \tfrac{1}{6}x - \tfrac{15}{6})\\
&= 6(x^2 + \tfrac{1}{6}x - \tfrac{90}{36})\\
&= 6(x + \tfrac{10}{6})(x - \tfrac{9}{6})\\
&= 6(x + \tfrac{5}{3})(x - \tfrac{3}{2})\\
&= (3x + 5)(2x - 3).
\end{aligned}$$

11. Resolve $9x^2 - 12x - 1$ into two factors.

$$\begin{aligned}9x^2 - 12x - 1 &= 9(x^2 - \tfrac{4}{3}x - \tfrac{1}{9})\\ &= 9[(x^2 - \tfrac{4}{3}x + \tfrac{4}{9}) - \tfrac{4}{9} - \tfrac{1}{9}]\\ &= 9[(x - \tfrac{2}{3})^2 - \tfrac{5}{9}]\\ &= 9(x - \tfrac{2}{3} + \tfrac{1}{3}\sqrt{5})(x - \tfrac{2}{3} - \tfrac{1}{3}\sqrt{5})\\ &= (3x - 2 + \sqrt{5})(3x - 2 - \sqrt{5}).\end{aligned}$$

12. Resolve $2x^2 + 10x - 9$ into two factors.

$$\begin{aligned}2x^2 + 10x - 9 &= 2(x^2 + 5x - \tfrac{9}{2})\\ &= 2[(x^2 + 5x + \tfrac{25}{4}) - \tfrac{25}{4} - \tfrac{9}{2}]\\ &= 2[(x + \tfrac{5}{2})^2 - \tfrac{43}{4}]\\ &= 2(x + \tfrac{5}{2} + \tfrac{1}{2}\sqrt{43})(x + \tfrac{5}{2} - \tfrac{1}{2}\sqrt{43}).\end{aligned}$$

13. Resolve $3x^2 - 7x - 5$ into two factors.

$$\begin{aligned}3x^2 - 7x - 5 &= 3(x^2 - \tfrac{7}{3}x - \tfrac{5}{3})\\ &= 3[(x^2 - \tfrac{7}{3}x + \tfrac{49}{36}) - \tfrac{49}{36} - \tfrac{5}{3}]\\ &= 3[(x - \tfrac{7}{6})^2 - \tfrac{109}{36}]\\ &= 3(x - \tfrac{7}{6} + \tfrac{1}{6}\sqrt{109})(x - \tfrac{7}{6} - \tfrac{1}{6}\sqrt{109}).\end{aligned}$$

14. Resolve $6x^2 - 5x + 5$ into two factors.

$$\begin{aligned}6x^2 - 5x + 5 &= 6(x^2 - \tfrac{5}{6}x + \tfrac{5}{6})\\ &= 6[(x^2 - \tfrac{5}{6}x + \tfrac{25}{144}) - \tfrac{25}{144} + \tfrac{5}{6}]\\ &= 6[(x - \tfrac{5}{12})^2 - (-\tfrac{95}{144})]\\ &= 6(x - \tfrac{5}{12} + \tfrac{1}{12}\sqrt{-95})(x - \tfrac{5}{12} - \tfrac{1}{12}\sqrt{-95}).\end{aligned}$$

Exercise 112. Page 275

Find all the roots of:

1. $x^2 + 11x + 30 = 0.$

$(x + 5)(x + 6) = 0.$

$\therefore x = -5, \text{ or } -6.$

2. $x^2 - 2x - 24 = 0.$

$(x - 6)(x + 4) = 0.$

$\therefore x = 6, \text{ or } -4.$

3. $x^2 - 10x + 21 = 0.$

$(x - 3)(x - 7) = 0.$

$\therefore x = 3, \text{ or } 7.$

4. $x^2 - 2x - 3 = 0.$

$(x - 3)(x + 1) = 0.$

$\therefore x = 3, \text{ or } -1.$

5. $36x^2 - 35x + 6 = 0.$

$(4x - 3)(9x - 2) = 0.$

$4x - 3 = 0,$

or $9x - 2 = 0.$

$4x = 3,$

or $9x = 2.$

$\therefore x = \tfrac{3}{4}, \text{ or } \tfrac{2}{9}.$

6. $(7x-8)(2x-3)=0.$

$7x-8=0$, or $2x-3=0.$

$7x=8$, or $2x=3.$

$\therefore x=\frac{8}{7}$, or $\frac{3}{2}.$

7. $(3x+2)(5x+9)=0.$

$3x+2=0$, or $5x+9=0.$

$3x=-2$, or $5x=-9.$

$\therefore x=-\frac{2}{3}$, or $-\frac{9}{5}.$

8. $(10-x)(6+5x)=0.$

$10-x=0$, or $6+5x=0.$

$-x=-10$, or $5x=-6.$

$\therefore x=10$, or $-\frac{6}{5}.$

9. $x^2-16x=0.$

$x(x-16)=0.$

$\therefore x=0$, or $16.$

10. $x^2+23x=0.$

$x(x+23)=0.$

$\therefore x=0$, or $-23.$

11. $4x^2+28x+49=0.$

$(2x+7)(2x+7)=0.$

$2x+7=0$, or $2x+7=0.$

$2x=-7$, or $2x=-7.$

$\therefore x=-\frac{7}{2}$, or $-\frac{7}{2}.$

12. $x^3-1=0.$

$$(x-1)(x^2+x+1)=0.$$
$$\therefore x=1, \text{ or } x^2+x+1=0.$$
$$x^2+x=-1.$$
$$4x^2+(\;)+1=-3.$$
$$2x+1=\pm\sqrt{-3}.$$
$$2x=-1\pm\sqrt{-3}.$$
$$\therefore x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$
$$\therefore x=1, \text{ or } -\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$

13. $x^3+1=0.$

$$(x+1)(x^2-x+1)=0.$$
$$\therefore x=-1, \text{ or } x^2-x+1=0.$$
$$x^2-x=-1.$$
$$4x^2-(\;)+1=-3.$$
$$2x-1=\pm\sqrt{-3}.$$
$$2x=1\pm\sqrt{-3}.$$
$$\therefore x=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$
$$\therefore x=-1, \text{ or } \tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$

14. $x^3-27=0.$

$$(x-3)(x^2+3x+9)=0.$$
$$\therefore x=3, \text{ or } x^2+3x+9=0.$$
$$x^2+3x=-9.$$
$$4x^2+(\;)+9=-27.$$

$$\begin{aligned}
2x+3 &= \pm 3\sqrt{-3}.\\
2x &= -3 \pm 3\sqrt{-3}.\\
\therefore x &= -\tfrac{3}{2} \pm \tfrac{3}{2}\sqrt{-3}.\\
\therefore x &= 3, \text{ or } -\tfrac{3}{2} \pm \tfrac{3}{2}\sqrt{-3}.
\end{aligned}$$

15. $x^4 - 16 = 0.$

$$\begin{aligned}
(x^2-4)(x^2+4) &= 0.\\
(x-2)(x+2)(x^2+4) &= 0.\\
\therefore x &= 2, \text{ or } -2, \text{ or } x^2+4=0.\\
x^2 &= -4.\\
\therefore x &= \pm 2\sqrt{-1}.\\
\therefore x &= \pm 2, \text{ or } \pm 2\sqrt{-1}.
\end{aligned}$$

16. $8x^3 - 27 = 0.$

$$\begin{aligned}
(2x-3)(4x^2+6x+9) &= 0.\\
\therefore 2x-3 &= 0, \text{ or } 4x^2+6x+9=0.
\end{aligned}$$

(i)

$$\begin{aligned}
2x-3 &= 0.\\
2x &= 3.\\
\therefore x &= \tfrac{3}{2}.
\end{aligned}$$

(ii)

$$\begin{aligned}
4x^2+6x+9 &= 0.\\
4x^2+6x &= -9.\\
16x^2+24x+9 &= -27.\\
4x+3 &= \pm 3\sqrt{-3}.\\
4x &= -3 \pm 3\sqrt{-3}.\\
\therefore x &= -\tfrac{3}{4} \pm \tfrac{3}{4}\sqrt{-3}.\\
\therefore x &= \tfrac{3}{2}, \text{ or } -\tfrac{3}{4} \pm \tfrac{3}{4}\sqrt{-3}.
\end{aligned}$$

17. $8x^3 + 27 = 0.$

$$\begin{aligned}
(2x+3)(4x^2-6x+9) &= 0.\\
\therefore 2x+3 &= 0, \text{ or } 4x^2-6x+9=0.
\end{aligned}$$

(i)

$$\begin{aligned}
2x+3 &= 0.\\
2x &= -3.\\
\therefore x &= -\tfrac{3}{2}.
\end{aligned}$$

(ii)

$$\begin{aligned}
4x^2-6x+9 &= 0.\\
4x^2-6x &= -9.\\
16x^2-24x+9 &= -27.\\
4x-3 &= \pm 3\sqrt{-3}.\\
4x &= 3 \pm 3\sqrt{-3}.\\
\therefore x &= \tfrac{3}{4} \pm \tfrac{3}{4}\sqrt{-3}.\\
\therefore x &= -\tfrac{3}{2}, \text{ or } \tfrac{3}{4} \pm \tfrac{3}{4}\sqrt{-3}.
\end{aligned}$$

24. Solve $$\frac{3x}{2}-\frac{3x-20}{18-2x}=2+\frac{3x^2-80}{2(x-1)}.$$

Clear of fractions,

$$\begin{aligned}&-3x^3+30x^2-27x-3x^2+23x-20\\&\quad=-4x^2+40x-36-3x^3+27x^2+80x-720.\end{aligned}$$

Transpose and combine, $4x^2-124x=-736.$

Complete the square, $4x^2-(\;)+961=225.$

Extract the root, $2x-31=\pm 15.$

$$\begin{aligned}2x&=46, \text{ or } 16.\\ \therefore x&=23, \text{ or } 8.\end{aligned}$$

25. Solve $$\frac{x}{2x-1}+\frac{24}{4x^2-1}=\frac{2(x-4)}{2x+1}-\frac{1}{9}.$$

Clear of fractions,

$$18x^2+9x+216=36x^2-162x+72-4x^2+1.$$

Transpose and combine, $14x^2-171x=143.$

Complete the square,

$$4\times 14^2x^2-(\;)+29241=37249.$$

Extract the root, $28x-171=\pm 193.$

$$\begin{aligned}28x&=364, \text{ or } -22.\\ \therefore x&=13, \text{ or } -\tfrac{11}{14}.\end{aligned}$$

Exercise 111. Page 272

1. Resolve $6x^2-7x+2$ into two factors.

$$\begin{aligned}&6x^2-7x+2\\&=6(x^2-\tfrac{7}{6}x+\tfrac{1}{3})\\&=6[(x^2-\tfrac{7}{6}x+\tfrac{49}{144})-\tfrac{49}{144}+\tfrac{1}{3}]\\&=6[(x-\tfrac{7}{12})^2-\tfrac{1}{144}]\\&=6(x-\tfrac{7}{12}+\tfrac{1}{12})(x-\tfrac{7}{12}-\tfrac{1}{12})\\&=6(x-\tfrac{1}{2})(x-\tfrac{2}{3})\\&=(2x-1)(3x-2).\end{aligned}$$

Another Method

$$\begin{aligned}&6x^2-7x+2\\&=6(x^2-\tfrac{7}{6}x+\tfrac{2}{6})\\&=6(x^2-\tfrac{7}{6}x+\tfrac{12}{36})\\&=6(x-\tfrac{3}{6})(x-\tfrac{4}{6})\\&=6(x-\tfrac{1}{2})(x-\tfrac{2}{3})\\&=(2x-1)(3x-2).\end{aligned}$$

2. Resolve $12x^2-7x-10$ into two factors.

$$\begin{aligned}&12x^2-7x-10\\&=12(x^2-\tfrac{7}{12}x-\tfrac{5}{6})\\&=12[(x^2-\tfrac{7}{12}x+\tfrac{49}{576})-\tfrac{49}{576}-\tfrac{5}{6}]\\&=12[(x-\tfrac{7}{24})^2-\tfrac{529}{576}]\\&=12(x-\tfrac{7}{24}+\tfrac{23}{24})(x-\tfrac{7}{24}-\tfrac{23}{24})\\&=12(x+\tfrac{2}{3})(x-\tfrac{5}{4})\\&=(3x+2)(4x-5).\end{aligned}$$

Another Method

$$\begin{aligned}&12x^2-7x-10\\&=12(x^2-\tfrac{7}{12}x-\tfrac{10}{12})\\&=12(x^2-\tfrac{7}{12}x-\tfrac{120}{144})\\&=12(x+\tfrac{8}{12})(x-\tfrac{15}{12})\\&=12(x+\tfrac{2}{3})(x-\tfrac{5}{4})\\&=(3x+2)(4x-5).\end{aligned}$$

3. Resolve $21x^2 - 13x + 2$ into two factors.

$$\begin{aligned}
&21x^2 - 13x + 2\\
&\quad = 21\left(x^2 - \tfrac{13}{21}x + \tfrac{2}{21}\right)\\
&\quad = 21\left[\left(x^2 - \tfrac{13}{21}x + \tfrac{169}{1764}\right) - \tfrac{169}{1764} + \tfrac{2}{21}\right]\\
&\quad = 21\left[\left(x - \tfrac{13}{42}\right)^2 - \tfrac{1}{1764}\right]\\
&\quad = 21\left(x - \tfrac{13}{42} + \tfrac{1}{42}\right)\left(x - \tfrac{13}{42} - \tfrac{1}{42}\right)\\
&\quad = 21\left(x - \tfrac{2}{7}\right)\left(x - \tfrac{1}{3}\right)\\
&\quad = (7x - 2)(3x - 1).
\end{aligned}$$

Another Method

$$\begin{aligned}
&21x^2 - 13x + 2\\
&\quad = 21\left(x^2 - \tfrac{13}{21}x + \tfrac{2}{21}\right)\\
&\quad = 21\left(x^2 - \tfrac{13}{21}x + \tfrac{42}{441}\right)\\
&\quad = 21\left(x - \tfrac{6}{21}\right)\left(x - \tfrac{7}{21}\right)\\
&\quad = 21\left(x - \tfrac{2}{7}\right)\left(x - \tfrac{1}{3}\right)\\
&\quad = (7x - 2)(3x - 1).
\end{aligned}$$

4. Resolve $3x^2 + 8x + 4$ into two factors.

$$\begin{aligned}
&3x^2 + 8x + 4\\
&\quad = 3\left(x^2 + \tfrac{8}{3}x + \tfrac{4}{3}\right)\\
&\quad = 3\left[\left(x^2 + \tfrac{8}{3}x + \tfrac{16}{9}\right) - \tfrac{16}{9} + \tfrac{4}{3}\right]\\
&\quad = 3\left[\left(x + \tfrac{4}{3}\right)^2 - \tfrac{4}{9}\right]\\
&\quad = 3\left(x + \tfrac{4}{3} + \tfrac{2}{3}\right)\left(x + \tfrac{4}{3} - \tfrac{2}{3}\right)\\
&\quad = 3(x + 2)\left(x + \tfrac{2}{3}\right)\\
&\quad = (x + 2)(3x + 2).
\end{aligned}$$

Another Method

$$\begin{aligned}
&3x^2 + 8x + 4\\
&\quad = 3\left(x^2 + \tfrac{8}{3}x + \tfrac{4}{3}\right)\\
&\quad = 3\left(x^2 + \tfrac{8}{3}x + \tfrac{12}{9}\right)\\
&\quad = 3\left(x + \tfrac{6}{3}\right)\left(x + \tfrac{2}{3}\right)\\
&\quad = 3(x + 2)\left(x + \tfrac{2}{3}\right)\\
&\quad = (x + 2)(3x + 2).
\end{aligned}$$

5. Resolve $6x^2 - 11x + 3$ into two factors.

$$\begin{aligned}
&6x^2 - 11x + 3\\
&\quad = 6\left(x^2 - \tfrac{11}{6}x + \tfrac{1}{2}\right)\\
&\quad = 6\left[\left(x^2 - \tfrac{11}{6}x + \tfrac{121}{144}\right) - \tfrac{121}{144} + \tfrac{1}{2}\right]\\
&\quad = 6\left[\left(x - \tfrac{11}{12}\right)^2 - \tfrac{49}{144}\right]\\
&\quad = 6\left(x - \tfrac{11}{12} + \tfrac{7}{12}\right)\left(x - \tfrac{11}{12} - \tfrac{7}{12}\right)\\
&\quad = 6\left(x - \tfrac{1}{3}\right)\left(x - \tfrac{3}{2}\right)\\
&\quad = (3x - 1)(2x - 3).
\end{aligned}$$

Another Method

$$\begin{aligned}
&6x^2 - 11x + 3\\
&\quad = 6\left(x^2 - \tfrac{11}{6}x + \tfrac{3}{6}\right)\\
&\quad = 6\left(x^2 - \tfrac{11}{6}x + \tfrac{18}{36}\right)\\
&\quad = 6\left(x - \tfrac{2}{6}\right)\left(x - \tfrac{9}{6}\right)\\
&\quad = 6\left(x - \tfrac{1}{3}\right)\left(x - \tfrac{3}{2}\right)\\
&\quad = (3x - 1)(2x - 3).
\end{aligned}$$

6. Resolve $3x^2 + x - 10$ into two factors.

$$\begin{aligned}
&3x^2 + x - 10\\
&\quad = 3\left(x^2 + \tfrac{1}{3}x - \tfrac{10}{3}\right)\\
&\quad = 3\left[\left(x^2 + \tfrac{1}{3}x + \tfrac{1}{36}\right) - \tfrac{1}{36} - \tfrac{10}{3}\right]\\
&\quad = 3\left[\left(x + \tfrac{1}{6}\right)^2 - \tfrac{121}{36}\right]\\
&\quad = 3\left(x + \tfrac{1}{6} + \tfrac{11}{6}\right)\left(x + \tfrac{1}{6} - \tfrac{11}{6}\right)\\
&\quad = 3(x + 2)\left(x - \tfrac{5}{3}\right)\\
&\quad = (x + 2)(3x - 5).
\end{aligned}$$

Another Method

$$\begin{aligned}
&3x^2 + x - 10\\
&\quad = 3\left(x^2 + \tfrac{1}{3}x - \tfrac{10}{3}\right)\\
&\quad = 3\left(x^2 + \tfrac{1}{3}x - \tfrac{30}{9}\right)\\
&\quad = 3\left(x + \tfrac{6}{3}\right)\left(x - \tfrac{5}{3}\right)\\
&\quad = 3(x + 2)\left(x - \tfrac{5}{3}\right)\\
&\quad = (x + 2)(3x - 5).
\end{aligned}$$

14. Solve $4x^2 - 4ax + a^2 - b^2 = 0.$

$$4x^2 - 4ax + a^2 = b^2.$$
$$2x - a = \pm b.$$
$$2x = a \pm b.$$
$$\therefore\ x = \tfrac{1}{2}(a + b), \text{ or } \tfrac{1}{2}(a - b).$$

15. Solve $(a - x)^2 + (x - b)^2 = (a - b)^2.$

$$a^2 - 2ax + x^2 + x^2 - 2bx + b^2 = a^2 - 2ab + b^2.$$
$$2x^2 - 2(a + b)x + 2ab = 0.$$
$$x^2 - (a + b)x + ab = 0.$$
$$(x - a)(x - b) = 0.$$
$$\therefore\ x = a, \text{ or } b.$$

16. Solve $ax^2 - (a^2 + 1)x + a = 0.$

$$ax^2 - (a^2 + 1)x = - a.$$
$$4a^2x^2 - (\ ) + (a^2 + 1)^2 = a^4 - 2a^2 + 1.$$
$$2ax - (a^2 + 1) = \pm (a^2 - 1).$$
$$2ax = 2a^2, \text{ or } 2.$$
$$\therefore\ x = a, \text{ or } \frac{1}{a}.$$

17. Solve $abx^2 - (a^2 + b^2)x + ab = 0.$

$$abx^2 - (a^2 + b^2)x = - ab.$$
$$4a^2b^2x^2 - (\ ) + (a^2 + b^2)^2 = a^4 - 2a^2b^2 + b^4.$$
$$2abx - (a^2 + b^2) = \pm (a^2 - b^2).$$
$$2abx = 2a^2, \text{ or } 2b^2.$$
$$\therefore\ x = \frac{a}{b}, \text{ or } \frac{b}{a}.$$

18. Solve $bcx - bd = adx - acx^2.$

$$acx^2 + bcx - adx = bd.$$
$$acx^2 + (bc - ad)x = bd.$$
$$4a^2c^2x^2 + (\ ) + (bc - ad)^2 = b^2c^2 + 2abcd + a^2d^2.$$
$$2acx + (bc - ad) = \pm (bc + ad).$$
$$2acx = 2ad, \text{ or } - 2bc.$$
$$\therefore\ x = \frac{d}{c}, \text{ or } -\frac{b}{a}.$$

19. Solve $$x^3 = a^3.$$

$$x^3 - a^3 = 0.$$

$$(x-a)(x^2+ax+a^2) = 0.$$

$$\therefore\ x - a = 0, \text{ or } x^2+ax+a^2 = 0.$$

(i) $$x - a = 0.$$

$$\therefore\ x = a.$$

(ii) $$x^2 + ax + a^2 = 0.$$

$$x^2 + ax = -a^2.$$

$$4x^2 + (\ ) + a^2 = -3a^2.$$

$$2x + a = \pm a\sqrt{-3}.$$

$$2x = -a \pm a\sqrt{-3}.$$

$$\therefore\ x = \frac{a}{2}(-1 \pm \sqrt{-3}).$$

$$\therefore\ x = a, \text{ or } \frac{a}{2}(-1 \pm \sqrt{-3}).$$

20. Solve $$x^3 = -b^3.$$

$$x^3 + b^3 = 0.$$

$$(x+b)(x^2-bx+b^2) = 0.$$

$$\therefore\ x + b = 0, \text{ or } x^2-bx+b^2 = 0.$$

(i) $$x + b = 0.$$

$$\therefore\ x = -b.$$

(ii) $$x^2 - bx + b^2 = 0.$$

$$x^2 - bx = -b^2.$$

$$4x^2 - (\ ) + b^2 = -3b^2.$$

$$2x - b = \pm b\sqrt{-3}.$$

$$2x = b \pm b\sqrt{-3}.$$

$$\therefore\ x = \frac{b}{2}(1 \pm \sqrt{-3}).$$

$$\therefore\ x = -b, \text{ or } \frac{b}{2}(1 \pm \sqrt{-3}).$$

21. Solve $$4ax^2 + 20a^2x + 9a^2 = 0.$$

$$4x^2 + 20ax + 9a = 0.$$

$$4x^2 + 20ax = -9a.$$

$$4x^2 + 20ax + 25a^2 = 25a^2 - 9a.$$

$$2x + 5a = \pm\sqrt{25a^2 - 9a}.$$

$$2x = -5a \pm \sqrt{25a^2 - 9a}.$$

$$\therefore\ x = \tfrac{1}{2}(-5a \pm \sqrt{25a^2 - 9a}).$$

22. Solve

$$\begin{aligned}
3x^2 + ax &= b.\\
36x^2 + (\ ) + a^2 &= 12b + a^2.\\
6x + a &= \pm\sqrt{12b + a^2}.\\
6x &= -a \pm \sqrt{12b + a^2}.\\
\therefore\ x &= \tfrac{1}{6}(-a \pm \sqrt{12b + a^2}).
\end{aligned}$$

23. Solve

$$\begin{aligned}
ax^2 - 2bx &= c.\\
a^2x^2 - (\ ) + b^2 &= ac + b^2.\\
ax - b &= \pm\sqrt{ac + b^2}.\\
ax &= b \pm \sqrt{ac + b^2}.\\
\therefore\ x &= \frac{1}{a}(b \pm \sqrt{ac + b^2}).
\end{aligned}$$

24. Solve

$$\begin{aligned}
ax^2 + 2bx + c &= 0.\\
ax^2 + 2bx &= -c.\\
a^2x^2 + (\ ) + b^2 &= b^2 - ac.\\
ax + b &= \pm\sqrt{b^2 - ac}.\\
ax &= -b \pm \sqrt{b^2 - ac}.\\
\therefore\ x &= \frac{1}{a}(-b \pm \sqrt{b^2 - ac}).
\end{aligned}$$

25. Solve $x^2 - (a+b)x + (a+c)(b-c) = 0.$

$$\begin{aligned}
x^2 - (a+b)x &= ac - bc - ab + c^2.\\
4x^2 - (\ ) + (a+b)^2 &= a^2 - 2ab + b^2 + 4ac - 4bc + 4c^2.\\
2x - (a+b) &= \pm(a - b + 2c).\\
2x &= 2a + 2c, \text{ or } 2b - 2c.\\
\therefore\ x &= a + c, \text{ or } b - c.
\end{aligned}$$

26. Solve $x^2 - (a-b)x - (a-1)(b-1) = 0.$

$$\begin{aligned}
x^2 - (a-b)x &= (a-1)(b-1).\\
x^2 - (a-b)x &= ab - a - b + 1.\\
4x^2 - (\ ) + (a-b)^2 &= a^2 + 2ab + b^2 - 4a - 4b + 4.\\
2x - (a-b) &= \pm(a + b - 2).\\
2x &= 2a - 2, \text{ or } 2 - 2b.\\
\therefore\ x &= a - 1, \text{ or } 1 - b.
\end{aligned}$$

27. Solve

$$\begin{aligned}
x^2 + 2ab(a^2 + b^2) &= (a+b)^2x.\\
x^2 - (a+b)^2x &= -2a^3b - 2ab^3.\\
4x^2 - (\ ) + (a+b)^4 &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.\\
2x - (a+b)^2 &= \pm(a^2 - 2ab + b^2).\\
2x - (a^2 + 2ab + b^2) &= \pm(a^2 - 2ab + b^2).\\
2x &= 2a^2 + 2b^2, \text{ or } 4ab.\\
\therefore\ x &= a^2 + b^2, \text{ or } 2ab.
\end{aligned}$$

28. Solve $(a+b+c)x^2-(2a+b+c)x+a=0$.

$$(a+b+c)x^2-(2a+b+c)x=-a.$$
$$4(a+b+c)^2x^2-(\ )+(2a+b+c)^2=b^2+2bc+c^2.$$
$$2(a+b+c)x-(2a+b+c)=\pm(b+c).$$
$$2(a+b+c)x=2a+2b+2c, \text{ or } 2a.$$
$$\therefore\ x=1, \text{ or } \frac{a}{a+b+c}.$$

29. Solve $$x+\frac{1}{x}=\frac{a-b}{a+b}+\frac{a+b}{a-b}.$$

Simplify the right member, $x+\frac{1}{x}=\frac{2(a^2+b^2)}{a^2-b^2}$.

$$(a^2-b^2)x^2+a^2-b^2=2(a^2+b^2)x.$$
$$(a^2-b^2)x^2-2(a^2+b^2)x=-a^2+b^2.$$
$$(a^2-b^2)^2x^2-(\ )+(a^2+b^2)^2=4a^2b^2.$$
$$(a^2-b^2)x-(a^2+b^2)=\pm 2ab.$$
$$(a^2-b^2)x=(a+b)^2, \text{ or } (a-b)^2.$$
$$\therefore\ x=\frac{a+b}{a-b}, \text{ or } \frac{a-b}{a+b}.$$

30. Solve $abx^2-(a+b)(ab+1)x+(ab+1)^2=0$.

$$abx^2-(a+b)(ab+1)x=-(ab+1)^2.$$
$$4a^2b^2x^2-(\ )+(a+b)^2(ab+1)^2=(a+b)^2(ab+1)^2-4ab(ab+1)^2.$$
$$4a^2b^2x^2-(\ )+(a+b)^2(ab+1)^2=(a^2-2ab+b^2)(ab+1)^2.$$
$$2abx-(a+b)(ab+1)=\pm(a-b)(ab+1).$$
$$2abx=(a+b)(ab+1)\pm(a-b)(ab+1)$$
$$=2a(ab+1), \text{ or } 2b(ab+1).$$
$$\therefore\ x=\frac{ab+1}{b}, \text{ or } \frac{ab+1}{a}.$$

31. Solve $$(a-x)^2+(b-x)^2=\tfrac{5}{2}(a-x)(b-x).$$

$$2(a-x)^2+2(b-x)^2=5(a-x)(b-x).$$
$$2(a-x)^2-5(a-x)(b-x)+2(b-x)^2=0.$$

Factor, $$[2(a-x)-(b-x)][(a-x)-2(b-x)]=0.$$
$$(2a-2x-b+x)(a-x-2b+2x)=0.$$
$$(2a-b-x)(a-2b+x)=0.$$
$$\therefore\ 2a-b-x=0, \text{ or } a-2b+x=0.$$
$$\therefore\ x=2a-b, \text{ or } 2b-a.$$

32. Solve $$\frac{x^3+3x^2+3x+1}{x^3-x^2-x+1}=\frac{a^2}{b^2}.$$

Factor, $$\frac{(x+1)^3}{(x-1)(x^2-1)}=\frac{a^2}{b^2},$$

or $$\frac{(x+1)^3}{(x-1)^2(x+1)}=\frac{a^2}{b^2}.$$

$$\frac{(x+1)^2}{(x-1)^2}=\frac{a^2}{b^2}.$$

$$\frac{x+1}{x-1}=\pm\frac{a}{b}.$$

$$\therefore\ bx+b=ax-a,\ \text{or}\ bx+b=-ax+a.$$

$$(a-b)x=a+b,\ \text{or}\ (a+b)x=a-b.$$

$$\therefore\ x=\frac{a+b}{a-b},\ \text{or}\ \frac{a-b}{a+b}.$$

33. Solve $$\frac{(a-x)^2+(x-b)^2}{(a-x)^2-(x-b)^2}=\frac{a^2+b^2}{a^2-b^2}.$$

$$\frac{a^2-2ax+x^2+x^2-2bx+b^2}{(a-x+x-b)(a-x-x+b)}=\frac{a^2+b^2}{(a+b)(a-b)}.$$

$$\frac{a^2+b^2-2(a+b)x+2x^2}{(a-b)(a+b-2x)}=\frac{a^2+b^2}{(a+b)(a-b)}.$$

Multiply by $a-b$,

$$\frac{(a^2+b^2)-2(a+b)x+2x^2}{a+b-2x}=\frac{a^2+b^2}{a+b}.$$

Clear of fractions,

$$(a+b)(a^2+b^2)-2(a+b)^2x+2(a+b)x^2$$
$$=(a+b)(a^2+b^2)-2(a^2+b^2)x.$$

Transpose and combine,

$$2(a+b)x^2-2(a+b)^2x+2(a^2+b^2)x=0.$$

Divide by 2, $$(a+b)x^2-(a+b)^2x+(a^2+b^2)x=0.$$

Combine, $$(a+b)x^2-2abx=0.$$

Factor, $$x(ax+bx-2ab)=0.$$

$$\therefore\ x=0,\ \text{or}\ ax+bx-2ab=0.$$

$$ax+bx=2ab.$$

$$\therefore\ x=\frac{2ab}{a+b}.$$

$$\therefore\ x=0,\ \text{or}\ \frac{2ab}{a+b}.$$

34. Solve
$$\frac{(a-x)^3+(x-b)^3}{(a-x)^2+(x-b)^2}=\frac{a^3-b^3}{a^2+b^2}.$$

Divide by $(a-x)+(x-b)$ or $a-b$,
$$\frac{(a-x)^2-(a-x)(x-b)+(x-b)^2}{(a-x)^2+(x-b)^2}=\frac{a^2+ab+b^2}{a^2+b^2}.$$

Clear of fractions,
$$(a^2+b^2)(a-x)^2-(a^2+b^2)(a-x)(x-b)+(a^2+b^2)(x-b)^2$$
$$=(a^2+b^2)(a-x)^2+(a^2+b^2)(x-b)^2+ab(a-x)^2+ab(x-b)^2.$$

Transpose and combine,
$$ab(a-x)^2+(a^2+b^2)(a-x)(x-b)+ab(x-b)^2=0.$$

Factor, $[a(a-x)+b(x-b)][b(a-x)+a(x-b)]=0.$
$$\therefore\ a(a-x)+b(x-b)=0, \text{ or } b(a-x)+a(x-b)=0.$$
$$a^2-ax+bx-b^2=0, \text{ or } ab-bx+ax-ab=0.$$
$$(a-b)x=a^2-b^2, \text{ or } 0.$$
$$\therefore\ x=a+b, \text{ or } 0.$$

35. Solve
$$x^2-a^2=\tfrac{3}{2}ax.$$
$$2x^2-2a^2=3ax.$$
$$2x^2-3ax-2a^2=0.$$
$$(x-2a)(2x+a)=0.$$
$$\therefore\ x=2a, \text{ or } -\frac{a}{2}.$$

36. Solve
$$a^2(b-x)^2=b^2(a-x)^2.$$
$$a^2b^2-2a^2bx+a^2x^2=a^2b^2-2ab^2x+b^2x^2.$$
$$a^2x^2-b^2x^2-2a^2bx+2ab^2x=0.$$
$$(a^2-b^2)x^2-2ab(a-b)x=0.$$

Divide by $a-b$, $(a+b)x^2-2abx=0.$

Factor, $x[(a+b)x-2ab]=0.$
$$\therefore\ x=0, \text{ or } (a+b)x-2ab=0.$$
$$\therefore\ x=0, \text{ or } \frac{2ab}{a+b}.$$

37. Solve
$$\frac{a}{x}+\frac{x}{a}=\frac{m}{n}.$$
$$a^2n+nx^2=amx.$$
$$nx^2-amx=-a^2n.$$
$$4n^2x^2-(\ )+a^2m^2=a^2m^2-4a^2n^2.$$
$$2nx-am=\pm a\sqrt{m^2-4n^2}.$$
$$2nx=am\pm a\sqrt{m^2-4n^2}.$$
$$\therefore\ x=\frac{a}{2n}\left(m\pm\sqrt{m^2-4n^2}\right).$$

38. Solve $$x^4 + (a - x)^4 = a^4.$$

$$\begin{aligned}
x^4 + a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4 &= a^4.\\
2x^4 - 4ax^3 + 6a^2x^2 - 4a^3x &= 0.\\
x^4 - 2ax^3 + 3a^2x^2 - 2a^3x &= 0.\\
x(x^3 - 2ax^2 + 3a^2x - 2a^3) &= 0.\\
x(x - a)(x^2 - ax + 2a^2) &= 0.
\end{aligned}$$

$$\therefore x = 0,\ x - a = 0, \text{ or } x^2 - ax + 2a^2 = 0.$$

(i) $$x = 0.$$

(ii) $$x - a = 0.$$
$$\therefore x = a.$$

(iii)
$$\begin{aligned}
x^2 - ax + 2a^2 &= 0.\\
x^2 - ax &= -2a^2.\\
4x^2 - (\ ) + a^2 &= -7a^2.\\
2x - a &= \pm a\sqrt{-7}.\\
2x &= a \pm a\sqrt{-7}.\\
\therefore x &= \frac{a}{2}(1 \pm \sqrt{-7}).\\
\therefore x &= 0,\ a, \text{ or } \frac{a}{2}(1 \pm \sqrt{-7}).
\end{aligned}$$

39. Solve $$2a^3x^3 = (ax - 6)^2.$$

$$\begin{aligned}
2a^3x^3 &= a^2x^2 - 12ax + 36.\\
2a^3x^3 - a^2x^2 + 12ax - 36 &= 0.\\
(ax - 2)(2a^2x^2 + 3ax + 18) &= 0.
\end{aligned}$$

$$\therefore ax - 2 = 0, \text{ or } 2a^2x^2 + 3ax + 18 = 0.$$

(i)
$$\begin{aligned}
ax - 2 &= 0.\\
ax &= 2.\\
\therefore x &= \frac{2}{a}.
\end{aligned}$$

(ii)
$$\begin{aligned}
2a^2x^2 + 3ax + 18 &= 0.\\
2a^2x^2 + 3ax &= -18.\\
16a^2x^2 + (\ ) + 9 &= -135.\\
4ax + 3 &= \pm 3\sqrt{-15}.\\
4ax &= -3 \pm 3\sqrt{-15}.\\
\therefore x &= \frac{3}{4a}(-1 \pm \sqrt{-15}).\\
\therefore x &= \frac{2}{a}, \text{ or } \frac{3}{4a}(-1 \pm \sqrt{-15}).
\end{aligned}$$

40. Solve
$$\frac{a-x}{x-b}+\frac{x-b}{a-x}=\frac{c}{d}.$$

$$\frac{a^2-2ax+x^2+x^2-2bx+b^2}{ax-ab-x^2+bx}=\frac{c}{d}.$$

$$\frac{2x^2-2(a+b)x+a^2+b^2}{-x^2+(a+b)x-ab}=\frac{c}{d}.$$

$$2dx^2-2d(a+b)x+a^2d+b^2d=-cx^2+c(a+b)x-abc.$$

$$(c+2d)x^2-(c+2d)(a+b)x=-a^2d-b^2d-abc.$$

$$\begin{aligned}4(c+2d)^2x^2-(\ \ )+(c+2d)^2(a+b)^2&\\ =(c+2d)^2(a+b)^2-4(a^2d+b^2d+abc)(c+2d)&\\ =(c+2d)[(c+2d)(a+b)^2-4(a^2d+b^2d+abc)]&\\ =(c+2d)(a^2c-2abc+b^2c-2a^2d+4abd-2b^2d)&\\ =(c+2d)(a-b)^2(c-2d).&\end{aligned}$$

$$2(c+2d)x-(c+2d)(a+b)=\pm(a-b)\sqrt{(c+2d)(c-2d)}.$$

$$2(c+2d)x=(c+2d)(a+b)\pm(a-b)\sqrt{(c+2d)(c-2d)}.$$

$$\therefore x=\frac{a+b}{2}\pm\frac{a-b}{2(c+2d)}\sqrt{(c+2d)(c-2d)}.$$

41. Solve
$$(a-1)^2x^2-a(x+b)=ax^2(a-2)-bx.$$

$$(a^2-2a+1)x^2-ax-ab=(a^2-2a)x^2-bx.$$

Transpose and combine,

$$x^2-ax+bx-ab=0.$$

$$(x-a)(x+b)=0.$$

$$\therefore x=a,\ \text{or}\ -b.$$

42. Solve
$$(a-b)x^2+\frac{ab}{a-b}=(a+b)x.$$

$$(a-b)^2x^2+ab=(a+b)(a-b)x.$$

$$(a-b)^2x^2-(a+b)(a-b)x=-ab.$$

$$4(a-b)^2x^2-4(a+b)(a-b)x+(a+b)^2=a^2-2ab+b^2.$$

$$2(a-b)x-(a+b)=\pm(a-b).$$

$$2(a-b)x=2a,\ \text{or}\ 2b.$$

$$\therefore x=\frac{a}{a-b},\ \text{or}\ \frac{b}{a-b}.$$

43. Solve $(x^2 + ax + a^2)(x + a) = (x^2 + a^2)(x - 2a) + 6a^3.$

$$x^3 + 2ax^2 + 2a^2x + a^3 = x^3 - 2ax^2 + a^2x - 2a^3 + 6a^3.$$
$$4ax^2 + a^2x - 3a^3 = 0.$$
$$4x^2 + ax - 3a^2 = 0.$$
$$(x + a)(4x - 3a) = 0.$$
$$\therefore x + a = 0, \text{ or } 4x - 3a = 0.$$
$$\therefore x = -a, \text{ or } \frac{3a}{4}.$$

44. Solve $4(a - x)^4 - 17(a - x)^2(x - b)^2 + 4(x - b)^4 = 0.$

Factor, $[4(a - x)^2 - (x - b)^2][(a - x)^2 - 4(x - b)^2] = 0.$

$$\therefore 4(a - x)^2 - (x - b)^2 = 0, \text{ or } (a - x)^2 - 4(x - b)^2 = 0.$$

(i) $4(a - x)^2 - (x - b)^2 = 0.$

$$4(a - x)^2 = (x - b)^2.$$
$$2(a - x) = \pm(x - b).$$
$$2a - 2x = x - b, \text{ or } -x + b.$$
$$3x = 2a + b, \text{ or } -x = -2a + b.$$
$$\therefore x = \frac{2a + b}{3}, \text{ or } 2a - b.$$

(ii) $(a - x)^2 - 4(x - b)^2 = 0.$

$$(a - x)^2 = 4(x - b)^2.$$
$$(a - x) = \pm 2(x - b).$$
$$a - x = 2x - 2b, \text{ or } -2x + 2b.$$
$$-3x = -2b - a, \text{ or } x = 2b - a.$$
$$\therefore x = \frac{2b + a}{3}, \text{ or } 2b - a.$$
$$\therefore x = 2a - b, 2b - a, \frac{2a + b}{3}, \text{ or } \frac{2b + a}{3}.$$

45. Solve $x(x - 2b) + a(b - x) + 2c(a - 2c) + b^2 = 0.$

$$x^2 - 2bx + ab - ax + 2ac - 4c^2 + b^2 = 0.$$
$$x^2 - (a + 2b)x = 4c^2 - 2ac - b^2 - ab.$$
$$4x^2 - (\ ) + (a + 2b)^2 = a^2 - 8ac + 16c^2.$$
$$2x - (a + 2b) = \pm(a - 4c).$$
$$2x = 2a + 2b - 4c, \text{ or } 2b + 4c.$$
$$\therefore x = a + b - 2c, \text{ or } b + 2c.$$

46. Solve $$(a-x)(b+x)=x(b-a+2x)-\frac{ab}{3}.$$

$$ab+ax-bx-x^2=bx-ax+2x^2-\frac{ab}{3}.$$

$$3x^2-2(a-b)x=\frac{4ab}{3}.$$

$$9x^2-6(a-b)x=4ab.$$

$$9x^2-6(a-b)x+(a-b)^2=a^2+2ab+b^2.$$

$$3x-(a-b)=\pm(a+b).$$

$$3x=2a, \text{ or } -2b.$$

$$\therefore x=\frac{2a}{3}, \text{ or } -\frac{2b}{3}.$$

47. Solve $$\frac{a-x}{x-b}+\frac{x-b}{a-x}=\frac{a}{b}+\frac{b}{a}.$$

$$\frac{a^2-2ax+x^2+x^2-2bx+b^2}{ax-x^2-ab+bx}=\frac{a^2+b^2}{ab}.$$

$$\frac{(a^2+b^2)-2(a+b)x+2x^2}{(a+b)x-x^2-ab}=\frac{a^2+b^2}{ab}.$$

$$ab(a^2+b^2)-2ab(a+b)x+2abx^2$$
$$=(a+b)(a^2+b^2)x-(a^2+b^2)x^2-ab(a^2+b^2).$$

Transpose and combine,

$$(a^2+2ab+b^2)x^2-(a^2+2ab+b^2)(a+b)x=-2ab(a^2+b^2).$$

$$(a+b)^2x^2-(a+b)^3x=-2a^3b-2ab^3.$$

$$4(a+b)^2x^2-4(a+b)^3x+(a+b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4.$$

$$2(a+b)x-(a+b)^2=\pm(a^2-2ab+b^2).$$

$$2(a+b)x=2a^2+2b^2, \text{ or } 4ab.$$

$$\therefore x=\frac{a^2+b^2}{a+b}, \text{ or } \frac{2ab}{a+b}.$$

48. Solve $$\frac{bx}{c}(acx+b)+\frac{3a^2x}{c}=\frac{6a^2+ab-2b^2}{c^2}.$$

$$abc^2x^2+b^2cx+3a^2cx=6a^2+ab-2b^2.$$

$$4a^2b^2c^2x^2+4abc(b^2+3a^2)x+(b^2+3a^2)^2$$
$$=9a^4+24a^3b+10a^2b^2-8ab^3+b^4.$$

$$2abcx+(b^2+3a^2)=\pm(3a^2+4ab-b^2).$$

$$2abcx=4ab-2b^2, \text{ or } -6a^2-4ab.$$

$$\therefore x=\frac{2a-b}{ac}, \text{ or } -\frac{3a+2b}{bc}.$$

49. Solve
$$\frac{ax+b}{a+bx}+\frac{cx+d}{c+dx}=\frac{ax-b}{a-bx}+\frac{cx-d}{c-dx}.$$

$$\frac{ax+b}{a+bx}-\frac{ax-b}{a-bx}=\frac{cx-d}{c-dx}-\frac{cx+d}{c+dx}.$$

$$\frac{a^2x+ab-abx^2-b^2x-a^2x-abx^2+ab+b^2x}{a^2-b^2x^2}$$
$$=\frac{c^2x-cd+cdx^2-d^2x-c^2x+cdx^2-cd+d^2x}{c^2-d^2x^2}.$$

$$\frac{2\,ab-2\,abx^2}{a^2-b^2x^2}=\frac{-2\,cd+2\,cdx^2}{c^2-d^2x^2}.$$

$$\frac{ab(1-x^2)}{a^2-b^2x^2}=\frac{-cd(1-x^2)}{c^2-d^2x^2}.$$

$$\therefore\ 1-x^2=0.$$
$$x^2=1.$$
$$x=\pm 1.$$

If $1-x^2$ is not 0, divide by $1-x^2$,

$$\frac{ab}{a^2-b^2x^2}=\frac{-cd}{c^2-d^2x^2}.$$
$$abc^2-abd^2x^2=-a^2cd+b^2cdx^2.$$
$$abd^2x^2+b^2cdx^2=abc^2+a^2cd.$$
$$bd(ad+bc)x^2=ac(bc+ad).$$
$$bdx^2=ac.$$
$$b^2d^2x^2=abcd.$$
$$bdx=\pm\sqrt{abcd}.$$
$$\therefore\ x=\pm\frac{1}{bd}\sqrt{abcd}.$$
$$\therefore\ x=\pm 1,\text{ or }\pm\frac{1}{bd}\sqrt{abcd}.$$

50. Solve
$$\frac{a}{x-a}+\frac{b}{x-b}-\frac{2\,c}{x-c}=0.$$

$$a(x-b)(x-c)+b(x-a)(x-c)-2\,c(x-a)(x-b)=0.$$
$$ax^2-acx-abx+abc+bx^2-bcx-abx+abc$$
$$-2\,cx^2+2\,bcx+2\,acx-2\,abc=0.$$
$$ax^2+bx^2-2\,cx^2-2\,abx+acx+bcx=0.$$
$$x(ax+bx-2\,cx-2\,ab+ac+bc)=0.$$
$$\therefore\ x=0,\text{ or }ax+bx-2\,cx-2\,ab+ac+bc=0.$$

$$ax + bx - 2\,cx = 2\,ab - ac - bc.$$

$$\therefore\ x = \frac{2\,ab - ac - bc}{a + b - 2\,c}.$$

$$\therefore\ x = 0, \text{ or } \frac{2\,ab - ac - bc}{a + b - 2\,c}.$$

51. Solve $$\frac{a - x}{a - b} + \frac{a - b}{a - x} = \frac{b - x}{6\,(a - b)} + \frac{6\,(a - b)}{b - x}.$$

$$\frac{a - x}{a - b} - \frac{b - x}{6\,(a - b)} = \frac{6\,(a - b)}{b - x} - \frac{a - b}{a - x}.$$

$$\frac{6\,a - 6\,x - b + x}{6\,(a - b)} = \frac{(a - b)\,(6\,a - 6\,x - b + x)}{(a - x)\,(b - x)}.$$

$$\frac{6\,a - b - 5\,x}{6\,(a - b)} = \frac{(a - b)\,(6\,a - b - 5\,x)}{(a - x)\,(b - x)}.$$

$$\therefore\ 6\,a - b - 5\,x = 0.$$

$$5\,x = 6\,a - b.$$

$$\therefore\ x = \frac{6\,a - b}{5}.$$

If $6\,a - b - 5\,x$ is not 0, divide by $6\,a - b - 5\,x$,

$$\frac{1}{6\,(a - b)} = \frac{a - b}{(a - x)\,(b - x)}.$$

$$ab - ax - bx + x^2 = 6\,a^2 - 12\,ab + 6\,b^2.$$

$$x^2 - (a + b)\,x = 6\,a^2 - 13\,ab + 6\,b^2.$$

$$4\,x^2 - (\ ) + (a + b)^2 = 25\,a^2 - 50\,ab + 25\,b^2.$$

$$2\,x - (a + b) = \pm\,5\,(a - b).$$

$$2\,x = 6\,a - 4\,b, \text{ or } 6\,b - 4\,a.$$

$$\therefore\ x = 3\,a - 2\,b, \text{ or } 3\,b - 2\,a.$$

$$\therefore\ x = 3\,a - 2\,b,\ 3\,b - 2\,a, \text{ or } \frac{6\,a - b}{5}.$$

52. Solve $$(x + a)\,(3\,x - b) = (3\,b - a)\,x - (a - b)^2 - ab.$$

$$3\,x^2 + 3\,ax - bx - ab = 3\,bx - ax - a^2 + 2\,ab - b^2 - ab.$$

$$3\,x^2 + 4\,ax - 4\,bx = -\,a^2 + 2\,ab - b^2.$$

$$3\,x^2 + 4\,(a - b)\,x = -\,(a - b)^2.$$

$$9\,x^2 + (\ ) + 4\,(a - b)^2 = (a - b)^2.$$

$$3\,x + 2\,(a - b) = \pm\,(a - b).$$

$$3\,x = b - a, \text{ or } 3\,b - 3\,a.$$

$$\therefore\ x = \frac{b - a}{3}, \text{ or } b - a.$$

53. Solve $\dfrac{a+b+(a-b)x^2}{2x} - \dfrac{2x}{(a-b)+(a+b)x^2} = 0.$

$$a^2-b^2+(a^2-2ab+b^2)x^2+(a^2+2ab+b^2)x^2+(a^2-b^2)x^4-4x^2=0.$$

$$(a^2-b^2)x^4+2(a^2+b^2-2)x^2=-(a^2-b^2).$$

$$(a^2-b^2)^2x^4+(\ )+(a^2+b^2-2)^2=4+4a^2b^2-4a^2-4b^2.$$

$$(a^2-b^2)x^2+(a^2+b^2-2)=\pm 2\sqrt{1+a^2b^2-a^2-b^2}.$$

$$(a^2-b^2)x^2=2-a^2-b^2\pm 2\sqrt{1+a^2b^2-a^2-b^2}.$$

Extract the square root,

$$\sqrt{a^2-b^2}\,x=\pm\sqrt{2-a^2-b^2\pm 2\sqrt{1+a^2b^2-a^2-b^2}}.$$

Two numbers whose sum is $2-a^2-b^2$, and whose product is $1+a^2b^2-a^2-b^2$ are $1-a^2$ and $1-b^2$.

$$\therefore \sqrt{2-a^2-b^2\pm 2\sqrt{1+a^2b^2-a^2-b^2}}$$
$$=\sqrt{1-a^2}\pm\sqrt{1-b^2}.$$

$$\therefore \sqrt{a^2-b^2}\,x=\pm(\sqrt{1-a^2}\pm\sqrt{1-b^2}).$$

$$\therefore x=\pm\frac{\sqrt{1-a^2}\pm\sqrt{1-b^2}}{\sqrt{a^2-b^2}}.$$

54. Solve $$b(a+x)^2=(a+b)^2-x^2-\frac{b^2(2a+b)}{b+1}.$$

$$a^2b+2abx+bx^2=a^2+2ab+b^2-x^2-\frac{2ab^2+b^3}{b+1}.$$

$$bx^2+x^2+2abx=a^2+2ab+b^2-a^2b-\frac{2ab^2+b^3}{b+1}.$$

$$(b+1)^2x^2+2ab(b+1)x$$
$$=a^2b+a^2+2ab^2+2ab+b^3+b^2-a^2b^2-a^2b-2ab^2-b^3.$$

$$(b+1)^2x^2+2ab(b+1)x=a^2+2ab+b^2-a^2b^2.$$

$$(b+1)^2x^2+2ab(b+1)x+a^2b^2=a^2+2ab+b^2.$$

$$(b+1)x+ab=\pm(a+b).$$

$$(b+1)x=a-ab+b,\text{ or }-(a+ab+b).$$

$$\therefore x=\frac{a-ab+b}{b+1},\text{ or }-\frac{a+ab+b}{b+1}.$$

55. Solve $x^4+(ab+1)^2=(a^2+b^2)(x^2+1)+2(a^2-b^2)x+1.$

$$x^4+a^2b^2+2ab+1=(a^2+b^2)x^2+a^2+b^2+2(a^2-b^2)x+1.$$

$$x^4+a^2b^2=(a^2+b^2)x^2+2(a^2-b^2)x+(a^2-2ab+b^2).$$

$$x^4+2abx^2+a^2b^2=(a^2+2ab+b^2)x^2+2(a^2-b^2)x+(a-b)^2.$$

$$x^4+2abx^2+a^2b^2=(a+b)^2x^2+2(a^2-b^2)x+(a-b)^2.$$

Extract the square root,

$$x^2 + ab = \pm [(a+b)x + (a-b)].$$

$$x^2 \mp (a+b)x = \pm (a-b) - ab.$$

(i) $$x^2 - (a+b)x = a - b - ab.$$

$$4x^2 - (\ ) + (a+b)^2 = (a-b)^2 + 4(a-b).$$

$$2x - (a+b) = \pm\sqrt{(a-b)^2 + 4(a-b)}.$$

$$2x = (a+b) \pm \sqrt{(a-b)^2 + 4(a-b)}.$$

$$\therefore x = \tfrac{1}{2}(a+b) \pm \tfrac{1}{2}\sqrt{(a-b)^2 + 4(a-b)}.$$

(ii) $$x^2 + (a+b)x = -a + b - ab.$$

$$4x^2 + (\ ) + (a+b)^2 = (a-b)^2 - 4(a-b).$$

$$2x + (a+b) = \pm\sqrt{(a-b)^2 - 4(a-b)}.$$

$$2x = -(a+b) \pm \sqrt{(a-b)^2 - 4(a-b)}.$$

$$\therefore x = -\tfrac{1}{2}(a+b) \pm \tfrac{1}{2}\sqrt{(a-b)^2 - 4(a-b)}.$$

$$\therefore x = \tfrac{1}{2}(a+b) \pm \tfrac{1}{2}\sqrt{(a-b)^2 + 4(a-b)},$$

$$\text{or } -\tfrac{1}{2}(a+b) \pm \tfrac{1}{2}\sqrt{(a-b)^2 - 4(a-b)}.$$

Exercise 115. Page 282

1. Determine, without solving, the character of the roots of the equation $x^2 + 7x + 12 = 0$.

Here $$a = 1,\ b = 7,\ c = 12.$$

$$\therefore b^2 - 4ac = 7^2 - 4 \times 1 \times 12 = 49 - 48 = 1.$$

Since $b^2 - 4ac$ is greater than 0, and is a perfect square, the roots of the equation $x^2 + 7x + 12 = 0$ are real, different, and rational.

2. Determine, without solving, the character of the roots of the equation $x^2 + 2x + 5 = 0$.

Here $$a = 1,\ b = 2,\ c = 5.$$

$$\therefore b^2 - 4ac = 2^2 - 4 \times 1 \times 5 = 4 - 20 = -16.$$

Since $b^2 - 4ac$ is less than 0, the roots of the equation $x^2 + 2x + 5 = 0$ are imaginary.

3. Determine, without solving, the character of the roots of the equation $x^2 - 4x - 2 = 0$.

Here $$a = 1,\ b = -4,\ c = -2.$$

$$\therefore b^2 - 4ac = (-4)^2 - 4 \times 1 \times (-2) = 16 + 8 = 24.$$

Since $b^2 - 4ac$ is greater than 0, and is not a perfect square, the roots of the equation $x^2 - 4x - 2 = 0$ are real, different, and irrational.

4. Determine, without solving, the character of the roots of the equation $5x^2 - 3x - 2 = 0$.

Here $a = 5,\ b = -3,\ c = -2.$

$$\therefore\ b^2 - 4ac = (-3)^2 - 4 \times 5 \times (-2) = 9 + 40 = 49.$$

Since $b^2 - 4ac$ is greater than 0, and is a perfect square, the roots of the equation $5x^2 - 3x - 2 = 0$ are real, different, and rational.

5. Determine, without solving, the character of the roots of the equation $9x^2 + 12x + 4 = 0$.

Here $a = 9,\ b = 12,\ c = 4.$

$$\therefore\ b^2 - 4ac = 12^2 - 4 \times 9 \times 4 = 144 - 144 = 0.$$

Since $b^2 - 4ac = 0$, the roots of the equation $9x^2 + 12x + 4 = 0$ are real and equal.

6. Determine, without solving, the character of the roots of the equation $4x^2 - 4x - 1 = 0$.

Here $a = 4,\ b = -4,\ c = -1.$

$$\therefore\ b^2 - 4ac = (-4)^2 - 4 \times 4 \times (-1) = 16 + 16 = 32.$$

Since $b^2 - 4ac$ is greater than 0, and is not a perfect square, the roots of the equation $4x^2 - 4x - 1 = 0$ are real, different, and irrational.

7. Determine, without solving, the character of the roots of the equation $x^2 - 2x - 4 = 0$.

Here $a = 1,\ b = -2,\ c = -4.$

$$\therefore\ b^2 - 4ac = (-2)^2 - 4 \times 1 \times (-4) = 4 + 16 = 20.$$

Since $b^2 - 4ac$ is greater than 0, and is not a perfect square, the roots of the equation $x^2 - 2x - 4 = 0$ are real, different, and irrational.

8. Determine, without solving, the character of the roots of the equation $7x^2 - 9x - 10 = 0$.

Here $a = 7,\ b = -9,\ c = -10.$

$$\therefore\ b^2 - 4ac = (-9)^2 - 4 \times 7 \times (-10) = 81 + 280 = 361.$$

Since $b^2 - 4ac$ is greater than 0, and is a perfect square, the roots of the equation $7x^2 - 9x - 10 = 0$ are real, different, and rational.

9. Determine, without solving, the character of the roots of the equation $25x^2 - 20x + 4 = 0$.

Here $a = 25,\ b = -20,\ c = 4.$

$$\therefore\ b^2 - 4ac = (-20)^2 - 4 \times 25 \times 4 = 400 - 400 = 0.$$

Since $b^2 - 4ac = 0$, the roots of the equation $25x^2 - 20x + 4 = 0$ are real and equal.

10. Determine, without solving, the character of the roots of the equation $3x^2 - 7x - 6 = 0$.

Here $a = 3,\ b = -7,\ c = -6.$

$$\therefore\ b^2 - 4ac = (-7)^2 - 4 \times 3 \times (-6) = 49 + 72 = 121.$$

Since $b^2 - 4ac$ is greater than 0, and is a perfect square, the roots of the equation $3x^2 - 7x - 6 = 0$ are real, different, and rational.

11. Determine, without solving, the character of the roots of the equation $x^2 + 7x + 18 = 0$.

Here $a = 1,\ b = 7,\ c = 18.$

$$\therefore\ b^2 - 4ac = 7^2 - 4 \times 1 \times 18 = 49 - 72 = -23.$$

Since $b^2 - 4ac$ is less than 0, the roots of the equation $x^2 + 7x + 18 = 0$ are imaginary.

12. Determine, without solving, the character of the roots of the equation $2x^2 - 13x + 15 = 0$.

Here $a = 2,\ b = -13,\ c = 15.$

$$\therefore\ b^2 - 4ac = (-13)^2 - 4 \times 2 \times 15 = 169 - 120 = 49.$$

Since $b^2 - 4ac$ is greater than 0, and is a perfect square, the roots of the equation $2x^2 - 13x + 15 = 0$ are real, different, and rational.

13. Determine, without solving, the character of the roots of the equation $3x^2 - 4x - 22 = 0$.

Here $a = 3,\ b = -4,\ c = -22.$

$$\therefore\ b^2 - 4ac = (-4)^2 - 4 \times 3 \times (-22) = 16 + 264 = 280.$$

Since $b^2 - 4ac$ is greater than 0, and is not a perfect square, the roots of the equation $3x^2 - 4x - 22 = 0$ are real, different, and irrational.

14. Determine, without solving, the character of the roots of the equation $2x^2 + 3x + 8 = 0$.

Here $a = 2,\ b = 3,\ c = 8.$

$$\therefore\ b^2 - 4ac = 3^2 - 4 \times 2 \times 8 = 9 - 64 = -55.$$

Since $b^2 - 4ac$ is less than 0, the roots of the equation $2x^2 + 3x + 8 = 0$ are imaginary.

15. Determine the values of m for which the two roots of the equation $(2m + 1)x^2 + (7m + 2)x + 6m + 1 = 0$ are equal.

Here $a = 2m + 1,\ b = 7m + 2,\ c = 6m + 1.$

$$\begin{aligned}\therefore\ b^2 - 4ac &= (7m + 2)^2 - 4(2m + 1)(6m + 1)\\ &= 49m^2 + 28m + 4 - 48m^2 - 32m - 4\\ &= m^2 - 4m.\end{aligned}$$

If the roots are to be equal, then

$$b^2 - 4ac = 0, \text{ or } m^2 - 4m = 0.$$

Factor, $m(m - 4) = 0.$

$$\therefore\ m = 0, \text{ or } 4.$$

16. Determine the values of m for which the two roots of the equation $(3m + 1)x^2 + (11m + 1)x + 8m + 9 = 0$ are equal.

Here $a = 3m + 1,\ b = 11m + 1,\ c = 8m + 9.$

$$\begin{aligned}\therefore\ b^2 - 4ac &= (11m + 1)^2 - 4(3m + 1)(8m + 9)\\ &= 121m^2 + 22m + 1 - 96m^2 - 140m - 36\\ &= 25m^2 - 118m - 35.\end{aligned}$$

If the roots are to be equal, then

$$b^2 - 4ac = 0, \text{ or } 25m^2 - 118m - 35 = 0.$$

Complete the square,

$$625m^2 - (\ ) + 3481 = 4356.$$

$$25m - 59 = \pm 66.$$

$$25m = 125, \text{ or } -7.$$

$$\therefore\ m = 5, \text{ or } -\tfrac{7}{25}.$$

17. Determine the values of m for which the two roots of the equation $2mx^2 + 7x^2 - 8mx + 2x + 5m + 4 = 0$ are equal.

We have $2mx^2 + 7x^2 - 8mx + 2x + 5m + 4 = 0.$

Combine, $(2m + 7)x^2 - (8m - 2)x + (5m + 4) = 0.$

Here $a = 2m + 7,\ b = -(8m - 2),\ c = 5m + 4.$

$$\begin{aligned}\therefore\ b^2 - 4ac &= [-(8m - 2)]^2 - 4(2m + 7)(5m + 4)\\ &= 64m^2 - 32m + 4 - 40m^2 - 172m - 112\\ &= 24m^2 - 204m - 108.\end{aligned}$$

If the roots are to be equal, then

$$b^2 - 4ac = 0, \text{ or } 24m^2 - 204m - 108 = 0.$$

Divide by 12, $2m^2 - 17m - 9 = 0.$

$$16m^2 - (\ ) + 289 = 361.$$
$$4m - 17 = \pm 19.$$
$$4m = 36, \text{ or } -2.$$
$$\therefore m = 9, \text{ or } -\tfrac{1}{2}.$$

18. Determine the values of m for which the two roots of the equation $mx^2 + 4m + 1 = 2x + 3mx + 2x^2$ are equal.

We have $mx^2 + 4m + 1 = 2x + 3mx + 2x^2.$

Transpose and combine,

$$(m - 2)x^2 - (3m + 2)x + (4m + 1) = 0.$$

Here $a = m - 2,\ b = -(3m + 2),\ c = 4m + 1.$

$$\begin{aligned}\therefore b^2 - 4ac &= [-(3m + 2)]^2 - 4(m - 2)(4m + 1)\\ &= 9m^2 + 12m + 4 - 16m^2 + 28m + 8\\ &= -7m^2 + 40m + 12.\end{aligned}$$

If the roots are to be equal, then

$$b^2 - 4ac = 0, \text{ or } -7m^2 + 40m + 12 = 0.$$
$$7m^2 - 40m = 12.$$
$$49m^2 - (\ ) + 400 = 484.$$
$$7m - 20 = \pm 22.$$
$$7m = 42, \text{ or } -2.$$
$$\therefore m = 6, \text{ or } -\tfrac{2}{7}.$$

19. Determine the values of m for which the two roots of the equation $3m + 2x + 5x^2 = 11mx + 3 - 11mx^2$ are equal.

We have $3m + 2x + 5x^2 = 11mx + 3 - 11mx^2.$

Transpose and combine,

$$(11m + 5)x^2 + (2 - 11m)x + (3m - 3) = 0.$$

Here $a = 11m + 5,\ b = 2 - 11m,\ c = 3m - 3.$

$$\begin{aligned}\therefore b^2 - 4ac &= (2 - 11m)^2 - 4(11m + 5)(3m - 3)\\ &= 4 - 44m + 121m^2 - 132m^2 + 72m + 60\\ &= -11m^2 + 28m + 64.\end{aligned}$$

If the roots are to be equal, then

$$b^2 - 4ac = 0, \text{ or } -11m^2 + 28m + 64 = 0.$$
$$11m^2 - 28m = 64.$$
$$121m - (\ ) + 196 = 900.$$
$$11m - 14 = \pm 30.$$
$$11m = 44, \text{ or } -16.$$
$$\therefore m = 4, \text{ or } -\tfrac{16}{11}.$$

Exercise 116. Page 285

1. Solve $\sqrt{x+7+\sqrt{5(x-2)}}=3.$

Square, $x+7+\sqrt{5(x-2)}=9.$

Transpose and combine, $\sqrt{5(x-2)}=2-x.$

Square, $5x-10=4-4x+x^2.$

Transpose and combine, $x^2-9x+14=0.$

Factor, $(x-2)(x-7)=0.$

$\therefore\ x=2, \text{ or } 7.$

But the value $x=7$ does not satisfy the given equation.

$\therefore\ x=2.$

2. Solve $\sqrt{14x-11}+\sqrt{3(2x-1)}=2\sqrt{2x+1}.$

Square, $14x-11+2\sqrt{3(14x-11)(2x-1)}+6x-3=8x+4.$

Transpose and combine,

$$2\sqrt{84x^2-108x+33}=18-12x.$$
$$\sqrt{84x^2-108x+33}=9-6x.$$

Square, $84x^2-108x+33=81-108x+36x^2.$

$$48x^2=48.$$
$$x^2=1.$$
$$\therefore\ x=\pm 1.$$

But the value $x=-1$ does not satisfy the given equation.

$\therefore\ x=1.$

3. Solve $\sqrt{a^2-x}+\sqrt{b^2+x}=a+b.$

Square,

$$a^2-x+2\sqrt{(a^2-x)(b^2+x)}+b^2+x=a^2+2ab+b^2.$$

Transpose and combine,

$$2\sqrt{a^2b^2-b^2x+a^2x-x^2}=2ab.$$
$$\sqrt{a^2b^2-b^2x+a^2x-x^2}=ab.$$

Square, $a^2b^2-b^2x+a^2x-x^2=a^2b^2.$

$$x^2-a^2x+b^2x=0.$$
$$x(x-a^2+b^2)=0.$$
$$\therefore\ x=0, \text{ or } a^2-b^2.$$

4. Solve $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}.$

Square, $a-x+2\sqrt{(a-x)(b-x)}+b-x=a+b-2x.$

$$2\sqrt{(a-x)(b-x)}=0.$$

$$\sqrt{(a-x)(b-x)} = 0.$$

Square, $(a-x)(b-x) = 0.$

$$\therefore\ x = a, \text{ or } b.$$

5. Solve $2\sqrt{5+2x} - \sqrt{13-6x} = \sqrt{37-6x}.$

Square, $20 + 8x - 4\sqrt{(5+2x)(13-6x)} + 13 - 6x = 37 - 6x.$

Transpose and combine, $8x - 4 = 4\sqrt{65 - 4x - 12x^2}.$

$$2x - 1 = \sqrt{65 - 4x - 12x^2}.$$

Square, $4x^2 - 4x + 1 = 65 - 4x - 12x^2.$

$$16x^2 = 64.$$

$$x^2 = 4.$$

$$\therefore\ x = \pm 2.$$

But the value $x = -2$ does not satisfy the given equation.

$$\therefore\ x = 2.$$

6. Solve $\sqrt{2(x-3)} = \sqrt[4]{x^2 - 4x + 11}.$

Square, $2x - 6 = \sqrt{x^2 - 4x + 11}.$

Square, $4x^2 - 24x + 36 = x^2 - 4x + 11.$

$$3x^2 - 20x = -25.$$

$$9x^2 - (\ ) + 100 = 25.$$

$$3x - 10 = \pm 5.$$

$$3x = 15, \text{ or } 5.$$

$$\therefore\ x = 5, \text{ or } \tfrac{5}{3}.$$

But the value $x = \frac{5}{3}$ does not satisfy the given equation.

$$\therefore\ x = 5.$$

7. Solve $\sqrt{2x+1} + \sqrt{x-3} = 2\sqrt{x}.$

Square, $2x + 1 + 2\sqrt{(2x+1)(x-3)} + x - 3 = 4x.$

Transpose and combine, $2\sqrt{2x^2 - 5x - 3} = x + 2.$

Square, $8x^2 - 20x - 12 = x^2 + 4x + 4.$

$$7x^2 - 24x = 16.$$

$$49x^2 - (\ ) + 144 = 256.$$

$$7x - 12 = \pm 16.$$

$$7x = 28, \text{ or } -4.$$

$$\therefore\ x = 4, \text{ or } -\tfrac{4}{7}.$$

But the value $x = -\frac{4}{7}$ does not satisfy the given equation.

$$\therefore\ x = 4.$$

8. Solve $\sqrt{5x-1}-\sqrt{8-2x}=\sqrt{x-1}.$

Square,

$5x-1-2\sqrt{(5x-1)(8-2x)}+8-2x=x-1.$

Transpose and combine, $2x+8=2\sqrt{-10x^2+42x-8}$

$$\begin{aligned}
x+4&=\sqrt{-10x^2+42x-8}.\\
x^2+8x+16&=-10x^2+42x-8.\\
11x^2-34x&=-24.\\
121x^2-(\ )+289&=25.\\
11x-17&=\pm 5.\\
11x&=22, \text{ or } 12.\\
\therefore x&=2, \text{ or } \tfrac{12}{11}.
\end{aligned}$$

But the value $x=\frac{12}{11}$ does not satisfy the given equation.

$$\therefore x=2.$$

9. Solve $\sqrt{4x-3}+\sqrt{5x+1}=\sqrt{15x+4}.$

Square,

$4x-3+2\sqrt{(4x-3)(5x+1)}+5x+1=15x+4.$

Transpose and combine,

$$\begin{aligned}
2\sqrt{20x^2-11x-3}&=6x+6.\\
\sqrt{20x^2-11x-3}&=3x+3.
\end{aligned}$$

Square, $20x^2-11x-3=9x^2+18x+9.$

$$\begin{aligned}
11x^2-29x&=12.\\
484x^2-(\ )+841&=1369.\\
22x-29&=\pm 37.\\
22x&=66, \text{ or } -8.\\
\therefore x&=3, \text{ or } -\tfrac{4}{11}.
\end{aligned}$$

But the value $x=-\frac{4}{11}$ does not satisfy the given equation.

$$\therefore x=3.$$

10. Solve $x\sqrt{x-a}+a\sqrt{x+a}=\sqrt{x^3+a^3}.$

Square,

$x^3-ax^2+2ax\sqrt{x^2-a^2}+a^2x+a^3=x^3+a^3.$

Transpose and combine, $2ax\sqrt{x^2-a^2}=ax^2-a^2x.$

Divide by a, $2x\sqrt{x^2-a^2}=x^2-ax.$

Square, $4x^4-4a^2x^2=x^4-2ax^3+a^2x^2.$

$$3x^4+2ax^3-5a^2x^2=0.$$

$$x^2(3x^2+2ax-5a^2)=0.$$
$$x^2(x-a)(3x+5a)=0.$$
$$\therefore\ x=0,\ 0,\ a,\ \text{or}\ -\frac{5a}{3}.$$

But the value $x=-\frac{5a}{3}$ does not satisfy the given equation.

$$\therefore\ x=0,\ \text{or}\ a.$$

11. Solve $\sqrt{a(x-b)}+\sqrt{b(x-a)}=x.$

Square,

$$ax-ab+2\sqrt{ab(x-b)(x-a)}+bx-ab=x^2.$$

Transpose and combine,

$$2\sqrt{ab(x-b)(x-a)}=x^2-ax-bx+2ab.$$
$$2\sqrt{ab(x-b)(x-a)}=x^2-(a+b)x+2ab.$$
$$2\sqrt{ab(x-b)(x-a)}=(x-a)(x-b)+ab.$$

Square,

$$4ab(x-b)(x-a)=(x-a)^2(x-b)^2+2ab(x-a)(x-b)+a^2b^2.$$
$$(x-a)^2(x-b)^2-2ab(x-a)(x-b)+a^2b^2=0.$$

Extract the square root,

$$(x-a)(x-b)-ab=0.$$
$$x^2-ax-bx+ab-ab=0.$$
$$x^2-ax-bx=0.$$
$$x(x-a-b)=0.$$
$$\therefore\ x=0,\ \text{or}\ a+b.$$

But the value $x=0$ does not satisfy the given equation.

$$\therefore\ x=a+b.$$

12. Solve $a\sqrt{c+x}-b\sqrt{c-x}=\sqrt{c(a^2+b^2)}.$

Square,

$$a^2c+a^2x-2ab\sqrt{c^2-x^2}+b^2c-b^2x=a^2c+b^2c.$$

Transpose and combine, $a^2x-b^2x=2ab\sqrt{c^2-x^2}.$

Square, $a^4x^2-2a^2b^2x^2+b^4x^2=4a^2b^2c^2-4a^2b^2x^2.$

$$(a^4+2a^2b^2+b^4)x^2=4a^2b^2c^2.$$
$$x^2=\frac{4a^2b^2c^2}{a^4+2a^2b^2+b^4}.$$
$$\therefore\ x=\pm\frac{2abc}{a^2+b^2}.$$

But the value $x=-\frac{2abc}{a^2+b^2}$ does not satisfy the given equation.

$$\therefore\ x=\frac{2abc}{a^2+b^2}.$$

13. Solve $\sqrt{2a - b + 2x} - \sqrt{10a - 9b - 6x} = 4\sqrt{a - b}$.

Square,

$$2a - b + 2x - 2\sqrt{(2a - b + 2x)(10a - 9b - 6x)} + 10a - 9b - 6x$$
$$= 16a - 16b.$$

Transpose and combine,

$$-4a + 6b - 4x = 2\sqrt{(2a - b + 2x)(10a - 9b - 6x)}.$$
$$3b - 2a - 2x = \sqrt{(2a - b + 2x)(10a - 9b - 6x)}.$$

Square, $9b^2 + 4a^2 + 4x^2 - 12ab - 12bx + 8ax$

$$= 20a^2 - 28ab + 8ax + 9b^2 - 12bx - 12x^2.$$
$$16x^2 = 16a^2 - 16ab.$$
$$x^2 = a^2 - ab.$$
$$\therefore\ x = \pm\sqrt{a(a - b)}.$$

But the value $x = -\sqrt{a(a - b)}$ does not satisfy the given equation.

$$\therefore\ x = \sqrt{a(a - b)}.$$

14. Solve $\sqrt{(a + x)(x + b)} + \sqrt{(a - x)(x - b)} = 2\sqrt{ax}$.

Square,

$$ax + x^2 + ab + bx + 2\sqrt{(a + x)(a - x)(x + b)(x - b)} + ax - ab - x^2 + bx$$
$$= 4ax.$$

Transpose and combine,

$$2\sqrt{(a^2 - x^2)(x^2 - b^2)} = 2ax - 2bx.$$
$$\sqrt{(a^2 - x^2)(x^2 - b^2)} = ax - bx.$$

Square, $$a^2x^2 - a^2b^2 - x^4 + b^2x^2 = a^2x^2 - 2abx^2 + b^2x^2.$$
$$x^4 - 2abx^2 + a^2b^2 = 0.$$

Extract the square root, $$x^2 - ab = 0.$$
$$x^2 = ab.$$
$$\therefore\ x = \pm\sqrt{ab}.$$

15. Solve $$2x^2 + a\sqrt{b(b + 4x)} = a(b + 2x).$$
$$2x^2 + a\sqrt{b^2 + 4bx} = ab + 2ax.$$
$$a\sqrt{b^2 + 4bx} = ab + 2ax - 2x^2.$$

Square, $a^2b^2 + 4a^2bx = a^2b^2 + 4a^2x^2 + 4x^4 - 4abx^2 + 4a^2bx - 8ax^3.$

$$4x^4 - 8ax^3 + 4a^2x^2 - 4abx^2 = 0.$$
$$x^4 - 2ax^3 + a^2x^2 - abx^2 = 0.$$
$$x^2(x^2 - 2ax + a^2 - ab) = 0.$$
$$\therefore\ x^2 = 0,$$

or $$x^2 - 2ax + a^2 - ab = 0.$$

(i)
$$x^2 = 0.$$
$$\therefore x = 0, \text{ or } 0.$$

(ii)
$$x^2 - 2ax + a^2 - ab = 0.$$
$$x^2 - 2ax + a^2 = ab.$$
$$x - a = \pm \sqrt{ab}.$$
$$\therefore x = a \pm \sqrt{ab}.$$
$$\therefore x = 0,\ 0, \text{ or } a \pm \sqrt{ab}.$$

But the value $x = a + \sqrt{ab}$ does not satisfy the given equation.
$$\therefore x = 0, \text{ or } a - \sqrt{ab}.$$

16. Solve
$$\sqrt{x-a} - \sqrt{x-b} = \sqrt{b-a}.$$

Square, $x - a - 2\sqrt{(x-a)(x-b)} + x - b = b - a.$

Transpose and combine,
$$2x - 2b = 2\sqrt{(x-a)(x-b)}.$$
$$x - b = \sqrt{(x-a)(x-b)}.$$

Square,
$$x^2 - 2bx + b^2 = x^2 - ax - bx + ab.$$
$$ax - bx = ab - b^2.$$
$$(a-b)x = b(a-b).$$
$$\therefore x = b.$$

17. Solve
$$2\sqrt{2a+b+2x} + \sqrt{10a+b-6x} = \sqrt{10a+9b-6x}.$$

Square,
$$8a + 4b + 8x + 4\sqrt{(2a+b+2x)(10a+b-6x)} + 10a + b - 6x$$
$$= 10a + 9b - 6x.$$
$$4\sqrt{(2a+b+2x)(10a+b-6x)} = -8a + 4b - 8x.$$
$$\sqrt{(2a+b+2x)(10a+b-6x)} = -2a + b - 2x.$$

Square,
$$20a^2 + 12ab + b^2 + 8ax - 4bx - 12x^2 = 4a^2 + b^2 + 4x^2 - 4ab + 8ax - 4bx.$$
$$-16x^2 = -16a^2 - 16ab.$$
$$x^2 = a^2 + ab.$$
$$\therefore x = \pm\sqrt{a(a+b)}.$$

18. Solve
$$\sqrt{a+b+x} - \sqrt{a+b-x} = \frac{x}{\sqrt{a}}.$$

Square,
$$a + b + x - 2\sqrt{(a+b+x)(a+b-x)} + a + b - x = \frac{x^2}{a}.$$

Transpose and combine,
$$2a + 2b - \frac{x^2}{a} = 2\sqrt{a^2 + 2ab + b^2 - x^2}.$$

Square,
$$4a^2 + 4b^2 + \frac{x^4}{a^2} - 4x^2 - \frac{4bx^2}{a} + 8ab = 4a^2 + 8ab + 4b^2 - 4x^2.$$

$$\frac{x^4}{a^2} - \frac{4\,bx^2}{a} = 0.$$

$$x^4 - 4\,abx^2 = 0.$$

$$x^2(x^2 - 4\,ab) = 0.$$

$$\therefore\ x^2 = 0, \text{ or } 4\,ab.$$

$$\therefore\ x = 0,\ 0, \text{ or } \pm 2\sqrt{ab}.$$

19. Solve $$\frac{a-x}{\sqrt{a-x}} + \frac{x-b}{\sqrt{x-b}} = \sqrt{a-b}.$$

Simplify, $$\sqrt{a-x} + \sqrt{x-b} = \sqrt{a-b}.$$

Square, $a - x + 2\sqrt{(a-x)(x-b)} + x - b = a - b.$

Transpose and combine,

$$2\sqrt{(a-x)(x-b)} = 0.$$

$$\sqrt{(a-x)(b-x)} = 0.$$

Square, $$(a-x)(b-x) = 0.$$

$$\therefore\ x = a, \text{ or } b.$$

20. Solve $$\sqrt{a-x} + \sqrt{-(a^2+ax)} = \frac{a}{\sqrt{a-x}}.$$

Clear of fractions,

$$a - x + \sqrt{-a(a+x)(a-x)} = a.$$

$$\sqrt{-a(a^2-x^2)} = x.$$

Square, $$-a^3 + ax^2 = x^2.$$

$$ax^2 - x^2 = a^3.$$

$$(a-1)x^2 = a^3.$$

$$x^2 = \frac{a^3}{a-1}.$$

$$\therefore\ x = \pm\frac{a}{a-1}\sqrt{a(a-1)}.$$

But the value $x = -\frac{a}{a-1}\sqrt{a(a-1)}$ does not satisfy the given equation

$$\therefore\ x = \frac{a}{a-1}\sqrt{a(a-1)}.$$

Exercise 117. Page 287

1. Solve $$\sqrt{x^2-7} = x^2 - 13.$$

Transpose, $$x^2 - \sqrt{x^2-7} = 13.$$

Add -7 to each member, $(x^2-7) - \sqrt{x^2-7} = 6.$

Complete the square, $4(x^2-7)-(\;)+1=25.$

$$2\sqrt{x^2-7}-1=\pm 5.$$
$$2\sqrt{x^2-7}=6, \text{ or } -4.$$
$$\sqrt{x^2-7}=3, \text{ or } -2.$$

Square, $x^2-7=9, \text{ or } 4.$

$$x^2=16, \text{ or } 11.$$
$$\therefore\ x=\pm 4, \text{ or } \pm\sqrt{11}.$$

But the values $x=\pm\sqrt{11}$ do not satisfy the given equation.

$$\therefore\ x=\pm 4.$$

2. Solve $x+c+\sqrt{x+c}=a.$

Complete the square, $4(x+c)+(\;)+1=4a+1.$

$$2\sqrt{x+c}+1=\pm\sqrt{4a+1}.$$
$$2\sqrt{x+c}=-1\pm\sqrt{4a+1}.$$

Square, $4(x+c)=2+4a\mp 2\sqrt{4a+1}.$

$$x+c=\tfrac{1}{2}+a\mp\tfrac{1}{2}\sqrt{4a+1}.$$
$$\therefore\ x=\tfrac{1}{2}+a-c\mp\tfrac{1}{2}\sqrt{4a+1}.$$

But the value $x=\frac{1}{2}+a-c+\frac{1}{2}\sqrt{4a+1}$ does not satisfy the given equation.

$$\therefore\ x=\tfrac{1}{2}+a-c-\tfrac{1}{2}\sqrt{4a+1}.$$

3. Solve $x^4-40x^2+144=0.$

Factor, $(x^2-36)(x^2-4)=0.$

$$\therefore\ x^2=36, \text{ or } 4.$$
$$\therefore\ x=\pm 6, \text{ or } \pm 2.$$

4. Solve $x^{\frac{3}{2}}+8x^{\frac{1}{2}}=9x.$

$$x^{\frac{3}{2}}-9x+8x^{\frac{1}{2}}=0.$$
$$x^{\frac{1}{2}}(x-9x^{\frac{1}{2}}+8)=0.$$
$$x^{\frac{1}{2}}(x^{\frac{1}{2}}-1)(x^{\frac{1}{2}}-8)=0.$$
$$\therefore\ x^{\frac{1}{2}}=0,\ 1, \text{ or } 8.$$
$$\therefore\ x=0,\ 1, \text{ or } 64.$$

5. Solve $2x^{\frac{1}{3}}-3x^{\frac{2}{3}}+x=0.$

$$x-3x^{\frac{2}{3}}+2x^{\frac{1}{3}}=0.$$
$$x^{\frac{1}{3}}(x^{\frac{2}{3}}-3x^{\frac{1}{3}}+2)=0.$$
$$x^{\frac{1}{3}}(x^{\frac{1}{3}}-1)(x^{\frac{1}{3}}-2)=0.$$
$$\therefore\ x^{\frac{1}{3}}=0,\ 1, \text{ or } 2.$$
$$\therefore\ x=0,\ 1, \text{ or } 8.$$

6. Solve $$\sqrt[4]{x^3} - 2\sqrt{x} + x = 0.$$

$$x^{\frac{3}{4}} - 2x^{\frac{1}{2}} + x = 0.$$
$$x + x^{\frac{3}{4}} - 2x^{\frac{1}{2}} = 0.$$
$$x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{\frac{1}{4}} - 2) = 0.$$
$$x^{\frac{1}{2}}(x^{\frac{1}{4}} - 1)(x^{\frac{1}{4}} + 2) = 0.$$
$$\therefore\ x^{\frac{1}{2}} = 0, \text{ or } x^{\frac{1}{4}} = 1, \text{ or } -2.$$
$$\therefore\ x = 0,\ 1, \text{ or } 16.$$

But the value $x = 16$ does not satisfy the given equation.

$$\therefore\ x = 0, \text{ or } 1.$$

7. Solve $$10x^4 - 156 = x^2.$$

$$10x^4 - x^2 = 156.$$
$$400x^4 - (\ ) + 1 = 6241.$$
$$20x^2 - 1 = \pm 79.$$
$$20x^2 = 80, \text{ or } -78.$$
$$x^2 = 4, \text{ or } -\tfrac{39}{10}.$$
$$\therefore\ x = \pm 2, \text{ or } \pm \tfrac{1}{10}\sqrt{-390}.$$

8. Solve $$\sqrt{x^3} - 3\sqrt[4]{x^3} = 40.$$

$$x^{\frac{3}{2}} - 3x^{\frac{3}{4}} = 40.$$
$$x^{\frac{3}{2}} - 3x^{\frac{3}{4}} - 40 = 0.$$
$$(x^{\frac{3}{4}} - 8)(x^{\frac{3}{4}} + 5) = 0.$$
$$\therefore\ x^{\frac{3}{4}} = 8, \text{ or } -5.$$
$$x^3 = 2^{12}, \text{ or } 625.$$
$$\therefore\ x = 16, \text{ or } 5\sqrt[3]{5}.$$

But the value $x = 5\sqrt[3]{5}$ does not satisfy the given equation.

$$\therefore\ x = 16.$$

9. Solve $$8x^6 + 63x^3 = 8.$$

$$256x^6 + (\ ) + 3969 = 4225.$$
$$16x^3 + 63 = \pm 65.$$
$$\therefore\ 16x^3 - 2 = 0, \text{ or } 16x^3 + 128 = 0.$$
$$8x^3 - 1 = 0, \text{ or } x^3 + 8 = 0.$$

Factor, $(2x - 1)(4x^2 + 2x + 1) = 0,$

or $(x + 2)(x^2 - 2x + 4) = 0.$

(i) $$2x - 1 = 0.$$
$$2x = 1.$$
$$\therefore\ x = \tfrac{1}{2}.$$

(ii)
$$4x^2+2x+1=0.$$
$$4x^2+2x=-1.$$
$$16x^2+(\ )+1=-3.$$
$$4x+1=\pm\sqrt{-3}.$$
$$4x=-1\pm\sqrt{-3}.$$
$$\therefore x=\tfrac{1}{4}(-1\pm\sqrt{-3}).$$

(iii)
$$x+2=0.$$
$$\therefore x=-2.$$

(iv)
$$x^2-2x+4=0.$$
$$x^2-2x=-4.$$
$$x^2-2x+1=-3.$$
$$x-1=\pm\sqrt{-3}.$$
$$\therefore x=1\pm\sqrt{-3}.$$
$$\therefore x=\tfrac{1}{2},\ -2,\ 1\pm\sqrt{-3},\text{ or }\tfrac{1}{4}(-1\pm\sqrt{-3}).$$

10. Solve
$$8x^{-6}+999x^{-3}=125.$$
$$256x^{-6}+(\ )+999^2=1002001.$$
$$16x^{-3}+999=\pm 1001.$$
$$16x^{-3}-2=0,\text{ or }16x^{-3}+2000=0.$$
$$8x^{-3}-1=0,\text{ or }x^{-3}+125=0.$$

Factor,
$$(2x^{-1}-1)(4x^{-2}+2x^{-1}+1)=0,$$
or
$$(x^{-1}+5)(x^{-2}-5x^{-1}+25)=0.$$

(i)
$$2x^{-1}-1=0.$$
$$2x^{-1}=1.$$
$$\frac{2}{x}=1.$$
$$\therefore x=2.$$

(ii)
$$4x^{-2}+2x^{-1}+1=0.$$
$$4x^{-2}+2x^{-1}=-1.$$
$$16x^{-2}+(\ )+1=-3.$$
$$4x^{-1}+1=\pm\sqrt{-3}.$$
$$4x^{-1}=-1\pm\sqrt{-3}.$$
$$\frac{4}{x}=-1\pm\sqrt{-3}.$$
$$x=\frac{4}{-1\pm\sqrt{-3}}.$$
$$\therefore x=-1\mp\sqrt{-3}.$$

(iii)

$$x^{-1} + 5 = 0.$$
$$x^{-1} = -5.$$
$$\frac{1}{x} = -5.$$
$$\therefore x = -\tfrac{1}{5}.$$

(iv)

$$x^{-2} - 5x^{-1} + 25 = 0.$$
$$x^{-2} - 5x^{-1} = -25.$$
$$4x^{-2} - (\ ) + 25 = -75.$$
$$2x^{-1} - 5 = \pm 5\sqrt{-3}.$$
$$2x^{-1} = 5 \pm 5\sqrt{-3}.$$
$$x^{-1} = \tfrac{5}{2} \pm \tfrac{5}{2}\sqrt{-3}.$$
$$\frac{1}{x} = \frac{5}{2} \pm \frac{5}{2}\sqrt{-3}.$$
$$x = \frac{1}{\tfrac{5}{2} \pm \tfrac{5}{2}\sqrt{-3}}.$$
$$\therefore x = \tfrac{1}{10}(1 \mp \sqrt{-3})$$
$$\therefore x = 2,\ -\tfrac{1}{5},\ -1 \pm \sqrt{-3},\text{ or }\tfrac{1}{10}(1 \pm \sqrt{-3}).$$

11. Solve $(3x - 5)^2 - 8(3x - 5) + 7 = 0.$

Factor, $(3x - 5 - 1)(3x - 5 - 7) = 0.$

$$(3x - 6)(3x - 12) = 0.$$
$$\therefore 3x = 6,\text{ or }12.$$
$$\therefore x = 2,\text{ or }4.$$

12. Solve $(2x - 3)^2 - 4(2x - 3) + 5 = 0.$

$$(2x - 3)^2 - 4(2x - 3) = -5.$$
$$(2x - 3)^2 - (\ ) + 4 = -1.$$
$$2x - 3 - 2 = \pm\sqrt{-1}.$$
$$2x = 5 \pm \sqrt{-1}.$$
$$\therefore x = \tfrac{1}{2}(5 \pm \sqrt{-1}).$$

13. Solve $(5x + 2)^2 - 6(5x + 2) + 9 = 0.$

Factor, $(5x + 2 - 3)(5x + 2 - 3) = 0.$

$$(5x - 1)(5x - 1) = 0.$$
$$5x = 1,\text{ or }1.$$
$$\therefore x = \tfrac{1}{5},\text{ or }\tfrac{1}{5}.$$

14. Solve $x^2 + 5 = 8x + 2\sqrt{x^2 - 8x + 40}.$

$$x^2 - 8x + 5 - 2\sqrt{x^2 - 8x + 40} = 0.$$

Add 35 to each member,

$$(x^2 - 8x + 40) - 2\sqrt{x^2 - 8x + 40} = 35.$$

$$(x^2 - 8x + 40) - 2\sqrt{x^2 - 8x + 40} + 1 = 36.$$

$$\sqrt{x^2 - 8x + 40} - 1 = \pm 6.$$

$$\sqrt{x^2 - 8x + 40} = 7, \text{ or } -5.$$

Square, $x^2 - 8x + 40 = 49, \text{ or } 25.$

$$x^2 - 8x - 9 = 0, \text{ or } x^2 - 8x + 15 = 0.$$

$$(x + 1)(x - 9) = 0, \text{ or } (x - 3)(x - 5) = 0.$$

$$\therefore x = -1, 9, 3, \text{ or } 5.$$

But the values $x = 3$ and $x = 5$ do not satisfy the given equation.

$$\therefore x = 9, \text{ or } -1.$$

15. Solve $2x^2 + 3\sqrt{x^2 - x + 1} = 2x + 3.$

$$2x^2 - 2x + 3\sqrt{x^2 - x + 1} = 3.$$

$$2(x^2 - x + 1) + 3\sqrt{x^2 - x + 1} = 5.$$

Complete the square,

$$16(x^2 - x + 1) + (\quad) + 9 = 49.$$

$$4\sqrt{x^2 - x + 1} + 3 = \pm 7.$$

$$4\sqrt{x^2 - x + 1} = 4, \text{ or } -10.$$

$$\sqrt{x^2 - x + 1} = 1, \text{ or } -\tfrac{5}{2}.$$

Square, $x^2 - x + 1 = 1, \text{ or } \tfrac{25}{4}.$

$$x^2 - x = 0, \text{ or } x^2 - x = \tfrac{21}{4}.$$

(i) $x^2 - x = 0.$

$$x(x - 1) = 0.$$

$$\therefore x = 0, \text{ or } 1.$$

(ii) $x^2 - x = \tfrac{21}{4}.$

$$4x^2 - 4x = 21.$$

$$4x^2 - 4x + 1 = 22.$$

$$2x - 1 = \pm\sqrt{22}.$$

$$2x = 1 \pm \sqrt{22}.$$

$$\therefore x = \tfrac{1}{2}(1 \pm \sqrt{22}).$$

$$\therefore x = 0, 1, \text{ or } \tfrac{1}{2}(1 \pm \sqrt{22}).$$

But the values $x = \frac{1}{2}(1 \pm \sqrt{22})$ do not satisfy the given equation.

$$\therefore x = 0, \text{ or } 1.$$

16. Solve $3\sqrt{3x^2-2x+4}=3x^2-2x-6.$

$$3x^2-2x-6-3\sqrt{3x^2-2x+4}=0.$$

Add 10 to each member,

$$(3x^2-2x+4)-3\sqrt{3x^2-2x+4}=10.$$
$$4(3x^2-2x+4)-(\ )+9=49.$$
$$2\sqrt{3x^2-2x+4}-3=\pm 7.$$
$$2\sqrt{3x^2-2x+4}=10, \text{ or } -4.$$
$$\sqrt{3x^2-2x+4}=5, \text{ or } -2.$$

Square, $3x^2-2x+4=25, \text{ or } 4.$

$$3x^2-2x=21, \text{ or } 0.$$
$$9x^2-(\ )+1=64, \text{ or } 0.$$
$$3x-1=\pm 8, \text{ or } 0.$$
$$3x=9,\ -7, \text{ or } 0.$$
$$\therefore\ x=3,\ -\tfrac{7}{3}, \text{ or } 0.$$

But the value $x=0$ does not satisfy the given equation.

$$\therefore\ x=3, \text{ or } -\tfrac{7}{3}.$$

17. Solve $2x^2+3x-7\sqrt{2x^2+3x-2}+8=0.$

Add -10 to each member,

$$(2x^2+3x-2)-7\sqrt{2x^2+3x-2}=-10.$$
$$4(2x^2+3x-2)-(\ )+49=9.$$
$$2\sqrt{2x^2+3x-2}-7=\pm 3.$$
$$2\sqrt{2x^2+3x-2}=10, \text{ or } 4.$$
$$\sqrt{2x^2+3x-2}=5, \text{ or } 2.$$

Square, $2x^2+3x-2=25, \text{ or } 4.$

$$2x^2+3x=27, \text{ or } 6.$$
$$16x^2+(\ )+9=225, \text{ or } 57.$$
$$4x+3=\pm 15, \text{ or } \pm\sqrt{57}.$$
$$4x=12,\ -18, \text{ or } -3\pm\sqrt{57}.$$
$$\therefore\ x=3,\ -\tfrac{9}{2}, \text{ or } \tfrac{1}{4}(-3\pm\sqrt{57}).$$

18. Solve $39-8x+\sqrt{7x^2+8x-19}-7x^2=0.$

$$7x^2+8x-\sqrt{7x^2+8x-19}=39.$$

Add -19 to each member,

$$(7x^2+8x-19)-\sqrt{7x^2+8x-19}=20.$$
$$4(7x^2+8x-19)-(\ )+1=81.$$
$$2\sqrt{7x^2+8x-19}-1=\pm 9.$$
$$2\sqrt{7x^2+8x-19}=10, \text{ or } -8.$$

$$\sqrt{7x^2+8x-19}=5, \text{ or } -4.$$

Square, $$7x^2+8x-19=25, \text{ or } 16.$$

$$7x^2+8x=44, \text{ or } 35.$$

$$49x^2+(\ )+16=324, \text{ or } 261.$$

$$7x+4=\pm 18, \text{ or } \pm 3\sqrt{29}.$$

$$7x=14,\ -22, \text{ or } -4\pm 3\sqrt{29}.$$

$$\therefore\ x=2,\ -\tfrac{22}{7}, \text{ or } \tfrac{1}{7}(-4\pm 3\sqrt{29}).$$

But the values $x=\frac{1}{7}(-4\pm 3\sqrt{29})$ do not satisfy the given equation.

$$\therefore\ x=2, \text{ or } -\tfrac{22}{7}.$$

19. Solve $$(2x^2-3x+1)^2=22x^2-33x+1.$$

$$(2x^2-3x+1)^2-(22x^2-33x)-1=0.$$

$$(2x^2-3x+1)^2-11(2x^2-3x+1)+10=0.$$

Factor, $$(2x^2-3x+1-1)(2x^2-3x+1-10)=0.$$

$$(2x^2-3x)(2x^2-3x-9)=0.$$

$$\therefore\ 2x^2-3x=0, \text{ or } 2x^2-3x-9=0.$$

(i) $$2x^2-3x=0.$$

$$x(2x-3)=0.$$

$$\therefore\ x=0, \text{ or } \tfrac{3}{2}.$$

(ii) $$2x^2-3x-9=0.$$

$$2x^2-3x=9.$$

$$16x^2-(\ )+9=81.$$

$$4x-3=\pm 9.$$

$$4x=12, \text{ or } -6.$$

$$\therefore\ x=3, \text{ or } -\tfrac{3}{2}.$$

$$\therefore\ x=0,\ \pm\tfrac{3}{2}, \text{ or } 3.$$

20. Solve $$x+2a\sqrt{2(a^2+b^2)-x}=3a^2+b^2.$$

$$-x-2a\sqrt{2(a^2+b^2)-x}=-3a^2-b^2.$$

Add $2a^2+2b^2$ to each member,

$$(2a^2+2b^2-x)-2a\sqrt{2a^2+2b^2-x}=b^2-a^2.$$

Complete the square,

$$(2a^2+2b^2-x)-(\ )+a^2=b^2.$$

$$\sqrt{2a^2+2b^2-x}-a=\pm b.$$

$$\sqrt{2a^2+2b^2-x}=a\pm b.$$

Square, $$2a^2+2b^2-x=a^2\pm 2ab+b^2.$$

$$x=a^2\mp 2ab+b^2.$$

$$\therefore\ x=(a-b)^2, \text{ or } (a+b)^2.$$

21. Solve $x^4 - 6x^3 + 7x^2 + 6x - 8 = 0$.

Attempt to extract the square root of the left member,

$$\begin{array}{r|l|l}
 & x^4 - 6x^3 + 7x^2 + 6x - 8 & \underline{x^2 - 3x - 1} \\
 & x^4 & \\
2x^2 - 3x & -6x^3 + 7x^2 & \\
 & \underline{-6x^3 + 9x^2} & \\
2x^2 - 6x - 1 & -2x^2 + 6x - 8 & \\
 & \underline{-2x^2 + 6x + 1} & \\
 & \qquad\qquad\quad -9 &
\end{array}$$

Add 9 to each member,

$$x^4 - 6x^3 + 7x^2 + 6x + 1 = 9.$$

Extract the square root,

$$x^2 - 3x - 1 = \pm 3.$$

$$x^2 - 3x - 4 = 0, \text{ or } x^2 - 3x + 2 = 0.$$

$$(x + 1)(x - 4) = 0, \text{ or } (x - 1)(x - 2) = 0.$$

$$\therefore x = -1, 4, 1, \text{ or } 2.$$

$$\therefore x = \pm 1, 2, \text{ or } 4.$$

22. Solve $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$.

Attempt to extract the square root of the left member,

$$\begin{array}{r|l|l}
 & x^4 - 2x^3 - 7x^2 + 8x + 12 & \underline{x^2 - x - 4} \\
 & x^4 & \\
2x^2 - x & -2x^3 - 7x^2 & \\
 & \underline{-2x^3 + x^2} & \\
2x^2 - 2x - 4 & -8x^2 + 8x + 12 & \\
 & \underline{-8x^2 + 8x + 16} & \\
 & \qquad\qquad\quad -4 &
\end{array}$$

Add 4 to each member,

$$x^4 - 2x^3 - 7x^2 + 8x + 16 = 4.$$

Extract the square root,

$$x^2 - x - 4 = \pm 2.$$

$$x^2 - x - 6 = 0, \text{ or } x^2 - x - 2 = 0.$$

$$(x - 3)(x + 2) = 0, \text{ or } (x - 2)(x + 1) = 0.$$

$$\therefore x = 3, -2, 2, \text{ or } -1.$$

$$\therefore x = 3, \pm 2, \text{ or } -1.$$

23. Solve $32x^4 - 48x^3 - 10x^2 + 21x + 5 = 0$.

Multiply by 2,

$$64x^4 - 96x^3 - 20x^2 + 42x + 10 = 0.$$

Attempt to extract the square root of the left member,

$$\begin{array}{r|l|l}
 & 64x^4 - 96x^3 - 20x^2 + 42x + 10 & \underline{8x^2 - 6x - \frac{7}{2}} \\
 & 64x^4 & \\
\hline
16x^2 - 6x & -96x^3 - 20x^2 & \\
 & \underline{-96x^3 + 36x^2} & \\
16x^2 - 12x - \frac{7}{2} & -56x^2 + 42x + 10 & \\
 & \underline{-56x^2 + 42x + \frac{49}{4}} & \\
 & \qquad\qquad\qquad - \frac{9}{4} &
\end{array}$$

Add $\frac{9}{4}$ to each member,

$$64x^4 - 96x^3 - 20x^2 + 42x + \tfrac{49}{4} = \tfrac{9}{4}.$$

Extract the square root,

$$8x^2 - 6x - \tfrac{7}{2} = \pm \tfrac{3}{2}.$$

$$8x^2 - 6x = 5, \text{ or } 2.$$

$$64x^2 - (\ ) + 9 = 49, \text{ or } 25.$$

$$8x - 3 = \pm 7, \text{ or } \pm 5.$$

$$8x = 10,\ -4,\ 8, \text{ or } -2.$$

$$\therefore\ x = \tfrac{5}{4},\ -\tfrac{1}{2},\ 1, \text{ or } -\tfrac{1}{4}.$$

$$\therefore\ x = 1,\ \tfrac{5}{4},\ -\tfrac{1}{2}, \text{ or } -\tfrac{1}{4}.$$

24. Solve $x^4 - 4(a+b)x^2 + 16(a-b)^2 = 0$.

$$x^4 - 4(a+b)x^2 = -16(a-b)^2.$$

$$x^4 - 4(a+b)x^2 + 4(a+b)^2 = -12a^2 + 40ab - 12b^2.$$

$$x^2 - 2(a+b) = \pm 2\sqrt{-3a^2 + 10ab - 3b^2}.$$

$$x^2 = 2(a+b) \pm 2\sqrt{(3a-b)(3b-a)}.$$

$$\therefore\ x = \pm\sqrt{2(a+b) \pm 2\sqrt{(3a-b)(3b-a)}}$$

$$= \sqrt{3a-b} \pm \sqrt{3b-a}, \text{ or } -\sqrt{3a-b} \pm \sqrt{3b-a}.$$

The square root of $2(a+b) \pm 2\sqrt{(3a-b)(3b-a)}$ is found by § 307 as follows:

Two numbers whose sum is $2(a+b)$ and product $(3a-b)(3b-a)$ are $3a-b$ and $3b-a$.

Then $2(a+b) \pm 2\sqrt{(3a-b)(3b-a)}$

$$= (3a-b) \pm 2\sqrt{(3a-b)(3b-a)} + (3b-a)$$

$$= (\sqrt{3a-b} \pm \sqrt{3b-a})^2.$$

$$\therefore\ \sqrt{2(a+b) \pm 2\sqrt{(3a-b)(3b-a)}} = \sqrt{3a-b} \pm \sqrt{3b-a}.$$

25. Solve $x^4 - 2(a^2 + 4ab - b^2)x^2 + (a - b)^4 = 0$.

$$x^4 - 2(a^2 + 4ab - b^2)x^2 = -(a - b)^4.$$
$$x^4 - (\ ) + (a^2 + 4ab - b^2)^2 = 12a^3b + 8a^2b^2 - 4ab^3.$$
$$x^2 - (a^2 + 4ab - b^2) = \pm 2\sqrt{ab(3a^2 + 2ab - b^2)}.$$
$$x^2 = a^2 + 4ab - b^2 \pm 2\sqrt{ab(a + b)(3a - b)}.$$
$$x^2 = [\sqrt{a(a + b)} \pm \sqrt{b(3a - b)}]^2.$$
$$\therefore x = \pm[\sqrt{a(a + b)} \pm \sqrt{b(3a - b)}].$$
$$\therefore x = \sqrt{a(a + b)} \pm \sqrt{b(3a - b)}, \text{ or } -\sqrt{a(a + b)} \pm \sqrt{b(3a - b)}.$$

Exercise 118. Page 288

1. Form the equation of which the roots are 5 and 4.

The equation is

$$(x - 5)(x - 4) = 0,$$
or $$x^2 - 9x + 20 = 0.$$

2. Form the equation of which the roots are 6 and -3.

The equation is

$$(x - 6)(x + 3) = 0,$$
or $$x^2 - 3x - 18 = 0.$$

3. Form the equation of which the roots are -3 and -5.

The equation is

$$(x + 3)(x + 5) = 0,$$
or $$x^2 + 8x + 15 = 0.$$

4. Form the equation of which the roots are $\frac{5}{2}$ and $\frac{8}{3}$.

The equation is

$$(x - \tfrac{5}{2})(x - \tfrac{8}{3}) = 0,$$
or $$(2x - 5)(3x - 8) = 0,$$
or $$6x^2 - 31x + 40 = 0.$$

5. Form the equation of which the roots are $-1\frac{1}{3}$ and $2\frac{1}{3}$.

The equation is

$$(x + 1\tfrac{1}{3})(x - 2\tfrac{1}{3}) = 0,$$
or $$(3x + 4)(3x - 7) = 0,$$
or $$9x^2 - 9x - 28 = 0.$$

6. Form the equation of which the roots are $-2\frac{2}{3}$ and $-2\frac{3}{4}$.

The equation is

$$(x + 2\tfrac{2}{3})(x + 2\tfrac{3}{4}) = 0,$$
or $$(3x + 8)(4x + 11) = 0,$$
or $$12x^2 + 65x + 88 = 0.$$

7. Form the equation of which the roots are $\frac{2}{9}$ and $-\frac{4}{9}$.

The equation is

$$(x - \tfrac{2}{9})(x + \tfrac{4}{9}) = 0,$$
or $$(9x - 2)(9x + 4) = 0,$$
or $$81x^2 + 18x - 8 = 0.$$

8. Form the equation of which the roots are 8 and $-\frac{5}{3}$.

The equation is

$$(x - 8)(x + \tfrac{5}{3}) = 0,$$
or $$(x - 8)(3x + 5) = 0,$$
or $$3x^2 - 19x - 40 = 0.$$

9. Form the equation of which the roots are $\frac{7}{11}$ and $-\frac{3}{11}$.

The equation is

$$(x - \tfrac{7}{11})(x + \tfrac{3}{11}) = 0,$$

or $(11x - 7)(11x + 3) = 0,$

or $121x^2 - 44x - 21 = 0.$

10. Form the equation of which the roots are 2, 4, and -3.

The equation is

$$(x - 2)(x - 4)(x + 3) = 0,$$

or $x^3 - 3x^2 - 10x + 24 = 0.$

11. Form the equation of which the roots are 5, -2, and -1.

The equation is

$$(x - 5)(x + 2)(x + 1) = 0,$$

or $x^3 - 2x^2 - 13x - 10 = 0.$

12. Form the equation of which the roots are $-\frac{1}{2}$, $-\frac{2}{3}$, and $-\frac{3}{4}$.

The equation is

$$(x + \tfrac{1}{2})(x + \tfrac{2}{3})(x + \tfrac{3}{4}) = 0,$$

or $(2x + 1)(3x + 2)(4x + 3) = 0,$

or $24x^3 + 46x^2 + 29x + 6 = 0.$

13. Form the equation of which the roots are $a + b$ and $a - b$.

The equation is

$$[x - (a + b)][x - (a - b)] = 0,$$

or $(x - a - b)(x - a + b) = 0,$

or $x^2 - 2ax + a^2 - b^2 = 0.$

14. Form the equation of which the roots are $a + 2b$ and $a - 3b$.

The equation is

$$[x - (a + 2b)][x - (a - 3b)] = 0,$$

or $(x - a - 2b)(x - a + 3b) = 0,$

or $x^2 - 2ax + bx + a^2 - ab - 6b^2 = 0.$

15. Form the equation of which the roots are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

The equation is $[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})] = 0,$

or $(x - 2 - \sqrt{3})(x - 2 + \sqrt{3}) = 0,$

or $x^2 - 4x + 1 = 0.$

16. Form the equation of which the roots are $\frac{8}{3} + \frac{2}{3}\sqrt{37}$ and $\frac{8}{3} - \frac{2}{3}\sqrt{37}$.

The equation is $[x - (\frac{8}{3} + \frac{2}{3}\sqrt{37})][x - (\frac{8}{3} - \frac{2}{3}\sqrt{37})] = 0,$

or $(3x - 8 - 2\sqrt{37})(3x - 8 + 2\sqrt{37}) = 0,$

or $9x^2 - 48x - 84 = 0,$

or $3x^2 - 16x - 28 = 0.$

17. Form the equation of which the roots are $2 + \sqrt{-1}$ and $2 - \sqrt{-1}$.

The equation is $[x - (2 + \sqrt{-1})][x - (2 - \sqrt{-1})] = 0,$

or $(x - 2 - \sqrt{-1})(x - 2 + \sqrt{-1}) = 0,$

or $x^2 - 4x + 5 = 0.$

18. Form the equation of which the roots are $-2 + \sqrt{-3}$ and $-2 - \sqrt{-3}$.

The equation is $[x - (-2 + \sqrt{-3})][x - (-2 - \sqrt{-3})] = 0,$

or $(x + 2 - \sqrt{-3})(x + 2 + \sqrt{-3}) = 0,$

or $x^2 + 4x + 7 = 0.$

Exercise 119. Page 291

1. In (2) Law 1 find t in terms of the other letters.

We have, by (2), Law 1,

$$s = \tfrac{1}{2}gt^2.$$
$$\therefore\ gt^2 = 2s.$$
$$t^2 = \frac{2s}{g}.$$
$$\therefore\ t = \sqrt{\frac{2s}{g}}.$$

2. In (3) Law 1 find v in terms of the other letters.

We have, by (3), Law 1,

$$v^2 = 2gs.$$
$$\therefore\ v = \sqrt{2gs}.$$

3. In Law 2, if h denotes the height and l the length of the inclined plane, find the three formulas that correspond to (1), (2), and (3) of Law 1. Find the values of t and the value of v in terms of g, h, l, and s.

We have, by Law 1,

$$v = gt; \qquad (1)$$
$$s = \tfrac{1}{2}gt^2; \qquad (2)$$
$$v^2 = 2gs. \qquad (3)$$

For motion down an inclined plane these formulas become respectively

$$v = \frac{hgt}{l}; \qquad (4)$$
$$s = \frac{hgt^2}{2l}; \qquad (5)$$
$$v^2 = \frac{2hgs}{l}. \qquad (6)$$

From (4), $hgt = lv$.

$$\therefore\ t = \frac{lv}{hg}.$$

From (5), $hgt^2 = 2ls$.

$$t^2 = \frac{2ls}{hg}.$$
$$\therefore\ t = \sqrt{\frac{2ls}{hg}}.$$

From (6), $v = \sqrt{\dfrac{2hgs}{l}}$.

4. In Law 3 find the values of t in terms of the other letters. Find the value of u in (3).

We have, by (1), (i), Law 3,

$$v = u + gt.$$
$$gt = v - u.$$
$$\therefore\ t = \frac{v-u}{g}.$$

We have, by (2), (i), Law 3,

$$s = ut + \tfrac{1}{2}gt^2.$$
$$\tfrac{1}{2}gt^2 + ut = s.$$
$$gt^2 + 2ut = 2s.$$
$$g^2t^2 + (\ ) + u^2 = 2gs + u^2.$$
$$gt + u = \sqrt{2gs + u^2}.$$
$$gt = -u + \sqrt{2gs + u^2}.$$
$$\therefore\ t = \frac{-u + \sqrt{2gs + u^2}}{g}.$$

We have, by (1), (ii), Law 3,

$$v = u - gt.$$
$$gt = u - v.$$
$$\therefore\ t = \frac{u-v}{g}.$$

We have, by (2), (ii), Law 3,

$$s = ut - \tfrac{1}{2}gt^2.$$
$$\tfrac{1}{2}gt^2 - ut = -s.$$
$$gt^2 - 2ut = -2s.$$
$$g^2t^2 - (\ ) + u^2 = u^2 - 2gs.$$
$$gt - u = \sqrt{u^2 - 2gs}.$$
$$gt = u + \sqrt{u^2 - 2gs}.$$
$$\therefore t = \frac{u + \sqrt{u^2 - 2gs}}{g}.$$

We have, by (3), (ii), Law 3,

$$s = \frac{u^2}{2g}.$$
$$u^2 = 2gs.$$
$$\therefore u = \sqrt{2gs}.$$

5. In Law 5 find the value of t in terms of a and s. Find the value of v in terms of a and s.

We have, by (2), Law 5,

$$s = \tfrac{1}{2}at^2.$$
$$at^2 = 2s.$$
$$t^2 = \frac{2s}{a}.$$
$$\therefore t = \sqrt{\frac{2s}{a}}.$$

We have, by (3), Law 5,

$$v^2 = 2as.$$
$$\therefore v = \sqrt{2as}.$$

6. Find in terms of the other letters the value of l and the value of g in Law 4.

We have, by Law 4,

$$t = \pi\sqrt{\frac{l}{g}}.$$
$$t^2 = \pi^2\frac{l}{g}.$$
$$gt^2 = \pi^2 l.$$
$$\therefore l = \frac{gt^2}{\pi^2},$$

and $$g = \frac{\pi^2 l}{t^2}.$$

7. Find in terms of the other letters the value of v and the value of t in Law 6.

We have, by (1), Law 6,

$$F = \frac{mv^2}{r}.$$
$$rF = mv^2.$$
$$v^2 = \frac{rF}{m}.$$
$$\therefore v = \sqrt{\frac{rF}{m}}.$$

We have, by (2), Law 6,

$$F = \frac{4\pi^2 mr}{t^2}.$$
$$Ft^2 = 4\pi^2 mr.$$
$$t^2 = \frac{4\pi^2 mr}{F}.$$
$$\therefore t = 2\pi\sqrt{\frac{mr}{F}}.$$

8. Find in terms of the other letters the value of d in Law 7.

We have, by Law 7,

$$F = \frac{kmm'}{d^2}.$$
$$Fd^2 = kmm'.$$
$$d^2 = \frac{kmm'}{F}.$$
$$\therefore d = \sqrt{\frac{kmm'}{F}}.$$

9. Find in terms of the other letters the values of v in Law 8.

We have, by (1), Law 8,

$$Ft = mv.$$
$$\therefore\ v = \frac{Ft}{m}.$$

We have, by (2), Law 8,

$$s = \tfrac{1}{2}vt.$$
$$2s = vt.$$
$$\therefore\ v = \frac{2s}{t}.$$

We have, by (3), Law 8,

$$Fs = \tfrac{1}{2}mv^2.$$
$$2Fs = mv^2.$$
$$v^2 = \frac{2Fs}{m}.$$
$$\therefore\ v = \sqrt{\frac{2Fs}{m}}.$$

10. Find in terms of the other letters the value of v in Law 9.

We have, by Law 9,

$$mh = \frac{mv^2}{2g}.$$
$$2gmh = mv^2.$$
$$v^2 = 2gh.$$
$$\therefore\ v = \sqrt{2gh}.$$

11. How long will it take a stone starting from rest to fall 1600 feet? What velocity will the stone acquire? What is the mean velocity?

By Formula (2), 1, § 350,

$$s = \tfrac{1}{2}gt^2.$$

Here $s = 1600$, and $g = 32$.

$$\therefore\ 1600 = \tfrac{1}{2} \times 32t^2.$$
$$16t^2 = 1600.$$
$$t^2 = 100.$$
$$\therefore\ t = 10.$$

By Formula (1),

$$v = gt$$
$$= 32 \times 10$$
$$= 320.$$

The mean velocity

$$= \tfrac{1}{2} \text{ of } 320 = 160.$$

Therefore, the time required is 10 seconds; the velocity acquired is 320 feet per second; the mean velocity is 160 feet per second.

12. A stone is thrown vertically upwards with a velocity of 160 feet per second. How high will it rise? How many seconds will it be in the air?

By Formula (3), (ii), 3, § 350,

$$s = \frac{u^2}{2g}.$$

Here $u = 160$, and $g = 32$.

$$\therefore\ s = \frac{160^2}{2 \times 32} = 400.$$

By Formula (2),

$$s = ut - \tfrac{1}{2}gt^2.$$

Here $s = 400$, $u = 160$, $g = 32$.

$$\therefore\ 400 = 160t - 16t^2.$$

Divide by 16,

$$25 = 10t - t^2.$$
$$t^2 - 10t + 25 = 0.$$
$$t - 5 = 0.$$
$$\therefore\ t = 5.$$

It will take the stone just as long to fall from the highest point as it takes to reach the highest point.

$$\therefore\ 2t = 10.$$

Therefore, the stone will rise 400 feet, and will be in the air 10 seconds.

13. With what initial velocity must a bullet be fired upwards that it may rise to a height of 6400 feet?

By Formula (3), (ii), 3, § 350,

$$s = \frac{u^2}{2g}.$$

Here $s = 6400$, and $g = 32$.

$$\therefore\ 6400 = \frac{u^2}{2 \times 32}.$$

$$u^2 = 64 \times 6400.$$

$$\therefore\ u = 8 \times 80 = 640.$$

Therefore, the required initial velocity is 640 feet per second.

14. A stone is thrown down a shaft 164 feet deep with an initial velocity of 50 feet per second. How many seconds will it take to reach the bottom?

By Formula (2), (i), 3, § 350,

$$s = ut + \tfrac{1}{2}gt^2.$$

Here $s = 164$, $u = 50$, $g = 32$.

$$\therefore\ 164 = 50t + \tfrac{1}{2} \times 32t^2.$$

$$16t^2 + 50t = 164.$$

$$256t^2 + (\ ) + 625 = 3249.$$

$$16t + 25 = \pm 57.$$

$$16t = 32, \text{ or } -82.$$

$$\therefore\ t = 2.$$

Therefore, the required time is 2 seconds.

15. A stone is dropped into a vertical shaft, and the sound of its striking the bottom is heard after 10 seconds. If the velocity of sound is 1120 feet per second, how deep is the shaft?

Let $t =$ the number of seconds the stone is falling.

Then $10 - t =$ the number of seconds required for the sound to travel a distance equal to the depth of the shaft,

and $1120(10 - t) =$ the number of feet in the depth of the shaft.

$$\therefore\ s = 1120(10 - t). \qquad (1)$$

By Formula (2), 1, § 350, $s = \tfrac{1}{2} \times 32t^2.$ (2)

Substitute in (2) the value of s from (1),

$$1120(10 - t) = \tfrac{1}{2} \times 32t^2.$$

$$11200 - 1120t = 16t^2.$$

Divide by 16, $t^2 + 70t = 700.$

$$t^2 + 70t + 1225 = 1925.$$

$$t + 35 = \pm 43.875.$$

$$\therefore\ t = 8.875, \text{ or } -78.875.$$

$$\begin{aligned} s &= 1120(10 - t) \\ &= 1120(10 - 8.875) \\ &= 1120 \times 1.125 \\ &= 1260. \end{aligned}$$

Therefore, the shaft is 1260 feet deep.

16. A stone is dropped into a vertical shaft 1024 feet deep, and the sound of its striking the bottom is heard after 8.9 seconds. Find the velocity of sound.

Let $x =$ the number of feet per second in the velocity of sound.

Then $t =$ the number of seconds required for the stone to reach the bottom,

and $8.9 - t =$ the number of seconds it takes sound to travel 1024 feet.

$$\therefore x = \frac{1024}{8.9 - t}.$$

By Formula (2), 1, § 350, $s = \frac{1}{2}gt^2.$

Here $s = 1024$, and $g = 32.$

$$\therefore 1024 = \tfrac{1}{2} \times 32t^2.$$

$$1024 = 16t^2.$$

$$t^2 = 64.$$

$$\therefore t = 8.$$

$$\therefore x = \frac{1024}{8.9 - 8} = \frac{1024}{0.9} = 1137\tfrac{7}{9}.$$

Therefore, the velocity of sound is $1137\frac{7}{9}$ feet per second.

17. When a balloon is just a mile above the ground and is rising at the rate of 22 feet per second, a stone is thrown vertically downwards with an initial velocity of 50 feet per second. How long will it take the stone to reach the ground?

By Formula (2), (i), 3, § 350, $s = ut + \frac{1}{2}gt^2.$

If the balloon was stationary, the value of u would be 50 feet; but since the balloon is *rising* at the rate of 22 feet per second, the value of u is 50 feet − 22 feet, or 28 feet.

Hence, $s = 5280$, $u = 28$, and $g = 32.$

$$\therefore 5280 = 28t + \tfrac{1}{2} \times 32t^2.$$

$$5280 = 28t + 16t^2.$$

$$16t^2 + 28t = 5280.$$

$$64t^2 + (\ ) + 49 = 21169.$$

$$8t + 7 = \pm 145.495.$$

$$8t = 138.495, \text{ or } -152.495.$$

$$\therefore t = 17.31.$$

Therefore, it will take the stone 17.31 seconds to reach the ground.

18. If the stone of the last example was thrown vertically *upwards*, how long would it take to reach the ground?

By Formula (3), (ii), 3, § 350, $s = \frac{u^2}{2g}$.

If the balloon was stationary, the value of u would be 50 feet; but since the balloon is *rising* at the rate of 22 feet per second, the value of u is 50 feet + 22 feet, or 72 feet.

Hence, $u = 72$, and $g = 32$.

$$\therefore\ s = \frac{72^2}{2 \times 32} = 81.$$

By Formula (2), (ii), 3, § 350, $s = ut - \frac{1}{2}gt^2$.

Here $s = 81$, $u = 72$, and $g = 32$.

$$\therefore\ 81 = 72t - \tfrac{1}{2} \times 32t^2.$$

$$16t^2 - 72t + 81 = 0.$$

Extract the square root, $4t - 9 = 0$.

$$4t = 9.$$

$$\therefore\ t = 2.25.$$

Hence, the time it takes the stone to rise to the highest point (81 feet above a mile) is 2.25 seconds.

It is now required to find the time it takes the stone to fall to the ground, or a distance of 81 feet + 5280 feet = 5361 feet.

By Formula (2), 1, § 350, $s = \frac{1}{2}gt^2$.

Here $s = 5361$, and $g = 32$.

$$\therefore\ 5361 = \tfrac{1}{2} \times 32t^2.$$

$$16t^2 = 5361.$$

$$4t = \pm 73.218.$$

$$\therefore\ t = 18.30.$$

Hence, it requires the stone 18.30 seconds to fall to the ground from the highest point it reaches.

Therefore, it takes the stone 2.25 seconds + 18.30 seconds, or 20.55 seconds, to reach the ground.

19. One body is allowed to slide down a smooth inclined plane 400 feet long and 144 feet high. Another body is allowed to fall vertically through the height of the plane. Find the velocity of each body or reaching the base of the plane, and the time required for each to fall.

The body allowed to fall vertically.

By Formula (3), 1, § 350, $v^2 = 2gs$.

Here $g = 32$, and $s = 144$.

$$\therefore\ v^2 = 2 \times 32 \times 144 = 64 \times 144.$$

$$\therefore\ v = \sqrt{64 \times 144} = 8 \times 12 = 96.$$

By Formula (1), 1, § 350, $v = gt.$

Here $v = 96$, and $g = 32.$

$$\therefore 96 = 32t.$$

$$\therefore t = 3.$$

The body allowed to slide down the inclined plane.

By 2, § 350 and by Formula (3), 1, § 350,

$$v^2 = \frac{2ghs}{l}.$$

Here $g = 32,\ h = 144,\ l = 400,\ s = 400.$

$$\therefore v^2 = 2 \times 32 \times \tfrac{144}{400} \times 400 = 64 \times 144.$$

$$\therefore v = \sqrt{64 \times 144} = 8 \times 12 = 96.$$

By 2, § 350, and by Formula (1), 1, § 350,

$$v = \frac{ght}{l}.$$

Here $v = 96,\ g = 32,\ h = 144,\ l = 400.$

$$\therefore 96 = 32 \times \tfrac{144}{400} \times t.$$

$$t = \frac{400 \times 96}{144 \times 32} = \frac{25}{3}.$$

$$\therefore t = 8\tfrac{1}{3}.$$

Therefore, the velocity of the falling body on reaching the base of the plane is 96 feet per second, and the velocity of the sliding body is 96 feet per second; the time required for the falling body is 3 seconds, and the time required for the sliding body is $8\frac{1}{3}$ seconds.

Exercise 120. Page 295

1. The difference between two numbers is 12, and the sum of the squares of the numbers is 1130. Find the numbers.

Let $x =$ the smaller number.

Then $x + 12 =$ the larger number.

$$\therefore x^2 + (x + 12)^2 = 1130.$$

$$x^2 + x^2 + 24x + 144 = 1130.$$

$$2x^2 + 24x = 986.$$

$$x^2 + 12x = 493.$$

$$x^2 + 12x + 36 = 529.$$

$$x + 6 = \pm 23.$$

$$\therefore x = 17, \text{ or } -29.$$

Then $x + 12 = 29$, or -17.

Therefore, the required numbers are 17 and 29, or -29 and -17.

2. Two men, A and B, can together do a piece of work in 20 days. If B requires 9 more days than A to do the whole work, how many days would it take each alone?

Let $x =$ the number of days it would take A alone.

Then $x + 9 =$ the number of days it would take B alone,

$\frac{1}{x} =$ the part A can do in 1 day,

and $\frac{1}{x+9} =$ the part B can do in 1 day.

$$\therefore \frac{20}{x} + \frac{20}{x+9} = 1.$$

$$20x + 180 + 20x = x^2 + 9x.$$

$$x^2 - 31x - 180 = 0.$$

$$(x - 36)(x + 5) = 0.$$

$$\therefore x = 36, \text{ or } -5.$$

Then $x + 9 = 45$, or 4.

Therefore, to do the whole work it would take A alone 36 days, and B alone 45 days.

3. The length of a rectangular lot is 19 feet longer than the width. If the width was 40 feet greater and the length was 33 feet greater, the area of the lot would be doubled. Find the dimensions of the lot.

Let $x =$ the number of feet in the width of the lot.

Then $x + 19 =$ the number of feet in the length of the lot,

$x + 40 =$ the number of feet in the width increased by 40 feet,

and $x + 52 =$ the number of feet in the length increased by 33 feet.

$$\therefore (x + 40)(x + 52) = 2(x)(x + 19).$$

$$x^2 + 92x + 2080 = 2x^2 + 38x.$$

$$x^2 - 54x = 2080.$$

$$x^2 - 54x + 729 = 2809.$$

$$x - 27 = \pm 53.$$

$$\therefore x = 80, \text{ or } -26.$$

Then $x + 19 = 99$, or -7.

Therefore, the length of the lot is 99 feet, and the width 80 feet.

4. If 11 is subtracted from a certain whole number, the result is the square of an integral number; if 24 is added to the same whole number, the result is the square of the integral number one larger than before. Find the number.

Let $x =$ the required number.

Then $x - 11 =$ the square of an integral number,

$x + 24 =$ the square of the next larger integral number,

$\sqrt{x - 11} =$ an integral number,

and $\sqrt{x + 24} =$ the next larger integral number.

But the difference between these integral numbers is 1.

$$\therefore\ \sqrt{x + 24} - \sqrt{x - 11} = 1.$$

$$\sqrt{x + 24} = 1 + \sqrt{x - 11}.$$

Square, $$x + 24 = 1 + 2\sqrt{x - 11} + x - 11.$$

$$34 = 2\sqrt{x - 11}.$$

$$17 = \sqrt{x - 11}.$$

Square, $$289 = x - 11.$$

$$\therefore\ x = 300.$$

Therefore, the required number is 300.

5. A company of gentlemen engaged a supper and agreed to pay \$80 for it. Four of the gentlemen failed to attend and each of the rest paid \$1 more than he expected to pay. How many were present at the supper?

Let $x =$ the number of gentlemen who agreed to attend.

Then $x - 4 =$ the number of gentlemen who did attend,

$\frac{80}{x} =$ the number of dollars each expected to pay,

$\frac{80}{x} + 1 =$ the number of dollars each did pay,

and $(x - 4)\left(\frac{80}{x} + 1\right) =$ the number of dollars paid for the supper.

$$\therefore\ (x - 4)\left(\frac{80}{x} + 1\right) = 80.$$

Clear of fractions, $$(x - 4)(80 + x) = 80x.$$

$$x^2 + 76x - 320 = 80x.$$

$$x^2 - 4x - 320 = 0.$$

$$(x - 20)(x + 16) = 0.$$

$$\therefore\ x = 20, \text{ or } -16.$$

Then $$x - 4 = 16.$$

Therefore, 16 gentlemen were present at the supper.

6. If the edges of a rectangular box were increased by 2 inches, 3 inches, and 4 inches respectively, the box would become a cube and its contents would be increased by 1008 cubic inches. Find the length of each edge of the box.

Let $x =$ the number of inches in an edge of the cube.
Then $x - 2 =$ the number of inches in one edge of the box,
$x - 3 =$ the number of inches in a second edge of the box,
$x - 4 =$ the number of inches in the third edge of the box,
$(x - 2)(x - 3)(x - 4) =$ the number of cubic inches in the contents of the box,
and $x^3 =$ the number of cubic inches in the contents of the cube.

$$\therefore\ (x - 2)(x - 3)(x - 4) = x^3 - 1008.$$

$$x^3 - 9x^2 + 26x - 24 = x^3 - 1008.$$

Transpose and combine, $9x^2 - 26x = 984.$

$$81x^2 - (\ ) + 169 = 9025.$$

$$9x - 13 = \pm 95.$$

$$9x = 108, \text{ or } -82.$$

$$\therefore\ x = 12.$$

Then $x - 2 = 10,$
$x - 3 = 9,$
$x - 4 = 8.$

Therefore, the lengths of the edges of the box are 10 inches, 9 inches, and 8 inches.

7. A railway train makes a run of 799 miles in a certain time. If the average speed was reduced $4\frac{1}{2}$ miles an hour, the run would take 1 hour 48 minutes longer. How long does it take the train to make the run?

Let $x =$ the number of hours the train requires for the run.
Then $\frac{799}{x} =$ the average number of miles the train makes per hour,
$\frac{799}{x} - \frac{9}{2} =$ the average number of miles the train would make per hour if the speed was reduced $4\frac{1}{2}$ miles per hour,
and $\dfrac{799}{\frac{799}{x} - \frac{9}{2}} =$ the number of hours the train would require for the run if the speed was reduced $4\frac{1}{2}$ miles per hour.

$$\therefore \frac{799}{\frac{799}{x}-\frac{9}{2}} - x = 1\tfrac{4}{5}.$$

Simplify fractions, $$\frac{1598x}{1598-9x} - x = \frac{9}{5}.$$

$$7990x - 7990x + 45x^2 = 14382 - 81x.$$

$$45x^2 + 81x = 14382.$$

$$5x^2 + 9x = 1598.$$

$$100x^2 + (\ ) + 81 = 32041.$$

$$10x + 9 = \pm 179.$$

$$10x = 170, \text{ or } -188.$$

$$\therefore x = 17.$$

Therefore, it takes the train 17 hours to make the run.

8. A body of soldiers can form a hollow square 4 men deep. If the outer side of the hollow square is diminished by 36 men, the soldiers can form a solid square. Find the number of soldiers.

Let $x =$ the number of soldiers in one side of the solid square.

Then $x^2 =$ the whole number of soldiers.

In the hollow square there are 8 rows of $x + 36$ men each and 8 rows of $x + 36 - 8$ men each.

$$\therefore 8(x + 36) + 8(x + 36 - 8) = x^2.$$

$$8x + 288 + 8x + 288 - 64 = x^2.$$

$$x^2 - 16x = 512.$$

$$x^2 - 16x + 64 = 576.$$

$$x - 8 = \pm 24.$$

$$\therefore x = 32, \text{ or } -16.$$

Then $x^2 = 1024.$

Therefore, the number of soldiers is 1024.

9. The difference between the cubes of two consecutive numbers is 1519. Find the numbers.

Let $x =$ the smaller number.

Then $x + 1 =$ the larger number.

$$\therefore (x + 1)^3 - x^3 = 1519.$$

$$x^3 + 3x^2 + 3x + 1 - x^3 = 1519.$$

$$3x^2 + 3x = 1518.$$

$$x^2 + x = 506.$$

$$4x^2 + (\ ) + 1 = 2025.$$

$$2x + 1 = \pm 45.$$
$$2x = 44, \text{ or } -46.$$
$$x = 22.$$

Then $x + 1 = 23.$

Therefore, the required numbers are 22 and 23.

10. Find the momentum and the kinetic energy of a mass of 5 pounds as it strikes the ground after falling 900 feet.

$$\text{The momentum} = \text{mass} \times \text{velocity}.$$

By Formula (3), 1, § 350, $v^2 = 2gs.$

Here $g = 32, \text{ and } s = 900.$

$$\therefore v^2 = 2 \times 32 \times 900 = 64 \times 900.$$
$$\therefore v = 8 \times 30 = 240.$$
$$\text{Momentum} = 5 \times 240, \text{ or } 1200, \text{ poundals}.$$

By 8, § 350,
$$\text{Kinetic energy} = \frac{mv^2}{2g} \text{ foot-pounds}$$
$$= \frac{5 \times 240^2}{2 \times 32} \text{ foot-pounds}$$
$$= 4500 \text{ foot-pounds}.$$

Therefore, the momentum is 1200 poundals, and the kinetic energy 4500 foot-pounds.

11. A mass of 10 pounds falls from rest 225 feet to the ground. Find its kinetic energy on reaching the ground.

By Formula (3), 1, § 350, $v^2 = 2gs.$

Here $g = 32, \text{ and } s = 225.$

$$\therefore v^2 = 2 \times 32 \times 225 = 64 \times 225.$$
$$\therefore v = 8 \times 15 = 120.$$

By 8, § 350,
$$\text{Kinetic energy} = \frac{mv^2}{2g} \text{ foot-pounds}$$
$$= \frac{10 \times 120^2}{2 \times 32} \text{ foot-pounds}$$
$$= 2250 \text{ foot-pounds}.$$

Therefore, the kinetic energy required is 2250 foot-pounds.

12. A ball weighing 2 pounds is fired vertically upwards with an initial velocity of 128 feet a second. Compute its kinetic energy at the end of each second of its ascent.

The ball rises until its velocity is 0.

By Formula (1), (ii), 3, § 350, $v = u - gt.$

When the velocity is 0, then $v = 0,\ u = 128,\ g = 32.$

$$\therefore\ 0 = 128 - 32t.$$

$$32t = 128.$$

$$\therefore\ t = 4.$$

Therefore, the ball rises for 4 seconds.

$$v = u - gt.$$

When $t = 1,\ v = 128 - 32 \times 1 = 128 - 32 = 96;$

when $t = 2,\ v = 128 - 32 \times 2 = 128 - 64 = 64;$

when $t = 3,\ v = 128 - 32 \times 3 = 128 - 96 = 32;$

when $t = 4,\ v = 128 - 32 \times 4 = 128 - 128 = 0.$

By 8, § 350,

when $t = 1$, kinetic energy $= \dfrac{2 \times 96^2}{2 \times 32} = 288$ foot-pounds;

when $t = 2$, kinetic energy $= \dfrac{2 \times 64^2}{2 \times 32} = 128$ foot-pounds;

when $t = 3$, kinetic energy $= \dfrac{2 \times 32^2}{2 \times 32} = 32$ foot-pounds;

when $t = 4$, kinetic energy $= \dfrac{2 \times 0^2}{2 \times 32} = 0$ foot-pounds.

Therefore, the kinetic energy at the end of 1 second is 288 foot-pounds; at the end of 2 seconds is 128 foot-pounds; at the end of 3 seconds is 32 foot-pounds; at the end of 4 seconds is 0 foot-pounds.

13. Find the effective horse power of a steam engine that raises 100 cubic feet of water per minute to a height of 132 feet. A cubic foot of water weighs $62\frac{1}{2}$ pounds.

$$\text{The horse power} = \frac{100 \times 62\frac{1}{2} \times 132}{33000}$$

$$= \frac{\cancel{100} \times \overset{25}{\cancel{125}} \times \overset{\cancel{66}}{\cancel{132}}}{\cancel{2} \times \underset{\underset{\cancel{5}}{\cancel{330}}}{\cancel{33000}}} = 25.$$

Therefore, the effective horse power of the engine is 25.

14. Find the time of vibration of a simple pendulum 1.63 feet long at a place where $g = 32.16$ feet.

By 4, § 350, $$t = \pi\sqrt{\frac{l}{g}}.$$

Here $$\pi = 3.1416,\ l = 1.63,\ g = 32.16.$$

$$\begin{aligned} \therefore\ t &= 3.1416 \times \sqrt{\frac{1.63}{32.16}} \\ &= 3.1416 \times \sqrt{0.050684} \\ &= 3.1416 \times 0.225 \\ &= 0.70686. \end{aligned}$$

Therefore, the required time of vibration is 0.707 second.

15. Find the time of vibration of a simple pendulum 60 centimeters long at a place where $g = 980$ centimeters.

By 4, § 350, $$t = \pi\sqrt{\frac{l}{g}}.$$

Here $$\pi = 3.1416,\ l = 60,\ g = 980.$$

$$\begin{aligned} \therefore\ t &= 3.1416 \times \sqrt{\tfrac{60}{980}} \\ &= 3.1416 \times \sqrt{\tfrac{3}{49}} \\ &= 3.1416 \times \tfrac{1}{7}\sqrt{3} \\ &= 0.4488 \times 1.7321 \\ &= 0.77736648. \end{aligned}$$

Therefore, the required time of vibration is 0.777 second.

16. Find the length in meters of a simple pendulum that vibrates once a second at a place where $g = 9.8$ meters.

By 4, § 350, $$t = \pi\sqrt{\frac{l}{g}}.$$

$$t^2 = \frac{\pi^2 l}{g}.$$

$$\therefore\ l = \frac{1}{\pi^2} \times gt^2.$$

Here $$g = 9.8,\ t = 1,\ \pi^2 = 3.1416^2 = 9.86965,$$

and $$\frac{1}{\pi^2} = \frac{1}{9.86965} = 0.10132.$$

$$\therefore\ l = 0.10132 \times 9.8 = 0.992936.$$

Therefore, the length of the required pendulum is 0.9929 meter.

17. Find the length in inches of a simple pendulum that vibrates once a second at New York, where $g = 32.16$ feet.

By Example 16,
$$\begin{aligned} l &= \frac{1}{\pi^2} \times gt^2 \\ &= 0.10132 \times 32.16 \text{ feet} \\ &= 0.10132 \times 32.16 \times 12 \text{ inches} \\ &= 39.1014144 \text{ inches.} \end{aligned}$$

Therefore, the length of the required pendulum is 39.101 inches.

18. If an iron ball is suspended by a fine wire 550 feet 8 inches long from the top of Washington monument, what will be the time of vibration if $g = 32.16$ feet?

By 4, § 350,
$$t = \pi\sqrt{\frac{l}{g}}.$$

Here
$$\pi = 3.1416,\ l = 550\tfrac{2}{3},\ g = 32.16.$$
$$\begin{aligned} \therefore\ t &= 3.1416 \times \sqrt{\frac{550\frac{2}{3}}{32.16}} \\ &= 3.1416 \times \sqrt{17.1227} \\ &= 3.1416 \times 4.138 \\ &= 12.9999408. \end{aligned}$$

Therefore, the required time of vibration is 13 seconds.

19. Two masses of 48 grams and 50 grams are attached by a cord passing over a pulley. Starting from rest, each mass moves 10 centimeters in 1 second. Find g in centimeters.

The moving force $= 50$ grams $- 48$ grams $= 2$ grams $= 2g$ dynes.

By 5, § 350,
$$a = \frac{2g}{50+48} = \frac{2g}{98}.$$

By Formula (2), 5, § 350,
$$s = \tfrac{1}{2}at^2.$$

Here
$$s = 10,\ a = \frac{2g}{98},\ t = 1.$$
$$\therefore\ 10 = \frac{1}{2} \times \frac{2g}{98} \times 1^2.$$
$$10 = \frac{g}{98}.$$
$$\therefore\ g = 980.$$

Therefore, $g = 980$ centimeters each second.

20. How many times a minute must a mass of 16 pounds revolve horizontally at the end of a wire 4 feet long that the tension of the wire may equal the weight of 800 pounds?

By Formula (2), 6, § 350, $F = \frac{4\pi^2 mr}{t^2}.$

$$t^2 = \frac{4\pi^2 mr}{F}.$$

$$\therefore\ t = 2\pi\sqrt{\frac{mr}{F}}.$$

Here $\pi = 3.1416,\ m = 16,\ r = 4,\ F = 800,\ g = 25600.$

$$\begin{aligned} \therefore\ t &= 2 \times 3.1416 \times \sqrt{\frac{16 \times 4}{25600}} \\ &= 2 \times 3.1416 \times \sqrt{\tfrac{1}{400}} \\ &= 2 \times 3.1416 \times \tfrac{1}{20} \\ &= 0.31416. \end{aligned}$$

Therefore, the time of revolution is 0.31416 second.

Hence, the number of revolutions per minute is $\frac{60}{0.31416} = 191.$

Therefore, the mass must revolve 191 times per minute.

21. A weight of $1\frac{1}{2}$ tons drops 24 feet on the head of a pile and drives the pile a distance of 9 inches. Find the resistance of the ground, neglecting the weight of the pile.

Let $x =$ the number of tons in the resistance of the ground.

By Formula (3), 1, § 350, $v^2 = 2gs.$

Here $g = 32$, and $s = 24.$

$$\therefore\ v^2 = 2 \times 32 \times 24 = 1536.$$

$$\therefore\ v = \sqrt{1536} = 16\sqrt{6}.$$

Hence, the velocity of the weight is $16\sqrt{6}$ feet per second.

By Formula (3), 5, § 350, $v^2 = 2as.$

Here $v = 16\sqrt{6},\ a = \frac{gx}{1\frac{1}{2}} = \frac{32x}{1\frac{1}{2}} = \frac{64x}{3}.$

$$s = 9 \text{ inches} = \tfrac{3}{4} \text{ foot}.$$

$$\therefore\ (16\sqrt{6})^2 = 2 \times \frac{64x}{3} \times \frac{3}{4}.$$

$$1536 = 32x.$$

$$\therefore\ x = 48.$$

Therefore, the average resistance of the ground is 48 tons.

22. Find the horse power of the engine that raises the weight of Example 21 back to its original position in 3 seconds.

$$1\tfrac{1}{2} \text{ tons} = 1\tfrac{1}{2} \times 2000 \text{ pounds} = 3000 \text{ pounds}.$$

Raising the weight 24 feet in 3 seconds is equivalent to raising it 20 × 24 feet, or 480 feet, in 1 minute.

$$\text{Horse power} = \frac{3000 \times 480}{33000} = 43\tfrac{7}{11}.$$

Therefore, the required horse power of the engine is $43\tfrac{7}{11}$.

Exercise 121. Page 301

1. Solve

$$\left.\begin{aligned} x^2 + y^2 &= 40 \\ x &= 3y \end{aligned}\right\} \quad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Substitute in (1) the value of x from (2),

$$9y^2 + y^2 = 40.$$
$$10y^2 = 40.$$
$$y^2 = 4.$$
$$\therefore\ y = \pm 2.$$

Substitute the value of y in (2), $x = \pm 6.$

Therefore, $x = 6$, and $y = 2$;
or $x = -6$, and $y = -2$.

2. Solve

$$\left.\begin{aligned} 3x - y &= 5 \\ xy - x &= 0 \end{aligned}\right\} \quad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $y = 3x - 5.$ (3)

Substitute the value of y in (2), $3x^2 - 5x - x = 0.$

$$3x^2 - 6x = 0.$$
$$x^2 - 2x = 0.$$

Factor, $x(x - 2) = 0.$

$$\therefore\ x = 0, \text{ or } 2.$$

Substitute the value of x in (3), $y = -5, \text{ or } 1.$

Therefore, $x = 0$, and $y = -5$;
or $x = 2$, and $y = 1$.

3. Solve

$$\left.\begin{aligned} xy &= 12 \\ 2x + 3y &= 18 \end{aligned}\right\} \quad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (2), $2x = 18 - 3y.$

$$\therefore\ x = \frac{18 - 3y}{2}. \quad (3)$$

Substitute the value of x in (1), $\frac{18y - 3y^2}{2} = 12.$

$$18y - 3y^2 = 24.$$
$$6y - y^2 = 8.$$
$$y^2 - 6y + 8 = 0.$$

Factor, $(y-2)(y-4) = 0.$

$$\therefore y = 2, \text{ or } 4.$$

Substitute the value of y in (3), $x = 6, \text{ or } 3.$

Therefore, $x = 6$, and $y = 2$;
or $x = 3$, and $y = 4$.

4. Solve
$$\left.\begin{aligned} 5x^2 + y &= 3xy \\ 2x - y &= 0 \end{aligned}\right\} \quad \begin{aligned}(1)\\(2)\end{aligned}$$

Transpose (2), $y = 2x.$ (3)
Substitute the value of y in (1), $5x^2 + 2x = 6x^2.$
Transpose and combine, $x^2 - 2x = 0.$
Factor, $x(x-2) = 0.$

$$\therefore x = 0, \text{ or } 2.$$

Substitute the value of x in 3, $y = 0, \text{ or } 4.$

Therefore, $x = 0$, and $y = 0$;
or $x = 2$, and $y = 4$.

5. Solve
$$\left.\begin{aligned} x^2 - xy + y^2 &= 7 \\ 2x - 3y &= 0 \end{aligned}\right\} \quad \begin{aligned}(1)\\(2)\end{aligned}$$

Transpose (2), $2x = 3y.$

$$\therefore x = \frac{3y}{2}. \quad (3)$$

Substitute the value of x in (1),

$$\frac{9y^2}{4} - \frac{3y^2}{2} + y^2 = 7.$$

Multiply by 4, $9y^2 - 6y^2 + 4y^2 = 28.$
Combine, $7y^2 = 28.$

$$\therefore y^2 = 4.$$
$$\therefore y = \pm 2.$$

Substitute the value of y in (3), $x = \pm 3.$

Therefore, $x = 3$, and $y = 2$;
or $x = -3$, and $y = -2$.

6. Solve
$$\left.\begin{aligned} x^2 - xy - 2y^2 &= 7 \\ x - y &= 3 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (2), $x = y + 3.$ (3)

Substitute the value of x in (1),
$$y^2 + 6y + 9 - y^2 - 3y - 2y^2 = 7.$$

Transpose and combine, $2y^2 - 3y - 2 = 0.$

Factor, $(y-2)(2y+1) = 0.$

$\therefore y = 2, \text{ or } -\frac{1}{2}.$

Substitute the value of y in (3), $x = 5, \text{ or } \frac{5}{2}.$

Therefore, $x = 5$, and $y = 2$;

or $x = \frac{5}{2}$, and $y = -\frac{1}{2}$.

7. Solve
$$\left.\begin{aligned} 6x^2 - 13xy + 6y^2 &= 26 \\ x - 2y + 1 &= 0 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (2), $x = 2y - 1.$ (3)

Substitute the value of x in (1),
$$6(2y-1)^2 - 13(2y-1)y + 6y^2 = 26.$$

Simplify, $24y^2 - 24y + 6 - 26y^2 + 13y + 6y^2 = 26.$

Transpose and combine, $4y^2 - 11y = 20.$

Complete the square, $64y^2 - (\ ) + 121 = 441.$

Extract the root, $8y - 11 = \pm 21.$

$8y = 32, \text{ or } -10.$

$\therefore y = 4, \text{ or } -\frac{5}{4}.$

Substitute the value of y in (3), $x = 7, \text{ or } -\frac{7}{2}.$

Therefore, $x = 7$, and $y = 4$;

or $x = -\frac{7}{2}$, and $y = -\frac{5}{4}$.

8. Solve
$$\left.\begin{aligned} x + xy &= 35 \\ y + xy &= 32 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Subtract (2) from (1), $x - y = 3.$

Transpose, $x = y + 3.$ (3)

Substitute the value of x in (2), $y + y^2 + 3y = 32.$

Transpose and combine, $y^2 + 4y - 32 = 0.$

Factor, $(y-4)(y+8) = 0.$

$\therefore y = 4, \text{ or } -8.$

Substitute the value of y in (3), $x = 7, \text{ or } -5.$

Therefore, $x = 7$, and $y = 4$;

or $x = -5$, and $y = -8$.

9. Solve $\left.\begin{aligned} 2x^2 - 3xy + 5y &= 5 \\ (x-2)(y-1) &= 0 \end{aligned}\right\}$ (1) (2)

From (2), $x - 2 = 0$, or $y - 1 = 0$.

$\therefore x = 2$, or $y = 1$. (3)

Substitute the value of x in (1), $8 - 6y + 5y - 5 = 0$.

Transpose and combine, $y = 3$.

Substitute in (1) the value of y from (3),

$$2x^2 - 3x + 5 - 5 = 0.$$

Combine, $2x^2 - 3x = 0$.

Factor, $x(2x - 3) = 0$.

$\therefore x = 0$, or $\frac{3}{2}$.

Therefore, $x = 2$, and $y = 3$;
$x = 0$, and $y = 1$;
or $x = \frac{3}{2}$, and $y = 1$.

10. Solve $\left.\begin{aligned} x^3 - y^3 &= 152 \\ x - y &= 2 \end{aligned}\right\}$ (1) (2)

Divide (1) by (2), $x^2 + xy + y^2 = 76$. (3)

Square (2), $x^2 - 2xy + y^2 = 4$. (4)

Subtract (4) from (3), $3xy = 72$.

$xy = 24$. (5)

Add (3) and (5), $x^2 + 2xy + y^2 = 100$.

$x + y = \pm 10$. (6)

Add (2) and (6), $2x = 12$, or -8.

$\therefore x = 6$, or -4.

Substitute the value of x in (5), $y = 4$, or -6.

Therefore, $x = 6$, and $y = 4$;
or $x = -4$, and $y = -6$.

11. Solve $\left.\begin{aligned} x^3 + y^3 &= 152 \\ x + y &= 8 \end{aligned}\right\}$ (1) (2)

Divide (1) by (2), $x^2 - xy + y^2 = 19$. (3)

Square (2), $x^2 + 2xy + y^2 = 64$. (4)

Subtract (3) from (4), $3xy = 45$.

$xy = 15$. (5)

Subtract (5) from (3), $x^2 - 2xy + y^2 = 4$.

$x - y = \pm 2$. (6)

Add (2) and (6), $2x = 10$, or 6.

$\therefore x = 5$, or 3.

Substitute the value of x in (5), $y = 3$, or 5.

Therefore, $x = 5$, and $y = 3$;
or $x = 3$, and $y = 5$.

12. Solve
$$\left.\begin{aligned} x^2 + xy &= 77 \\ xy + y^2 &= 44 \end{aligned}\right\}\qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Add (1) and (2), $x^2 + 2xy + y^2 = 121.$

Extract the root, $x + y = \pm 11.$ (3)

Subtract (2) from (1), $x^2 - y^2 = 33.$

Factor, $(x + y)(x - y) = 33.$

$$x - y = \frac{33}{x + y}. \qquad (4)$$

Substitute the value of $x + y$ in (4), $x - y = \pm 3.$ (5)

Add (3) and (5), $2x = \pm 14.$

$$x = \pm 7.$$

Substitute the value of x in (3), $y = \pm 4.$

Therefore, $x = 7$, and $y = 4$;

or $x = -7$, and $y = -4$.

13. Solve
$$\left.\begin{aligned} x^2 - xy &= 45 \\ xy - y^2 &= -36 \end{aligned}\right\}\qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Subtract (2) from (1), $x^2 - 2xy + y^2 = 81.$

Extract the root, $x - y = \pm 9.$ (3)

Factor (2), $y(x - y) = -36.$

$$y = -\frac{36}{x - y}. \qquad (4)$$

Substitute in (4) the value of $x - y$ from (3), $y = \mp 4.$

Substitute the value of y in (3), $x = \mp 5.$

Therefore, $x = 5$, and $y = -4$;

or $x = -5$, and $y = 4$.

14. Solve
$$\left.\begin{aligned} x^2 + xy &= 55 \\ y^2 + xy &= 66 \end{aligned}\right\}\qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Add (1) and (2), $x^2 + 2xy + y^2 = 121.$

Extract the root, $x + y = \pm 11.$ (3)

Factor (1), $x(x + y) = 55.$ (4)

Substitute in (4) the value of $x + y$, $\pm 11x = 55.$

$$\therefore x = \pm 5.$$

Substitute the value of x in (3), $y = \pm 6.$

Therefore, $x = 5$, and $y = 6$;

or $x = -5$, and $y = -6$.

15. Solve

$$\left.\begin{array}{r} x^2 - y^2 = 40 \\ xy = 21 \end{array}\right\} \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

Multiply (1) by 21, $21x^2 - 21y^2 = 840.$ (3)

Multiply (2) by 40, $40xy = 840.$ (4)

Subtract (4) from (3), $21x^2 - 40xy - 21y^2 = 0.$

Factor, $(3x - 7y)(7x + 3y) = 0.$

$$\therefore 3x = 7y, \text{ or } 7x = -3y.$$

$$\therefore x = \frac{7y}{3}, \text{ or } -\frac{3y}{7}. \qquad (5)$$

Substitute the value of x in (2),

$$\frac{7y^2}{3} = 21, \text{ or } -\frac{3y^2}{7} = 21.$$

$$7y^2 = 63, \text{ or } -3y^2 = 147.$$

$$y^2 = 9, \text{ or } -49.$$

$$y = \pm 3, \text{ or } \pm 7\sqrt{-1}.$$

Substitute the value of y in (5), $x = \pm 7, \text{ or } \mp 3\sqrt{-1}.$

Therefore, $x = 7$, and $y = 3$;

$x = -7$, and $y = -3$;

$x = 3\sqrt{-1}$, and $y = -7\sqrt{-1}$;

or $x = -3\sqrt{-1}$, and $y = 7\sqrt{-1}$.

16. Solve

$$\left.\begin{array}{r} x^2 + y^2 = 250 \\ x - y = 4 \end{array}\right\} \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

Square (2), $x^2 - 2xy + y^2 = 16.$ (3)

Subtract (3) from (1), $2xy = 234.$ (4)

Add (1) and (4), $x^2 + 2xy + y^2 = 484.$

Extract the root, $x + y = \pm 22.$ (5)

Add (5) and (2), $2x = 26, \text{ or } -18.$

$$\therefore x = 13, \text{ or } -9.$$

Subtract (2) from (5), $2y = 18, \text{ or } -26.$

$$\therefore y = 9, \text{ or } -13.$$

Therefore, $x = 13$, and $y = 9$;

or $x = -9$, and $y = -13$.

17. Solve

$$\left.\begin{array}{r} x - y = 5 \\ xy = 36 \end{array}\right\} \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

Square (1), $x^2 - 2xy + y^2 = 25.$ (3)

Multiply (2) by 4, $4xy = 144.$ (4)

Add (3) and (4), $x^2 + 2xy + y^2 = 169.$

Extract the root, $x + y = \pm 13.$ (5)

Add (1) and (5), $2x = 18, \text{ or } -8.$

$\therefore\ x = 9, \text{ or } -4.$

Subtract (1) from (5), $2y = 8, \text{ or } -18.$

$\therefore\ y = 4, \text{ or } -9.$

Therefore, $x = 9, \text{ and } y = 4;$

or $x = -4, \text{ and } y = -9.$

18. Solve

$$\left.\begin{aligned} x^2 - xy + y^2 &= 39 \\ 2x^2 - 3xy + 2y^2 &= 43 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Subtract (1) from (2), $x^2 - 2xy + y^2 = 4.$

Extract the root, $x - y = \pm 2.$

$\therefore\ x = y \pm 2.$ (3)

Substitute the value of x in (1),

$$y^2 \pm 4y + 4 - y^2 \mp 2y + y^2 = 39.$$

$$y^2 + 2y - 35 = 0, \text{ or } y^2 - 2y - 35 = 0.$$

Factor, $(y - 5)(y + 7) = 0, \text{ or } (y - 7)(y + 5) = 0.$

$\therefore\ y = 5, -7, 7, \text{ or } -5.$

Substitute the value of y in (3), $x = 7, -5, 5, \text{ or } -7.$

Therefore, $x = 7, \text{ and } y = 5;$

$x = -7, \text{ and } y = -5;$

$x = 5, \text{ and } y = 7;$

or $x = -5, \text{ and } y = -7.$

19. Solve

$$\left.\begin{aligned} x + xy + y &= 5 \\ x^2 + xy + y^2 &= 7 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Add (1) and (2), $x^2 + 2xy + y^2 + x + y = 12.$

Transpose, $(x + y)^2 + (x + y) - 12 = 0.$

Factor, $(x + y - 3)(x + y + 4) = 0.$

$\therefore\ x + y = 3, \text{ or } -4.$ (3)

Square (3), $x^2 + 2xy + y^2 = 9, \text{ or } 16.$ (4)

Subtract (2) from (4), $xy = 2, \text{ or } 9.$

$3xy = 6, \text{ or } 27.$ (5)

Subtract (5) from (2), $x^2 - 2xy + y^2 = 1, \text{ or } -20.$

Extract the root, $x - y = \pm 1, \text{ or } \pm 2\sqrt{-5}.$ (6)

Add (3) and (6), $2x = 4, 2, \text{ or } -4 \pm 2\sqrt{-5}.$

$\therefore\ x = 2, 1, \text{ or } -2 \pm \sqrt{-5}.$

Subtract (6) from (3), $2y = 2, 4, \text{ or } -4 \mp 2\sqrt{-5}.$

$\therefore\ y = 1, 2, \text{ or } -2 \mp \sqrt{-5}.$

Therefore, $x = 2, \text{ and } y = 1;$

$x = 1, \text{ and } y = 2;$

$x = -2 + \sqrt{-5}, \text{ and } y = -2 - \sqrt{-5};$

or $x = -2 - \sqrt{-5}, \text{ and } y = -2 + \sqrt{-5}.$

20. Solve
$$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{3}{2}\qquad(1)\\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{5}{4}\qquad(2)\end{aligned}\right\}$$

Square (1),
$$\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{9}{4}.\tag{3}$$

Subtract (2) from (3),
$$\frac{2}{xy}=1.\tag{4}$$

Subtract (4) from (2),
$$\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{4}.$$

Extract the root,
$$\frac{1}{x}-\frac{1}{y}=\pm\frac{1}{2}.\tag{5}$$

Add (1) and (5),
$$\frac{2}{x}=2, \text{ or } 1.$$
$$\therefore x=1, \text{ or } 2.$$

Subtract (5) from (1),
$$\frac{2}{y}=1, \text{ or } 2.$$
$$\therefore y=2, \text{ or } 1.$$

Therefore, $x=1$, and $y=2$;
or $x=2$, and $y=1$.

21. Solve
$$\left.\begin{aligned}\frac{x+1}{y+1}&=2\qquad(1)\\ \frac{x^2+1}{y^2+1}&=5\qquad(2)\end{aligned}\right\}$$

From (1),
$$x+1=2y+2.$$
$$\therefore x=2y+1.\tag{3}$$

From (2),
$$x^2+1=5y^2+5.\tag{4}$$

Substitute the value of x in (4), $4y^2+4y+1+1=5y^2+5.$
$$y^2-4y+3=0.$$

Factor,
$$(y-1)(y-3)=0.$$
$$\therefore y=1, \text{ or } 3.$$

Substitute the value of y in (3), $x=3$, or 7.

Therefore, $x=3$, and $y=1$;
or $x=7$, and $y=3$.

22. Solve
$$\left.\begin{aligned}x-y&=3\qquad(1)\\ x^3-y^3&=387\qquad(2)\end{aligned}\right\}$$

Divide (2) by (1),
$$x^2+xy+y^2=129.\tag{3}$$

Square (1),
$$x^2-2xy+y^2=9.\tag{4}$$

Subtract (4) from (3), $3xy = 120.$

$xy = 40.$ (5)

Add (3) and (5), $x^2 + 2xy + y^2 = 169.$

$x + y = \pm 13.$ (6)

Add (1) and (6), $2x = 16, \text{ or } -10.$

$\therefore x = 8, \text{ or } -5.$

Substitute the value of x in (5), $y = 5, \text{ or } -8.$

Therefore, $x = 8$, and $y = 5$;

or $x = -5$, and $y = -8$.

23. Solve

$$\left.\begin{aligned} x^2 + y^2 + x + y &= 18 \\ x^2 - y^2 + x - y &= 6 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Subtract (2) from (1), $2y^2 + 2y = 12.$

Divide by 2, $y^2 + y = 6.$

Transpose, $y^2 + y - 6 = 0.$

Factor, $(y - 2)(y + 3) = 0.$

$\therefore y = 2, \text{ or } -3.$

Substitute the value of y in (1), $x^2 + 4 + x + 2 = 18,$

or $x^2 + 9 + x - 3 = 18.$

Transpose and combine, $x^2 + x - 12 = 0,$

or $x^2 + x - 12 = 0.$

Factor, $(x - 3)(x + 4) = 0.$

$\therefore x = 3, \text{ or } -4.$

Therefore, $x = 3$, and $y = 2$;

$x = 3$, and $y = -3$;

$x = -4$, and $y = 2$;

or $x = -4$, and $y = -3$.

24. Solve

$$\left.\begin{aligned} 2x^2 - 5xy + 3x - 2y &= 10 \\ 5xy - 2x^2 + 7x - 8y &= 10 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Add (1) and (2), $10x - 10y = 20.$

$x - y = 2.$

$\therefore x = y + 2.$ (3)

Substitute the value of x in (1),

$$2(y + 2)^2 - 5(y + 2)y + 3(y + 2) - 2y = 10.$$

Expand,

$$2y^2 + 8y + 8 - 5y^2 - 10y + 3y + 6 - 2y = 10.$$

Transpose and combine, $3y^2 + y - 4 = 0.$
Factor, $(y - 1)(3y + 4) = 0.$
$\therefore y = 1, \text{ or } -\frac{4}{3}.$
Substitute the value of y in (3), $x = 3, \text{ or } \frac{2}{3}.$

Therefore, $x = 3$, and $y = 1$;
or $x = \frac{2}{3}$, and $y = -\frac{4}{3}$.

25. Solve
$$\left.\begin{aligned} 5x + y + 3 &= 2xy \\ xy - 2x + y &= 9 \end{aligned}\right\} \quad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose (1), $5x + y - 2xy = -3.$ (3)
Transpose (2), $-2x + y + xy = 9.$ (4)
Multiply (4) by 2, $-4x + 2y + 2xy = 18.$ (5)
Add (3) and (5), $x + 3y = 15.$
Transpose, $x = 15 - 3y.$ (6)
Substitute the value of x in (2),
$$15y - 3y^2 - 30 + 6y + y = 9.$$
Transpose and combine, $3y^2 - 22y = -39.$
Complete the square, $9y^2 - (\ ) + 121 = 4.$
Extract the root, $3y - 11 = \pm 2.$
$3y = 13, \text{ or } 9.$
$\therefore y = 4\frac{1}{3}, \text{ or } 3.$
Substitute the value of y in (6), $x = 2, \text{ or } 6.$

Therefore, $x = 2$, and $y = 4\frac{1}{3}$;
or $x = 6$, and $y = 3$.

26. Solve
$$\left.\begin{aligned} x^3 + y^3 &= 37 \\ x^2 - xy + y^2 &= 37 \end{aligned}\right\} \quad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Divide (1) by (2), $x + y = 1.$ (3)
Square (3), $x^2 + 2xy + y^2 = 1.$ (4)
Subtract (2) from (4), $3xy = -36.$
$xy = -12.$ (5)
Subtract (5) from (2), $x^2 - 2xy + y^2 = 49.$
Extract the root, $x - y = \pm 7.$ (6)
Add (3) and (6), $2x = 8, \text{ or } -6.$
$\therefore x = 4, \text{ or } -3.$
Subtract (6) from (3), $2y = -6, \text{ or } 8.$
$\therefore y = -3, \text{ or } 4.$

Therefore, $x = 4$, and $y = -3$;
or $x = -3$, and $y = 4$.

27. Solve
$$\left.\begin{aligned} x^2 + y &= 5(x - y) \\ x + y^2 &= 2(x - y) \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Subtract (2) from (1), $x^2 - y^2 - x + y = 3(x - y).$

$$x^2 - y^2 - (x - y) = 3(x - y).$$

$$(x + y)(x - y) - 4(x - y) = 0.$$

Factor, $(x - y)(x + y - 4) = 0.$

$$\therefore x = y, \text{ or } 4 - y. \qquad (3)$$

Substitute the value of x in (2),

$$y + y^2 = 0, \text{ or } 4 - y + y^2 = 8 - 4y.$$

$$y^2 + y = 0, \text{ or } y^2 + 3y - 4 = 0.$$

Factor, $y(y + 1) = 0, \text{ or } (y - 1)(y + 4) = 0.$

$$\therefore y = 0, -1, 1, \text{ or } -4.$$

Substitute the value of y in (3), $x = 0, -1, 3, \text{ or } 8.$

Therefore, $x = 0$, and $y = 0$;
$x = -1$, and $y = -1$;
$x = 3$, and $y = 1$;
$x = 8$, and $y = -4$.

28. Solve
$$\left.\begin{aligned} x^2 + y^2 + x + y &= 58 \\ xy + 14 &= 0 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (2) by 2, $2xy + 28 = 0.$

$$\therefore 2xy = -28. \qquad (3)$$

Add (1) and (3),

$$x^2 + 2xy + y^2 + x + y = 30.$$

$$(x + y)^2 + (x + y) - 30 = 0.$$

Factor, $(x + y - 5)(x + y + 6) = 0.$

$$\therefore x + y = 5, \text{ or } -6. \qquad (4)$$

Square (4), $x^2 + 2xy + y^2 = 25, \text{ or } 36. \qquad (5)$

Multiply (3) by 2, $4xy = -56. \qquad (6)$

Subtract (6) from (5), $x^2 - 2xy + y^2 = 81, \text{ or } 92.$

Extract the root, $x - y = \pm 9, \text{ or } \pm 2\sqrt{23}. \qquad (7)$

Add (4) and (7), $2x = 14, -4, \text{ or } -6 \pm 2\sqrt{23}.$

$$\therefore x = 7, -2, \text{ or } -3 \pm \sqrt{23}.$$

Subtract (7) from (4), $2y = -4, 14, \text{ or } -6 \mp 2\sqrt{23}.$

$$\therefore y = -2, 7, \text{ or } -3 \mp \sqrt{23}.$$

Therefore, $x = 7$, and $y = -2$;
$x = -2$, and $y = 7$;
$x = -3 + \sqrt{23}$, and $y = -3 - \sqrt{23}$;
or $x = -3 - \sqrt{23}$, and $y = -3 + \sqrt{23}$.

29. Solve
$$\left.\begin{aligned} x^2 + y^2 &= xy + 103 \quad &(1)\\ x + y &= xy - 79 \quad &(2)\end{aligned}\right\}$$

Square (2), $x^2 + 2xy + y^2 = x^2y^2 - 158xy + 6241.$ (3)

Subtract (1) from (3), $2xy = x^2y^2 - 159xy + 6138.$

$x^2y^2 - 161xy = -6138.$

Complete the square,

$4x^2y^2 - (\ ) + 161^2 = 1369.$

Extract the root, $2xy - 161 = \pm 37.$

$2xy = 198,$ or $124.$

$\therefore xy = 99,$ or $62.$ (4)

Substitute the value of xy in (2), $x + y = 20,$ or $-17.$ (5)

Square (5), $x^2 + 2xy + y^2 = 400,$ or $289.$ (6)

Multiply (4) by 4, $4xy = 396,$ or $248.$ (7)

Subtract (7) from (6), $x^2 - 2xy + y^2 = 4,$ or $41.$

Extract the root, $x - y = \pm 2,$ or $\pm\sqrt{41}.$ (8)

Add (5) and (8), $2x = 22, 18,$ or $-17 \pm \sqrt{41}.$

$\therefore x = 11, 9,$ or $\frac{1}{2}(-17 \pm \sqrt{41}).$

Subtract (8) from (5), $2y = 18, 22,$ or $-17 \mp \sqrt{41}.$

$\therefore y = 9, 11,$ or $\frac{1}{2}(-17 \mp \sqrt{41}).$

Therefore, $x = 11,$ and $y = 9;$

$x = 9,$ and $y = 11;$

$x = \frac{1}{2}(-17 + \sqrt{41}),$ and $y = \frac{1}{2}(-17 - \sqrt{41});$

or $x = \frac{1}{2}(-17 - \sqrt{41}),$ and $y = \frac{1}{2}(-17 + \sqrt{41}).$

30. Solve
$$\left.\begin{aligned} x^2 + y^2 &= 57 + xy \quad &(1)\\ x + y &= xy - 41 \quad &(2)\end{aligned}\right\}$$

Square (2), $x^2 + 2xy + y^2 = x^2y^2 - 82xy + 1681.$ (3)

Subtract (1) from (3), $2xy = x^2y^2 - 83xy + 1624.$

$x^2y^2 - 85xy = -1624.$

Complete the square, $4x^2y^2 - (\ ) + 85^2 = 729.$

Extract the root, $2xy - 85 = \pm 27.$

$2xy = 112,$ or $58.$

$\therefore xy = 56,$ or $29.$ (4)

Substitute the value of xy in (2), $x + y = 15,$ or $-12.$ (5)

Square (5), $x^2 + 2xy + y^2 = 225,$ or $144.$ (6)

Multiply (4) by 4, $4xy = 224,$ or $116.$ (7)

Subtract (7) from (6), $x^2 - 2xy + y^2 = 1,$ or $28.$

Extract the root, $x - y = \pm 1,$ or $\pm 2\sqrt{7}.$ (8)

Add (5) and (8), $2x = 16,\ 14,\ \text{or}\ -12 \pm 2\sqrt{7}.$

$\therefore\ x = 8,\ 7,\ \text{or}\ -6 \pm \sqrt{7}.$

Subtract (8) from (5), $2y = 14,\ 16,\ \text{or}\ -12 \mp 2\sqrt{7}.$

$\therefore\ y = 7,\ 8,\ \text{or}\ -6 \mp \sqrt{7}.$

Therefore, $x = 8$, and $y = 7$;
$x = 7$, and $y = 8$;
$x = -6 + \sqrt{7}$, and $y = -6 - \sqrt{7}$;
or $x = -6 - \sqrt{7}$, and $y = -6 + \sqrt{7}$.

31. Solve

$$\left.\begin{aligned} x + xy + y &= 11 \\ x^2 + x^2y^2 + y^2 &= 49 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Transpose (1), $x + y = 11 - xy.$ (3)

Square (3), $x^2 + 2xy + y^2 = 121 - 22xy + x^2y^2.$ (4)

Transpose (2), $x^2 + y^2 = 49 - x^2y^2.$ (5)

Subtract (5) from (4), $2xy = 72 - 22xy + 2x^2y^2.$

$$2x^2y^2 - 24xy + 72 = 0.$$

$$x^2y^2 - 12xy + 36 = 0.$$

Extract the root, $xy - 6 = 0.$

$\therefore\ xy = 6.$ (6)

Substitute the value of xy in (3), $x + y = 5.$ (7)

Square (7), $x^2 + 2xy + y^2 = 25.$ (8)

Multiply (6) by 4, $4xy = 24.$ (9)

Subtract (9) from (8), $x^2 - 2xy + y^2 = 1.$

Extract the root, $x - y = \pm 1.$ (10)

Add (7) and (10), $2x = 6,\ \text{or}\ 4.$

$\therefore\ x = 3,\ \text{or}\ 2.$

Substitute the value of x in (7), $y = 2,\ \text{or}\ 3.$

Therefore, $x = 3$, and $y = 2$;
or $x = 2$, and $y = 3$.

32. Solve

$$\left.\begin{aligned} x^2 + y^2 - 12 &= x + y \\ xy + 8 &= 2(x + y) \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (2) by 2, $2xy + 16 = 4(x + y).$ (3)

Add (1) and (3), $x^2 + 2xy + y^2 + 4 = 5(x + y).$

Factor and transpose,

$$(x + y)^2 - 5(x + y) + 4 = 0.$$

Factor, $(x + y - 1)(x + y - 4) = 0.$

$\therefore\ x + y = 1,\ \text{or}\ 4.$ (4)

Substitute the value of $x + y$ in (2),

$$xy + 8 = 2,\ \text{or}\ 8.$$

$\therefore\ xy = -6,\ \text{or}\ 0.$ (5)

Square (4), $x^2 + 2xy + y^2 = 1, \text{ or } 16.$ (6)
Multiply (5) by 4, $4xy = -24, \text{ or } 0.$ (7)
Subtract (7) from (6), $x^2 - 2xy + y^2 = 25, \text{ or } 16.$
Extract the root, $x - y = \pm 5, \text{ or } \pm 4.$ (8)
Add (4) and (8), $2x = 6, -4, 8, \text{ or } 0.$
$\therefore x = 3, -2, 4, \text{ or } 0.$
Subtract (8) from (4), $2y = -4, 6, 0, \text{ or } 8.$
$\therefore y = -2, 3, 0, \text{ or } 4.$

Therefore, $x = 3$, and $y = -2$;
$x = -2$, and $y = 3$;
$x = 4$, and $y = 0$;
$x = 0$, and $y = 4$.

33. Solve

$$\left.\begin{aligned} x^2 - xy + y^2 &= 37 \\ x^2 - y^2 &= 40 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Put vx for y in (1), $x^2 - vx^2 + v^2x^2 = 37.$
$x^2(1 - v + v^2) = 37.$

$$\therefore x^2 = \frac{37}{1 - v + v^2}. \qquad (3)$$

Put vx for y in (2), $x^2 - v^2x^2 = 40.$
$x^2(1 - v^2) = 40.$

$$\therefore x^2 = \frac{40}{1 - v^2}. \qquad (4)$$

Equate the values of x^2,

$$\frac{37}{1 - v + v^2} = \frac{40}{1 - v^2}.$$

$37 - 37v^2 = 40 - 40v + 40v^2.$
$77v^2 - 40v = -3.$
Complete the square, $77^2v^2 - (\ ) + 400 = 169.$
Extract the root, $77v - 20 = \pm 13.$
$77v = 33, \text{ or } 7.$
$\therefore v = \frac{3}{7}, \text{ or } \frac{1}{11}.$
Substitute the value of v in (4), $x^2 = 49, \text{ or } \frac{121}{3}.$
$\therefore x = \pm 7, \text{ or } \pm \frac{11}{3}\sqrt{3}.$
Since $y = vx$, $y = \pm 3, \text{ or } \pm \frac{1}{3}\sqrt{3}.$

Therefore, $x = 7$, and $y = 3$;
$x = -7$, and $y = -3$;
$x = \frac{11}{3}\sqrt{3}$, and $y = \frac{1}{3}\sqrt{3}$;
or $x = -\frac{11}{3}\sqrt{3}$, and $y = -\frac{1}{3}\sqrt{3}$.

34. Solve
$$\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 21 \\ x^2 + xy + y^2 &= 7 \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Divide (1) by (2), $x^2 - xy + y^2 = 3.$ (3)

Subtract (3) from (2), $2xy = 4.$

$\therefore xy = 2.$ (4)

Add (2) and (4), $x^2 + 2xy + y^2 = 9.$

Extract the root, $x + y = \pm 3.$ (5)

Subtract (4) from (3), $x^2 - 2xy + y^2 = 1.$

Extract the root, $x - y = \pm 1.$ (6)

Add (5) and (6), $2x = \pm 4, \text{ or } \pm 2.$

$\therefore x = \pm 2, \text{ or } \pm 1.$

Substitute the value of x in (4), $y = \pm 1, \text{ or } \pm 2.$

Therefore, $x = 2$, and $y = 1$;
$x = -2$, and $y = -1$;
$x = 1$, and $y = 2$;
or $x = -1$, and $y = -2$.

35. Solve
$$\left.\begin{aligned} x^2 - 5xy + y^2 + 5 &= 0 \\ xy - x - y + 1 &= 0 \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Multiply (2) by 7, $7xy - 7x - 7y + 7 = 0.$ (3)

Add (1) and (3), $x^2 + 2xy + y^2 - 7x - 7y + 12 = 0.$

$(x + y)^2 - 7(x + y) + 12 = 0.$

Factor, $(x + y - 3)(x + y - 4) = 0.$

$\therefore x + y = 3, \text{ or } 4.$ (4)

Substitute the value of $x + y$ in (2), $xy = 2, \text{ or } 3.$ (5)

Square (4), $x^2 + 2xy + y^2 = 9, \text{ or } 16.$ (6)

Multiply (5) by 4, $4xy = 8, \text{ or } 12.$ (7)

Subtract (7) from (6), $x^2 - 2xy + y^2 = 1, \text{ or } 4.$

Extract the root, $x - y = \pm 1, \text{ or } \pm 2.$ (8)

Add (4) and (8), $2x = 4, 2, 6, \text{ or } 2.$

$\therefore x = 2, 1, 3, \text{ or } 1.$

Substitute the value of x in (5), $y = 1, 2, 1, \text{ or } 3.$

Therefore, $x = 2$, and $y = 1$;
$x = 1$, and $y = 2$;
$x = 3$, and $y = 1$;
or $x = 1$, and $y = 3$.

36. Solve
$$\left.\begin{aligned} x^2 + 2xy - y^2 &= 41 \\ 3x^2 + 2xy + 2y^2 &= 103 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Add (1) and (2), $4x^2 + 4xy + y^2 = 144.$

Extract the root, $2x + y = \pm 12.$

$$\therefore\ y = \pm 12 - 2x. \qquad (3)$$

Substitute the value of y in (1),

$$x^2 + 2x(\pm 12 - 2x) - (\pm 12 - 2x)^2 = 41.$$
$$x^2 \pm 24x - 4x^2 - 144 \pm 48x - 4x^2 = 41.$$

Transpose and combine, $7x^2 \mp 72x = -185.$

Complete the square, $49x^2 \mp (\ ) + 36^2 = 1.$

Extract the root, $7x \mp 36 = \pm 1.$

$$7x = \pm 37, \text{ or } \pm 35.$$
$$\therefore\ x = \pm \tfrac{37}{7}, \text{ or } \pm 5.$$

Substitute the value of x in (3), $y = \pm \tfrac{10}{7}, \text{ or } \pm 2.$

Therefore, $x = \tfrac{37}{7}$, and $y = \tfrac{10}{7}$;
$x = -\tfrac{37}{7}$, and $y = -\tfrac{10}{7}$;
$x = 5$, and $y = 2$;
or $x = -5$, and $y = -2$.

37. Solve
$$\left.\begin{aligned} 2x^2 - xy + 5y^2 &= 10(x + y) \\ x^2 + 4xy + 3y^2 &= 14(x + y) \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (1) by 7, $14x^2 - 7xy + 35y^2 = 70(x + y).$ (3)

Multiply (2) by 5, $5x^2 + 20xy + 15y^2 = 70(x + y).$ (4)

Subtract (4) from (3), $9x^2 - 27xy + 20y^2 = 0.$

Factor, $(3x - 5y)(3x - 4y) = 0.$

$$\therefore\ x = \frac{5y}{3}, \text{ or } \frac{4y}{3}. \qquad (5)$$

Substitute the value of x in (2),

$$\frac{25y^2}{9} + \frac{20y^2}{3} + 3y^2 = \frac{112y}{3}, \text{ or } \frac{16y^2}{9} + \frac{16y^2}{3} + 3y^2 = \frac{98y}{3}.$$
$$25y^2 + 60y^2 + 27y^2 = 336y, \text{ or } 16y^2 + 48y^2 + 27y^2 = 294y.$$
$$112y^2 - 336y = 0, \text{ or } 91y^2 - 294y = 0.$$
$$y^2 - 3y = 0, \text{ or } 13y^2 - 42y = 0.$$

Factor, $y(y - 3) = 0, \text{ or } y(13y - 42) = 0.$

$$\therefore\ y = 0, \text{ or } 3, \text{ or } y = 0, \text{ or } \tfrac{42}{13}.$$

Substitute the value of y in (5),

$$x = 0, \text{ or } 5, \text{ or } x = 0, \text{ or } \tfrac{56}{13}.$$

Therefore, $x = 0$, and $y = 0$;
$x = 5$, and $y = 3$;
or $x = \tfrac{56}{13}$, and $y = \tfrac{42}{13}$.

38. Solve
$$\left.\begin{aligned} 8x + 8y - x^2 - xy &= 10 \\ 5x + 5y - xy - y^2 &= 20 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Factor (2), $5(x + y) - y(x + y) = 20.$

$$\therefore (5 - y)(x + y) = 20. \qquad (3)$$

Add (1) and (2), $13x + 13y - x^2 - 2xy - y^2 = 30.$

$$13(x + y) - (x + y)^2 = 30.$$

Factor, $(13 - x + y)(x + y) = 30.$

$$\therefore x + y = \frac{30}{13 - x - y}. \qquad (4)$$

Substitute the value of $x + y$ in (3), $\dfrac{30(5 - y)}{13 - x - y} = 20.$

Divide by 10, $\dfrac{3(5 - y)}{13 - x - y} = 2.$

Clear of fractions, $15 - 3y = 26 - 2x - 2y.$

Transpose and combine, $y = 2x - 11. \qquad (5)$

Substitute the value of y in (1),

$$8x + 16x - 88 - x^2 - 2x^2 + 11x = 10.$$

Transpose and combine, $3x^2 - 35x = - 98.$

Complete the square, $36x^2 - (\ ) + 1225 = 49.$

Extract the root, $6x - 35 = \pm 7.$

$$6x = 42, \text{ or } 28.$$

$$\therefore x = 7, \text{ or } 4\tfrac{2}{3}.$$

Substitute the value of x in (5), $y = 3, \text{ or } -1\tfrac{2}{3}.$

Therefore, $x = 7$, and $y = 3$;

or $x = 4\frac{2}{3}$, and $y = -1\frac{2}{3}$.

39. Solve
$$\left.\begin{aligned} 4x^2 - 9y^2 &= 0 \\ 4x^2 + y^2 &= 8(x + y) \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Transpose (1), $4x^2 = 9y^2. \qquad (3)$

Extract the square root, $2x = \pm 3y.$

$$\therefore x = \pm \frac{3y}{2}. \qquad (4)$$

Substitute the value of x in (2), $9y^2 + y^2 = 8\left(\pm \dfrac{3y}{2} + y\right).$

Simplify, $10y^2 = 20y, \text{ or } 10y^2 = -4y.$

$$y^2 = 2y, \text{ or } 5y^2 = -2y.$$

Transpose, $y^2 - 2y = 0$, or $5y^2 + 2y = 0$.
Factor, $y(y - 2) = 0$, or $y(5y + 2) = 0$.
$\therefore y = 0, 2, 0$, or $-\frac{2}{5}$.
Substitute the value of y in (4), $x = 0, 3, 0$, or $\frac{3}{5}$.

Therefore, $x = 0$, and $y = 0$;
$x = 0$, and $y = 0$;
$x = 3$, and $y = 2$;
or $x = \frac{3}{5}$, and $y = -\frac{2}{5}$.

40. Solve

$$\left.\begin{aligned} x^2 - 5y^2 - 3x - y + 22 &= 0 \\ (x - 3)(y - 2) &= y^2 - 3y + 2 \end{aligned}\right\} \begin{aligned} &(1) \\ &(2) \end{aligned}$$

From (2), $(x - 3)(y - 2) = (y - 1)(y - 2)$.
Transpose,
$(x - 3)(y - 2) - (y - 1)(y - 2) = 0$.
Factor, $(y - 2)(x - 3 - y + 1) = 0$,
or $(y - 2)(x - y - 2) = 0$.
$\therefore y - 2 = 0$, (3)
or $x - y - 2 = 0$. (4)
From (3), $y = 2$.
Substitute the value of y in (1),
$x^2 - 20 - 3x - 2 + 22 = 0$.
Combine, $x^2 - 3x = 0$.
Factor, $x(x - 3) = 0$.
$\therefore x = 0$, or 3.
From (4), $x = y + 2$. (5)
Substitute the value of x in (1),
$y^2 + 4y + 4 - 5y^2 - 3y - 6 - y + 22 = 0$.
Transpose and combine, $4y^2 = 20$.
$\therefore y^2 = 5$.
$\therefore y = \pm\sqrt{5}$.
Substitute the value of y in (5), $x = 2 \pm \sqrt{5}$.

Therefore, $x = 0$, and $y = 2$;
$x = 3$, and $y = 2$;
$= 2 + \sqrt{5}$, and $y = \sqrt{5}$;
or $x = 2 - \sqrt{5}$, and $y = -\sqrt{5}$.

41. Solve
$$\left.\begin{aligned} x^2+y^2&=xy+37\\ x+y&=xy-17\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

From (1), $xy = x^2 + y^2 - 37.$ (3)

From (2), $xy = x + y + 17.$ (4)

Equate the values of xy, $x^2 + y^2 - 37 = x + y + 17.$

$x^2 + y^2 = x + y + 54.$ (5)

Multiply (4) by 2, $2xy = 2x + 2y + 34.$ (6)

Add (5) and (6), $x^2 + 2xy + y^2 = 3x + 3y + 88.$

Transpose and factor,

$$(x+y)^2 - 3(x+y) - 88 = 0.$$

Factor, $(x + y - 11)(x + y + 8) = 0.$

$\therefore x + y = 11, \text{ or } -8.$ (7)

Substitute the value of $x + y$ in (4), $xy = 28, \text{ or } 9.$

$2xy = 56, \text{ or } 18.$ (8)

Substitute the value of $x + y$ in (5), $x^2 + y^2 = 65, \text{ or } 46.$ (9)

Subtract (8) from (9), $x^2 - 2xy + y^2 = 9, \text{ or } 28.$

$x - y = \pm 3, \text{ or } \pm 2\sqrt{7}.$ (10)

Add (7) and (10), $2x = 14, 8, \text{ or } -8 \pm 2\sqrt{7}.$

$\therefore x = 7, 4, \text{ or } -4 \pm \sqrt{7}.$

Substitute the value of x in (7), $y = 4, 7, \text{ or } -4 \mp \sqrt{7}.$

Therefore, $x = 7$, and $y = 4$;
$x = 4$, and $y = 7$;
$x = -4 + \sqrt{7}$, and $y = -4 - \sqrt{7}$;
or $x = -4 - \sqrt{7}$, and $y = -4 + \sqrt{7}$.

42. Solve
$$\left.\begin{aligned} x^4+y^4&=97\\ x+y&=5\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Raise (2) to the 4th power,

$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 625.$ (3)

Subtract (1) from (3),

$$4x^3y + 6x^2y^2 + 4xy^3 = 528.$$
$$2x^3y + 3x^2y^2 + 2xy^3 = 264.$$

$xy(2x^2 + 3xy + 2y^2) = 264.$ (4)

Square (2), $x^2 + 2xy + y^2 = 25.$ (5)

$$2x^2 + 4xy + 2y^2 = 50.$$

$2x^2 + 2y^2 = 50 - 4xy.$ (6)

Substitute the value of $2x^2 + 2y^2$ in (4),

$$xy(50 - xy) = 264.$$
$$50xy - x^2y^2 = 264.$$
$$x^2y^2 - 50xy + 264 = 0.$$

Factor, $(xy-44)(xy+6)=0.$

$\therefore\ xy=44, \text{ or } 6.$

$4\,xy=176, \text{ or } 24. \qquad (7)$

Subtract (7) from (5), $x^2-2\,xy+y^2=-151, \text{ or } 1.$

Extract the root, $x-y=\pm\sqrt{-151}, \text{ or } \pm 1. \qquad (8)$

Add (2) and (8), $2\,x=5\pm\sqrt{-151},\ 6, \text{ or } 4.$

$\therefore\ x=\frac{1}{2}(5\pm\sqrt{-151}),\ 3, \text{ or } 2.$

Substitute the value of x in (2), $y=\frac{1}{2}(5\mp\sqrt{-151}),\ 2, \text{ or } 3.$

Therefore, $x=3$, and $y=2$;

$x=2$, and $y=3$;

$x=\frac{1}{2}(5+\sqrt{-151})$, and $y=\frac{1}{2}(5-\sqrt{-151})$;

or $x=\frac{1}{2}(5-\sqrt{-151})$, and $y=\frac{1}{2}(5+\sqrt{-151})$.

43. Solve

$$\left.\begin{aligned} x^4+y^4&=272 \qquad (1)\\ x-y&=\ \ 2 \qquad (2)\end{aligned}\right\}$$

Raise (2) to the 4th power,

$$x^4-4\,x^3y+6\,x^2y^2-4\,xy^3+y^4=16. \qquad (3)$$

Subtract (3) from (1),

$$4\,x^3y-6\,x^2y^2+4\,xy^3=256.$$

$$2\,x^3y-3\,x^2y^2+2\,xy^3=128.$$

$$xy\,(2\,x^2-3\,xy+2\,y^2)=128. \qquad (4)$$

Square (2), $x^2-2\,xy+y^2=4. \qquad (5)$

$$2\,x^2-4\,xy+2\,y^2=8.$$

$$2\,x^2+2\,y^2=8+4\,xy.$$

Substitute the value of $2\,x^2+2\,y^2$ in (4),

$$xy\,(8+xy)=128.$$

$$8\,xy+x^2y^2=128.$$

$$x^2y^2+8\,xy-128=0.$$

Factor, $(xy-8)(xy+16)=0.$

$\therefore\ xy=8, \text{ or } -16.$

$4\,xy=32, \text{ or } -64. \qquad (6)$

Add (5) and (6), $x^2+2\,xy+y^2=36, \text{ or } -60.$

Extract the root, $x+y=\pm 6, \text{ or } \pm 2\sqrt{-15}. \qquad (7)$

Add (2) and (7), $2\,x=8,\ -4, \text{ or } 2\pm 2\sqrt{-15}.$

$\therefore\ x=4,\ -2, \text{ or } 1\pm\sqrt{-15}.$

Substitute the value of x in (2), $y=2,\ -4, \text{ or } -1\pm\sqrt{-15}.$

Therefore, $x=4$, and $y=2$;

$x=-2$, and $y=-4$;

$x=1+\sqrt{-15}$, and $y=-1+\sqrt{-15}$;

or $x=1-\sqrt{-15}$, and $y=-1-\sqrt{-15}$.

44 Solve
$$\left.\begin{aligned} x^5+y^5&=33\\ x+y&=\ \,3\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Divide (1) by (2),
$$x^4-x^3y+x^2y^2-xy^3+y^4=11. \qquad (3)$$

Raise (2) to the 4th power,
$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=81. \qquad (4)$$

Subtract (3) from (4),
$$5x^3y+5x^2y^2+5xy^3=70.$$
$$x^3y+x^2y^2+xy^3=14.$$
$$xy(x^2+xy+y^2)=14. \qquad (5)$$

Square (2), $$x^2+2xy+y^2=9. \qquad (6)$$
$$x^2+y^2=9-2xy.$$

Substitute the value of x^2+y^2 in (5),
$$xy(9-xy)=14.$$
$$9xy-x^2y^2=14.$$
$$x^2y^2-9xy+14=0.$$

Factor, $$(xy-2)(xy-7)=0.$$
$$\therefore\ xy=2,\text{ or }7.$$
$$4xy=8,\text{ or }28. \qquad (7)$$

Subtract (7) from (6), $x^2-2xy+y^2=1$, or -19.

Extract the root, $$x-y=\pm 1,\text{ or }\pm\sqrt{-19}. \qquad (8)$$

Add (2) and (8), $$2x=4,\ 2,\text{ or }3\pm\sqrt{-19}.$$
$$\therefore\ x=2,\ 1,\text{ or }\tfrac{1}{2}(3\pm\sqrt{-19}).$$

Substitute the value of x in (2), $y=1,\ 2,\text{ or }\tfrac{1}{2}(3\mp\sqrt{-19})$.

Therefore, $x=2$, and $y=1$;
$x=1$, and $y=2$;
$x=\tfrac{1}{2}(3+\sqrt{-19})$, and $y=\tfrac{1}{2}(3-\sqrt{-19})$;
or $x=\tfrac{1}{2}(3-\sqrt{-19})$, and $y=\tfrac{1}{2}(3+\sqrt{-19})$.

45. Solve
$$\left.\begin{aligned} x^5-y^5&=211\\ x-y&=1\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Divide (1) by (2), $$x^4+x^3y+x^2y^2+xy^3+y^4=211. \qquad (3)$$

Raise (2) to the 4th power,
$$x^4-4x^3y+6x^2y^2-4xy^3+y^4=1. \qquad (4)$$

Subtract (4) from (3), $$5x^3y-5x^2y^2+5xy^3=210.$$
$$x^3y-x^2y^2+xy^3=42.$$

$$xy(x^2 - xy + y^2) = 42. \qquad (5)$$

Square (2), $x^2 - 2xy + y^2 = 1.$ (6)

$$x^2 + y^2 = 1 + 2xy.$$

Substitute the value of $x^2 + y^2$ in (5),

$$xy(1 + xy) = 42.$$

$$xy + x^2y^2 = 42.$$

$$x^2y^2 + xy - 42 = 0.$$

Factor, $(xy - 6)(xy + 7) = 0.$

$$\therefore\ xy = 6, \text{ or } -7.$$

$$4xy = 24, \text{ or } -28. \qquad (7)$$

Add (6) and (7), $x^2 + 2xy + y^2 = 25, \text{ or } -27.$

Extract the root, $x + y = \pm 5, \text{ or } \pm 3\sqrt{-3}.$ (8)

Add (2) and (8), $2x = 6, -4, \text{ or } 1 \pm 3\sqrt{-3}.$

$$\therefore\ x = 3, -2, \text{ or } \tfrac{1}{2}(1 \pm 3\sqrt{-3}).$$

Substitute the value of x in (2), $y = 2, -3, \text{ or } \tfrac{1}{2}(-1 \pm 3\sqrt{-3}).$

Therefore, $x = 3$, and $y = 2$;

$x = -2$, and $y = -3$;

$x = \tfrac{1}{2}(1 + 3\sqrt{-3})$, and $y = \tfrac{1}{2}(-1 + 3\sqrt{-3})$;

or $x = \tfrac{1}{2}(1 - 3\sqrt{-3})$, and $y = \tfrac{1}{2}(-1 - 3\sqrt{-3})$.

46. Solve

$$\left.\begin{aligned} x^4 + y^4 &= 162 \\ x^2 + y^2 &= 3xy - 9 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Square (2), $x^4 + 2x^2y^2 + y^4 = 9x^2y^2 - 54xy + 81.$

$$x^4 - 7x^2y^2 + y^4 = 81 - 54xy. \qquad (3)$$

Subtract (3) from (1), $7x^2y^2 = 81 + 54xy.$

$$7x^2y^2 - 54xy = 81.$$

Complete the square,

$$49x^2y^2 - (\ ) + 729 = 1296.$$

$$7xy - 27 = \pm 36.$$

$$7xy = 63, \text{ or } -9.$$

$$\therefore\ xy = 9, \text{ or } -\tfrac{9}{7}. \qquad (4)$$

From (2), $x^2 + 2xy + y^2 = 5xy - 9.$ (5)

Substitute in (5) the value of xy,

$$x^2 + 2xy + y^2 = 36, \text{ or } -\tfrac{108}{7}.$$

Extract the root, $x + y = \pm 6, \text{ or } \pm \tfrac{6}{7}\sqrt{-21}.$ (6)

From (2), $x^2 - 2xy + y^2 = xy - 9.$ (7)

Substitute in (7) the value of xy from (4),

$$x^2 - 2xy + y^2 = 0, \text{ or } -\tfrac{72}{7}.$$

Extract the root, $x - y = 0, 0, \text{ or } \pm \tfrac{6}{7}\sqrt{-14}.$ (8)

Add (6) and (8), $2x = 6, -6, \text{ or } \pm \tfrac{6}{7}(\sqrt{-21} + \sqrt{-14}).$

$\therefore x = 3, -3, \text{ or } \pm \tfrac{3}{7}(\sqrt{-21} + \sqrt{-14}).$

Subtract (8) from (6), $2y = 6, -6, \text{ or } \pm \tfrac{6}{7}(\sqrt{-21} - \sqrt{-14}).$

$\therefore y = 3, -3, \text{ or } \pm \tfrac{3}{7}(\sqrt{-21} - \sqrt{-14}).$

Therefore,

$x = 3$, and $y = 3$;

$x = -3$, and $y = -3$;

$x = \tfrac{3}{7}(\sqrt{-21} + \sqrt{-14})$, and $y = \tfrac{3}{7}(\sqrt{-21} - \sqrt{-14})$;

or $x = -\tfrac{3}{7}(\sqrt{-21} + \sqrt{-14})$, and $y = -\tfrac{3}{7}(\sqrt{-21} - \sqrt{-14})$.

47. Solve

$$\left.\begin{aligned} \frac{x+y}{xy-1} &= \frac{7}{11} \qquad (1)\\ \frac{y-x}{xy+1} &= \frac{1}{13} \qquad (2)\end{aligned}\right\}$$

Clear (1) of fractions, $11x + 11y = 7xy - 7.$ (3)

Clear (2) of fractions, $13y - 13x = xy + 1.$ (4)

Multiply (4) by 7, $91y - 91x = 7xy + 7.$ (5)

Subtract (3) from (5), $80y - 102x = 14.$

Divide by 2, $40y - 51x = 7.$

$$40y = 51x + 7.$$

$$\therefore y = \frac{51x + 7}{40}. \qquad (6)$$

Substitute the value of y in (3),

$$11x + \frac{561x + 77}{40} = \frac{357x^2 + 49x}{40} - 7.$$

Multiply by 40, $440x + 561x + 77 = 357x^2 + 49x - 280.$

Transpose and combine,

$$357x^2 - 952x - 357 = 0.$$

Divide by 119, $3x^2 - 8x - 3 = 0.$

Factor, $(x - 3)(3x + 1) = 0.$

$\therefore x = 3, \text{ or } -\tfrac{1}{3}.$

Substitute the value of x in (6), $y = 4, \text{ or } -\tfrac{1}{4}.$

Therefore, $x = 3$, and $y = 4$;

or $x = -\tfrac{1}{3}$, and $y = -\tfrac{1}{4}$.

48. Solve

$$\left.\begin{aligned}\frac{x+x^2}{y+y^2}&=\frac{14}{3}\\ \frac{y+x^2}{x+y^2}&=\frac{13}{4}\end{aligned}\right\}\qquad\begin{aligned}&(1)\\ &(2)\end{aligned}$$

Clear (1) of fractions, $3x+3x^2=14y+14y^2.$ (3)

Clear (2) of fractions, $4y+4x^2=13x+13y^2.$ (4)

Multiply (3) by 4, $12x^2+12x-56y^2-56y=0.$ (5)

Multiply (4) by (3), $12x^2-39x-39y^2+12y=0.$ (6)

Subtract (6) from (5), $51x-17y^2-68y=0.$

Divide by 17, $3x-y^2-4y=0.$

$$3x=y^2+4y.$$

$$\therefore\ x=\frac{y^2+4y}{3}.\qquad(7)$$

Substitute the value of x in (3),

$$y^2+4y+\frac{y^4+8y^3+16y^2}{3}=14y+14y^2.$$

$$3y^2+12y+y^4+8y^3+16y^2=42y+42y^2.$$

$$y^4+8y^3-23y^2-30y=0.$$

$$y(y^3+8y^2-23y-30)=0.$$

$$y(y+1)(y^2+7y-30)=0.$$

$$y(y+1)(y-3)(y+10)=0.$$

$$\therefore\ y=0,\ -1,\ 3,\text{ or }-10.$$

Substitute the value of y in (7), $x=0,\ -1,\ 7,\text{ or }20.$

The values $x=0$ and $y=0$, and $x=-1$ and $y=-1$, are found to be extraneous values and are introduced when the value of x, found in (7), is substituted in (3).

Therefore, $x=7$, and $y=3$;

or $x=20$, and $y=-10$.

49. Solve

$$\left.\begin{aligned}x^2+y^2&=x^2y^2+1\\ x+y&=2xy-2\end{aligned}\right\}\qquad\begin{aligned}&(1)\\ &(2)\end{aligned}$$

Square (2), $x^2+2xy+y^2=4x^2y^2-8xy+4.$ (3)

Subtract (1) from (3), $2xy=3x^2y^2-8xy+3.$

$$3x^2y^2-10xy+3=0.$$

Factor, $(xy-3)(3xy-1)=0.$

$$\therefore\ xy=3,\text{ or }\tfrac{1}{3}.\qquad(4)$$

Substitute the value of xy in (1), $x^2 + y^2 = 10$, or $\frac{10}{9}$. (5)

Substitute the value of xy in (2), $x + y = 4$, or $-\frac{4}{3}$. (6)

Multiply (4) by 2, $2xy = 6$, or $\frac{2}{3}$. (7)

Subtract (7) from (5), $x^2 - 2xy + y^2 = 4$, or $\frac{4}{9}$.

Extract the root, $x - y = \pm 2$, or $\pm \frac{2}{3}$. (8)

Add (6) and (8), $2x = 6,\ 2,\ -\frac{2}{3}$, or -2.

$\therefore\ x = 3,\ 1,\ -\frac{1}{3}$, or -1.

Substitute the value of x in (6), $y = 1,\ 3,\ -1$, or $-\frac{1}{3}$.

Therefore, $x = 3$, and $y = 1$;
$x = 1$, and $y = 3$;
$x = -\frac{1}{3}$, and $y = -1$;
or $x = -1$, and $y = -\frac{1}{3}$.

50. Solve

$$\left.\begin{aligned} x^3 + y^3 &= -147 - 3xy \\ x^2 + y^2 &= xy + 39 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Transpose (2), $x^2 - xy + y^2 = 39$. (3)

Divide (1) by (3), $x + y = \dfrac{-49 - xy}{13}$. (4)

Square (4), $x^2 + 2xy + y^2 = \dfrac{2401 + 98xy + x^2y^2}{169}$. (5)

Subtract (3) from (5), $3xy = \dfrac{2401 + 98xy + x^2y^2}{169} - 39$.

$507xy = 2401 + 98xy + x^2y^2 - 6591$.

$x^2y^2 - 409xy - 4190 = 0$.

Factor, $(xy - 419)(xy + 10) = 0$.

$\therefore\ xy = 419$, or -10. (6)

Substitute the value of xy in (4),

$x + y = -36$, or -3. (7)

Subtract (6) from (3), $x^2 - 2xy + y^2 = -380$, or 49.

Extract the root, $x - y = \pm 2\sqrt{-95}$, or ± 7. (8)

Add (7) and (8), $2x = -36 \pm 2\sqrt{-95},\ 4$, or -10.

$\therefore\ x = -18 \pm \sqrt{-95},\ 2$, or -5.

Subtract (8) from (7), $2y = -36 \mp 2\sqrt{-95},\ -10$, or 4.

$\therefore\ y = -18 \mp \sqrt{-95},\ -5$, or 2.

Therefore, $x = 2$, and $y = -5$;
$x = -5$, and $y = 2$;
$x = -18 + \sqrt{-95}$, and $y = -18 - \sqrt{-95}$;
or $x = -18 - \sqrt{-95}$, and $y = -18 + \sqrt{-95}$.

51. Solve
$$\left.\begin{aligned} x+y&=8 \\ x^4+y^4&=706 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Raise (1) to the 4th power,
$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=4096. \qquad (3)$$

Subtract (2) from (3),
$$4x^3y+6x^2y^2+4xy^3=3390.$$
$$2x^3y+3x^2y^2+2xy^3=1695.$$
$$xy(2x^2+3xy+2y^2)=1695. \qquad (4)$$

Square (1), $x^2+2xy+y^2=64.$ (5)
$$2x^2+4xy+2y^2=128.$$
$$2x^2+2y^2=128-4xy.$$

Substitute the value of $2x^2+2y^2$ in (4),
$$xy(128-xy)=1695.$$
$$128xy-x^2y^2=1695.$$
$$x^2y^2-128xy=-1695.$$

Complete the square, $x^2y^2-(\ )+4096=2401.$

Extract the root, $xy-64=\pm 49.$
$$\therefore xy=113, \text{ or } 15.$$
$$4xy=452, \text{ or } 60. \qquad (6)$$

Subtract (6) from (5), $x^2-2xy+y^2=-388, \text{ or } 4.$

Extract the root, $x-y=\pm 2\sqrt{-97}, \text{ or } \pm 2.$ (7)

Add (1) and (7), $2x=8\pm 2\sqrt{-97}, 10, \text{ or } 6.$
$$\therefore x=4\pm\sqrt{-97}, 5, \text{ or } 3.$$

Substitute the value of x in (1), $y=4\mp\sqrt{-97}, 3, \text{ or } 5.$

Therefore, $x=5$, and $y=3$;
$x=3$, and $y=5$;
$x=4+\sqrt{-97}$, and $y=4-\sqrt{-97}$;
or $x=4-\sqrt{-97}$, and $y=4+\sqrt{-97}$.

52. Solve
$$\left.\begin{aligned} x+y&=5 \\ x^5+y^5&=275 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Divide (2) by (1), $x^4-x^3y+x^2y^2-xy^3+y^4=55.$ (3)

Raise (1) to the 4th power,
$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=625. \qquad (4)$$

Subtract (3) from (4), $5x^3y+5x^2y^2+5xy^3=570.$
$$x^3y+x^2y^2+xy^3=114.$$
$$xy(x^2+xy+y^2)=114. \qquad (5)$$

Square (1), $x^2+2xy+y^2=25.$ (6)
$$x^2+y^2=25-2xy.$$

Substitute the value of $x^2 + y^2$ in (5),

$$xy(25 - xy) = 114.$$
$$25xy - x^2y^2 = 114.$$
$$x^2y^2 - 25xy + 114 = 0.$$
$$(xy - 6)(xy - 19) = 0.$$
$$\therefore\ xy = 6, \text{ or } 19.$$
$$4xy = 24, \text{ or } 76. \qquad (7)$$

Subtract (7) from (6), $x^2 - 2xy + y^2 = 1, \text{ or } -51.$

Extract the root, $x - y = \pm 1, \text{ or } \pm\sqrt{-51}.$ (8)

Add (1) and (8), $2x = 6, 4, \text{ or } 5 \pm \sqrt{-51}.$

$$\therefore\ x = 3, 2, \text{ or } \tfrac{1}{2}(5 \pm \sqrt{-51}).$$

Substitute the value of x in (1), $y = 2, 3, \text{ or } \tfrac{1}{2}(5 \mp \sqrt{-51}).$

Therefore, $x = 3$, and $y = 2$;
$x = 2$, and $y = 3$;
$x = \tfrac{1}{2}(5 + \sqrt{-51})$, and $y = \tfrac{1}{2}(5 - \sqrt{-51})$;
or $x = \tfrac{1}{2}(5 - \sqrt{-51})$, and $y = \tfrac{1}{2}(5 + \sqrt{-51})$.

53. Solve

$$\left.\begin{aligned} x^3 + 3xy^2 + 171 &= 0 \qquad (1)\\ 3x^2y + y^3 + 172 &= 0 \qquad (2)\end{aligned}\right\}$$

Subtract (2) from (1),

$$x^3 - 3x^2y + 3xy^2 - y^3 - 1 = 0.$$
$$(x - y)^3 - 1 = 0.$$

Factor,

$$(x - y - 1)[(x - y)^2 + (x - y) + 1] = 0.$$
$$\therefore\ x - y - 1 = 0,$$

or

$$(x - y)^2 + (x - y) + 1 = 0.$$

(i) $x - y = 1.$

(ii) $(x - y)^2 + (x - y) = -1.$

$$4(x - y)^2 + (\ ) + 1 = -3.$$
$$2(x - y) + 1 = \pm\sqrt{-3}.$$
$$2(x - y) = -1 \pm \sqrt{-3}.$$
$$\therefore\ x - y = \tfrac{1}{2}(-1 \pm \sqrt{-3}).$$
$$\therefore\ x - y = 1, \text{ or } \tfrac{1}{2}(-1 \pm \sqrt{-3}). \qquad (3)$$

Add (1) and (2),

$$x^3 + 3x^2y + 3xy^2 + y^3 + 343 = 0.$$
$$(x + y)^3 + 343 = 0.$$

Factor,

$$(x + y + 7)[(x + y)^2 - 7(x + y) + 49] = 0.$$
$$\therefore\ x + y + 7 = 0,$$

or

$$(x + y)^2 - 7(x + y) + 49 = 0.$$

(i) $$x + y = -7.$$

(ii) $$(x + y)^2 - 7(x + y) = -49.$$

$$4(x + y)^2 - (\ ) + 49 = -147.$$

$$2(x + y) - 7 = \pm 7\sqrt{-3}.$$

$$2(x + y) = 7 \pm 7\sqrt{-3}.$$

$$\therefore\ x + y = \tfrac{7}{2}(1 \pm \sqrt{-3}).$$

$$\therefore\ x + y = -7, \text{ or } \tfrac{7}{2}(1 \pm \sqrt{-3}). \qquad (4)$$

Add (3) and (4),

$$2x = -6,\ \tfrac{1}{2}(-15 \pm \sqrt{-3}),\ \tfrac{1}{2}(9 \pm 7\sqrt{-3}),\ 3 \pm 4\sqrt{-3}, \text{ or } 3(1 \pm \sqrt{-3}).$$

$$\therefore\ x = -3,\ \tfrac{1}{4}(-15 \pm \sqrt{-3}),\ \tfrac{1}{4}(9 \pm 7\sqrt{-3}),\ \tfrac{1}{2}(3 \pm 4\sqrt{-3}), \text{ or } \tfrac{3}{2}(1 \pm \sqrt{-3}).$$

Subtract (3) from (4),

$$2y = -8,\ \tfrac{1}{2}(-13 \mp \sqrt{-3}),\ \tfrac{1}{2}(5 \pm 7\sqrt{-3}),\ 4 \pm 3\sqrt{-3}, \text{ or } 4(1 \pm \sqrt{-3}).$$

$$\therefore\ y = -4,\ \tfrac{1}{4}(-13 \mp \sqrt{-3}),\ \tfrac{1}{4}(5 \pm 7\sqrt{-3}),\ \tfrac{1}{2}(4 \pm 3\sqrt{-3}), \text{ or } 2(1 \pm \sqrt{-3}).$$

Therefore, $x = -3$, and $y = -4$;

$x = \tfrac{1}{4}(-15 + \sqrt{-3})$, and $y = \tfrac{1}{4}(-13 - \sqrt{-3})$;

$x = \tfrac{1}{4}(-15 - \sqrt{-3})$, and $y = \tfrac{1}{4}(-13 + \sqrt{-3})$;

$x = \tfrac{1}{4}(9 + 7\sqrt{-3})$, and $y = \tfrac{1}{4}(5 + 7\sqrt{-3})$;

$x = \tfrac{1}{4}(9 - 7\sqrt{-3})$, and $y = \tfrac{1}{4}(5 - 7\sqrt{-3})$;

$x = \tfrac{1}{2}(3 + 4\sqrt{-3})$, and $y = \tfrac{1}{2}(4 + 3\sqrt{-3})$;

$x = \tfrac{1}{2}(3 - 4\sqrt{-3})$, and $y = \tfrac{1}{2}(4 - 3\sqrt{-3})$;

$x = \tfrac{3}{2}(1 + \sqrt{-3})$, and $y = 2(1 + \sqrt{-3})$;

or $x = \tfrac{3}{2}(1 - \sqrt{-3})$, and $y = 2(1 - \sqrt{-3})$.

54. Solve

$$\left.\begin{aligned} x^4 + y^4 &= 97 \\ x^2 + y^2 &= 49 - x^2y^2 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

From (2), $$x^2 + x^2y^2 + y^2 = 49.$$

$$2x^2 + 2x^2y^2 + 2y^2 = 98. \qquad (3)$$

Add (1) and (3),

$$x^4 + 2x^2y^2 + y^4 + 2x^2 + 2y^2 = 195.$$

$$(x^2 + y^2)^2 + 2(x^2 + y^2) = 195.$$

Complete the square,

$$(x^2 + y^2)^2 + (\ ) + 1 = 196.$$

Extract the root, $$x^2 + y^2 + 1 = \pm 14.$$

$$x^2 + y^2 = 13, \text{ or } -15. \qquad (4)$$

Substitute the value of x^2+y^2 in (2),

$$x^2y^2=36, \text{ or } 64.$$

$$\therefore\ xy=\pm 6, \text{ or } \pm 8.$$

$$2xy=\pm 12, \text{ or } \pm 16. \qquad (5)$$

Add (4) and (5), $x^2+2xy+y^2=25,\ 1,\ 1, \text{ or } -31.$

$$x+y=\pm 5,\ \pm 1,\ \pm 1, \text{ or } \pm\sqrt{-31}. \qquad (6)$$

Subtract (5) from (4),

$$x^2-2xy+y^2=1,\ 25,\ -31, \text{ or } 1.$$

$$x-y=\pm 1,\ \pm 5,\ \pm\sqrt{-31}, \text{ or } \pm 1. \qquad (7)$$

Add (6) and (7),

$$2x=6,\ 4,\ -4,\ -6,\ 6,\ -4,\ 4,\ -6,\ 1+\sqrt{-31},\ 1-\sqrt{-31},\ -1+\sqrt{-31},\ -1-\sqrt{-31};\ 1+\sqrt{-31},\ -1+\sqrt{-31},\ 1-\sqrt{-31}, \text{ or } -1-\sqrt{-31}.$$

$$\therefore\ x=3,\ 2,\ -2,\ -3,\ 3,\ -2,\ 2,\ -3,\ \tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31},\ \tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31},\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31},\ -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31},\ \tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31},\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31},\ \tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}, \text{ or } -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}.$$

Substitute the value of x in (6),

$$y=2,\ 3,\ -3,\ -2,\ -2,\ 3,\ -3,\ 2,\ \tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31},\ \tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31},\ -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31},\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31},\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31},\ \tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31},\ -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}, \text{ or } \tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}.$$

Therefore, $x=3$, and $y=2$;
$x=2$, and $y=3$;
$x=-2$, and $y=-3$;
$x=-3$, and $y=-2$;
$x=3$, and $y=-2$;
$x=-2$, and $y=3$;
$x=2$, and $y=-3$;
$x=-3$, and $y=2$;
$x=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31}$, and $y=\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}$;
$x=\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}$, and $y=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31}$;
$x=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31}$, and $y=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}$;
$x=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}$, and $y=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31}$;
$x=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31}$, and $y=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31}$;
$x=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31}$, and $y=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-31}$;
$x=\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}$, and $y=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}$;
or $x=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}$, and $y=\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-31}$.

55. Solve

$$\left.\begin{aligned} 16x^2 + 16y &= 17 \\ 4x + 4y^2 &= 5 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

From (1), $16x^2 = 17 - 16y.$ (3)

From (2), $4x = 5 - 4y^2.$

Square, $16x^2 = 25 - 40y^2 + 16y^4.$ (4)

Equate the values of $16x^2$,

$$25 - 40y^2 + 16y^4 = 17 - 16y.$$

$$16y^4 - 40y^2 + 16y + 8 = 0.$$

Divide by 8, $2y^4 - 5y^2 + 2y + 1 = 0.$

$$(y-1)(2y^3 + 2y^2 - 3y - 1) = 0.$$

$$(y-1)^2(2y^2 + 4y + 1) = 0.$$

$$\therefore\ y = 1, \text{ or } 1, \text{ or } 2y^2 + 4y + 1 = 0.$$

Transpose, $2y^2 + 4y = -1.$

Complete the square, $4y^2 + 8y + 4 = 2.$

Extract the root, $2y + 2 = \pm\sqrt{2}.$

$$2y = -2 \pm \sqrt{2}.$$

$$y = \tfrac{1}{2}(-2 \pm \sqrt{2}).$$

$$\therefore\ y = 1,\ 1, \text{ or } \tfrac{1}{2}(-2 \pm \sqrt{2}).$$

Substitute the value of y in (2), $4x = 1,\ 1, \text{ or } -1 \pm 4\sqrt{2}.$

$$\therefore\ x = \tfrac{1}{4},\ \tfrac{1}{4}, \text{ or } \tfrac{1}{4}(-1 \pm 4\sqrt{2}).$$

Therefore, $x = \frac{1}{4}$, and $y = 1$;

$x = \frac{1}{4}$, and $y = 1$;

$x = \frac{1}{4}(-1 + 4\sqrt{2})$, and $y = \frac{1}{2}(-2 + \sqrt{2})$;

or $x = \frac{1}{4}(-1 - 4\sqrt{2})$, and $y = \frac{1}{2}(-2 - \sqrt{2})$.

56. Solve

$$\left.\begin{aligned} x + xy + y &= 5 \\ x^3 + x^3y^3 + y^3 &= 17 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

From (1), $x + y = 5 - xy.$ (3)

Cube (3), $x^3 + 3x^2y + 3xy^2 + y^3 = 125 - 75xy + 15x^2y^2 - x^3y^3.$ (4)

From (2), $x^3 + y^3 = 17 - x^3y^3.$ (5)

Subtract (5) from (4),

$$3x^2y + 3xy^2 = 108 - 75xy + 15x^2y^2.$$

Divide by 3, $xy(x + y) = 36 - 25xy + 5x^2y^2.$ (6)

Substitute in (6) the value of $x + y$ from (3),

$$5xy - x^2y^2 = 36 - 25xy + 5x^2y^2.$$

$$6x^2y^2 - 30xy + 36 = 0.$$

$$x^2y^2 - 5xy + 6 = 0.$$

Factor, $(xy - 2)(xy - 3) = 0.$

$$\therefore xy = 2, \text{ or } 3. \qquad (7)$$

Substitute the value of xy in (3), $x + y = 3$, or 2. (8)

Square (8), $x^2 + 2xy + y^2 = 9$, or 4. (9)

Multiply (7) by 4, $4xy = 8$, or 12. (10)

Subtract (10) from (9), $x^2 - 2xy + y^2 = 1$, or -8.

Extract the root, $x - y = \pm 1$, or $\pm 2\sqrt{-2}$. (11)

Add (8) and (11), $2x = 4, 2$, or $2 \pm 2\sqrt{-2}$.

$$\therefore x = 2, 1, \text{ or } 1 \pm \sqrt{-2}.$$

Subtract (11) from (8), $2y = 2, 4$, or $2 \mp 2\sqrt{-2}$.

$$\therefore y = 1, 2, \text{ or } 1 \mp \sqrt{-2}.$$

Therefore, $x = 2$, and $y = 1$;

$x = 1$, and $y = 2$;

$x = 1 + \sqrt{-2}$, and $y = 1 - \sqrt{-2}$;

or $x = 1 - \sqrt{-2}$, and $y = 1 + \sqrt{-2}$.

57. Solve

$$\left.\begin{aligned} xy &= a \\ \frac{x}{y} &= b \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

From (1), $x = \dfrac{a}{y}$. (3)

From (2), $x = by$. (4)

Equate the values of x, $by = \dfrac{a}{y}$.

$$by^2 = a.$$

$$\therefore y^2 = \frac{a}{b}.$$

$$\therefore y = \pm \frac{1}{b}\sqrt{ab}.$$

Substitute the value of y in (4), $x = \pm\sqrt{ab}$.

Therefore, $x = \sqrt{ab}$, and $y = \dfrac{1}{b}\sqrt{ab}$;

or $x = -\sqrt{ab}$, and $y = -\dfrac{1}{b}\sqrt{ab}$.

58. Solve

$$\left.\begin{aligned} x^2 + xy &= a \\ y^2 + xy &= b \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Add (1) and (2), $x^2 + 2xy + y^2 = a + b.$

Extract the square root, $x + y = \pm\sqrt{a + b}.$ (3)

Factor (1), $x(x + y) = a.$

$$\therefore\ x = \frac{a}{x + y}. \qquad (4)$$

Factor (2), $y(x + y) = b.$

$$\therefore\ y = \frac{b}{x + y}. \qquad (5)$$

Substitute in (4) the value of $x + y$ from (3), $x = \pm\dfrac{a}{\sqrt{a + b}}.$

Substitute in (5) the value of $x + y$ from (3), $y = \pm\dfrac{b}{\sqrt{a + b}}.$

Therefore, $x = \dfrac{a}{\sqrt{a + b}}$, and $y = \dfrac{b}{\sqrt{a + b}}$;

or $x = -\dfrac{a}{\sqrt{a + b}}$, and $y = -\dfrac{b}{\sqrt{a + b}}.$

59. Solve

$$\left.\begin{aligned} x^2y &= a \\ xy^2 &= b \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Divide (1) by (2), $\dfrac{x}{y} = \dfrac{a}{b}.$

$$\therefore\ x = \frac{ay}{b}. \qquad (3)$$

Substitute the value of x in (1), $\dfrac{a^2y^3}{b^2} = a.$

$$\therefore\ y^3 = \frac{b^2}{a}.$$

$$\therefore\ y = \sqrt[3]{\frac{b^2}{a}} = \frac{1}{a}\sqrt[3]{a^2b^2}.$$

Substitute the value of y in (3), $x = \dfrac{1}{b}\sqrt[3]{a^2b^2}.$

Therefore, $x = \dfrac{1}{b}\sqrt[3]{a^2b^2}$, and $y = \dfrac{1}{a}\sqrt[3]{a^2b^2}$;

$$x = \frac{1}{2b}\sqrt[3]{a^2b^2}(-1 + \sqrt{-3}), \text{ and } y = \frac{1}{2a}\sqrt[3]{a^2b^2}(-1 + \sqrt{-3});$$

or $$x = \frac{1}{2b}\sqrt[3]{a^2b^2}(-1 - \sqrt{-3}), \text{ and } y = \frac{1}{2a}\sqrt[3]{a^2b^2}(-1 - \sqrt{-3}).$$

60. Solve
$$\left.\begin{aligned} x+y&=a(x^2+y^2)\\ x-y&=b(x^2+y^2)\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Divide (1) by (2), $\dfrac{x+y}{x-y}=\dfrac{a}{b}.$

Clear of fractions, $bx+by=ax-ay.$

Transpose, $ax-bx=ay+by.$

$$\therefore\ x=\frac{a+b}{a-b}y. \qquad (3)$$

Substitute the value of x in (1),
$$\frac{a+b}{a-b}y+y=a\left(\frac{a^2+2ab+b^2}{a^2-2ab+b^2}y^2+y^2\right).$$

Simplify, $\dfrac{2a}{a-b}y=\dfrac{2a(a^2+b^2)}{a^2-2ab+b^2}y^2.$

Multiply by $\dfrac{a-b}{2a}$, $y=\dfrac{a^2+b^2}{a-b}y^2.$

Transpose, $\dfrac{a^2+b^2}{a-b}y^2-y=0.$

Factor, $y\left(\dfrac{a^2+b^2}{a-b}y-1\right)=0.$

$$\therefore\ y=0,\text{ or }\frac{a^2+b^2}{a-b}y=1.$$

$$\therefore\ y=0,\text{ or }\frac{a-b}{a^2+b^2}.$$

Substitute the value of y in (3), $x=0,\text{ or }\dfrac{a+b}{a^2+b^2}.$

Therefore, $x=0$, and $y=0$;

or $x=\dfrac{a+b}{a^2+b^2}$, and $y=\dfrac{a-b}{a^2+b^2}.$

61. Solve
$$\left.\begin{aligned} x+y&=a\\ x^3+y^3&=b\end{aligned}\right\}\qquad\begin{matrix}(1)\\(2)\end{matrix}$$

Divide (2) by (1), $x^2-xy+y^2=\dfrac{b}{a}.$ (3)

Square (1), $x^2+2xy+y^2=a^2.$ (4)

Subtract (3) from (4), $3xy=a^2-\dfrac{b}{a},$

or $3xy=\dfrac{a^3-b}{a}.$

Divide by 3, $$xy = \frac{a^3 - b}{3a}. \tag{5}$$

Subtract (5) from (3), $$x^2 - 2xy + y^2 = \frac{b}{a} - \frac{a^3 - b}{3a},$$

or $$x^2 - 2xy + y^2 = \frac{4b - a^3}{3a}.$$

Extract the square root, $$x - y = \pm\sqrt{\frac{4b - a^3}{3a}}. \tag{6}$$

Add (1) and (6), $$2x = a \pm \sqrt{\frac{4b - a^3}{3a}}.$$

$$\therefore\ x = \frac{1}{2}\left(a \pm \sqrt{\frac{4b - a^3}{3a}}\right).$$

Subtract (6) from (1), $$2y = a \mp \sqrt{\frac{4b - a^3}{3a}}.$$

$$\therefore\ y = \frac{1}{2}\left(a \mp \sqrt{\frac{4b - a^3}{3a}}\right).$$

Therefore, $x = \frac{1}{2}\left(a + \sqrt{\frac{4b - a^3}{3a}}\right)$, and $y = \frac{1}{2}\left(a - \sqrt{\frac{4b - a^3}{3a}}\right)$;

or $x = \frac{1}{2}\left(a - \sqrt{\frac{4b - a^3}{3a}}\right)$, and $y = \frac{1}{2}\left(a + \sqrt{\frac{4b - a^3}{3a}}\right)$.

62. Solve
$$\left.\begin{aligned} \frac{a - x}{b - y} + \frac{b - y}{a - x} &= \frac{34}{15} \quad (1)\\ x - y &= 3(a - b) \quad (2) \end{aligned}\right\}$$

From (2), $$x = 3a - 3b + y. \tag{3}$$

Substitute the value of x in (1),

$$\frac{3b - 2a - y}{b - y} + \frac{b - y}{3b - 2a - y} = \frac{34}{15}.$$

$$9b^2 + 4a^2 + y^2 - 12ab - 6by + 4ay + b^2 - 2by + y^2$$
$$= \tfrac{34}{15}(3b^2 - 2ab - 4by + 2ay + y^2).$$

$$2y^2 + 4ay - 8by + 4a^2 - 12ab + 10b^2$$
$$= \tfrac{34}{15}(3b^2 - 2ab - 4by + 2ay + y^2).$$

$$30y^2 + 60ay - 120by + 60a^2 - 180ab + 150b^2$$
$$= 102b^2 - 68ab - 136by + 68ay + 34y^2.$$

$$4y^2 + 8ay - 16by = 60a^2 - 112ab + 48b^2.$$
$$y^2 + 2(a - 2b)y = 15a^2 - 28ab + 12b^2.$$
$$y^2 + (\ ) + (a - 2b)^2 = 16a^2 - 32ab + 16b^2.$$
$$y + (a - 2b) = \pm(4a - 4b).$$
$$\therefore\ y = 3a - 2b, \text{ or } 6b - 5a.$$

Substitute the value of y in (3), $x = 6a - 5b$, or $3b - 2a$.

Therefore, $x = 6a - 5b$, and $y = 3a - 2b$;
or $x = 3b - 2a$, and $y = 6b - 5a$.

63. Solve
$$\left.\begin{aligned} x + y &= a \\ \frac{x}{b-y} + \frac{b-y}{x} &= \frac{5}{2} \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

From (1), $y = a - x.$ (3)

Substitute the value of y in (2),
$$\frac{x}{b-a+x} + \frac{b-a+x}{x} = \frac{5}{2}.$$

Clear of fractions,
$$2x^2 + 2(b-a)^2 + 4(b-a)x + 2x^2 = 5(b-a)x + 5x^2.$$
$$x^2 + (b-a)x - 2(b-a)^2 = 0.$$

Factor, $[x + 2(b-a)][x - (b-a)] = 0.$
$$\therefore\ x = 2a - 2b, \text{ or } b - a.$$

Substitute the value of x in (3), $y = 2b - a$, or $2a - b$.

Therefore, $x = 2a - 2b$, and $y = 2b - a$;
or $x = b - a$, and $y = 2a - b$.

64. Solve
$$\left.\begin{aligned} x^3 + xy^2 &= a \\ y^3 + x^2y &= b \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Divide (1) by (2), $\dfrac{x}{y} = \dfrac{a}{b}.$
$$\therefore\ x = \frac{ay}{b}. \qquad (3)$$

Substitute the value of x in (2), $y^3 + \dfrac{a^2y^3}{b^2} = b.$
$$b^2y^3 + a^2y^3 = b^3.$$
$$y^3 = \frac{b^3}{a^2 + b^2}.$$

$$\therefore\ y = \frac{b}{a^2 + b^2}\sqrt[3]{(a^2 + b^2)^2}.$$

Substitute the value of y in (3), $x = \dfrac{a}{a^2 + b^2}\sqrt[3]{(a^2 + b^2)^2}.$

Therefore, $x = \dfrac{a}{a^2 + b^2}\sqrt[3]{(a^2 + b^2)^2}$, and $y = \dfrac{b}{a^2 + b^2}\sqrt[3]{(a^2 + b^2)^2}$;

$x = \dfrac{a\omega}{a^2 + b^2}\sqrt[3]{(a^2 + b^2)^2}$, and $y = \dfrac{b\omega}{a^2 + b^2}\sqrt[3]{(a^2 + b^2)^2}$;

or $x = \dfrac{a\omega^2}{a^2 + b^2}\sqrt[3]{(a^2 + b^2)^2}$, and $y = \dfrac{b\omega^2}{a^2 + b^2}\sqrt[3]{(a^2 + b^2)^2}$;

where $\omega = \frac{1}{2}(-1 + \sqrt{-3})$, and $\omega^2 = \frac{1}{2}(-1 - \sqrt{-3})$.

65. Solve

$$\left.\begin{aligned} x + y^2 &= ax \\ x^2 + y &= by \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

From (1), $$y^2 = ax - x.$$

$$\therefore\ y = \pm\sqrt{ax - x}. \qquad (3)$$

From (2), $$y = \frac{x^2}{b - 1}. \qquad (4)$$

Equate the values of y, $$\frac{x^2}{b - 1} = \pm\sqrt{ax - x}.$$

Square, $$\frac{x^4}{(b - 1)^2} = ax - x. \qquad (5)$$

The equation is satisfied if $x = 0.$

Substitute the value of x in (4), $y = 0.$

If x is not 0, divide (5) by x, $$\frac{x^3}{(b - 1)^2} = a - 1.$$

$$x^3 = (a - 1)(b - 1)^2.$$

$$\therefore\ x = \sqrt[3]{(a - 1)(b - 1)^2}.$$

Substitute the value of x in (4), $y = \sqrt[3]{(a - 1)^2(b - 1)}.$

Therefore, $x = 0$, and $y = 0$;

$x = \sqrt[3]{(a - 1)(b - 1)^2}$, and $y = \sqrt[3]{(a - 1)^2(b - 1)}$;

$x = \omega\sqrt[3]{(a - 1)(b - 1)^2}$, and $y = \omega^2\sqrt[3]{(a - 1)^2(b - 1)}$;

or $x = \omega^2\sqrt[3]{(a - 1)(b - 1)^2}$, and $y = \omega\sqrt[3]{(a - 1)^2(b - 1)}$;

where $\omega = \frac{1}{2}(-1 + \sqrt{-3})$, and $\omega^2 = \frac{1}{2}(-1 - \sqrt{-3})$.

66. Solve

$$\left.\begin{aligned} x + y^2 &= ay^2 \\ x^2 + y &= bx^2 \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

From (1), $$x = (a-1)y^2. \qquad (3)$$

Square, $$x^2 = (a-1)^2 y^4. \qquad (4)$$

Transpose (2), $$bx^2 - x^2 = y.$$

$$\therefore\ x^2 = \frac{y}{b-1}. \qquad (5)$$

From (4) and (5), $$(a-1)^2 y^4 = \frac{y}{b-1}. \qquad (6)$$

The equation is satisfied if $y = 0$.

Substitute the value of y in (3), $x = 0$.

If y is not 0, divide (6) by y, $$(a-1)^2 y^3 = \frac{1}{b-1}.$$

$$\therefore\ y^3 = \frac{1}{(a-1)^2(b-1)}.$$

Extract the cube root, $$y = \frac{\sqrt[3]{(a-1)(b-1)^2}}{(a-1)(b-1)}.$$

Substitute the value of y in (3), $$x = \frac{\sqrt[3]{(a-1)^2(b-1)}}{(a-1)(b-1)}.$$

Therefore, $x = 0$, and $y = 0$;

$$x = \frac{\sqrt[3]{(a-1)^2(b-1)}}{(a-1)(b-1)}, \text{ and } y = \frac{\sqrt[3]{(a-1)(b-1)^2}}{(a-1)(b-1)};$$

$$x = \frac{\omega\sqrt[3]{(a-1)^2(b-1)}}{(a-1)(b-1)}, \text{ and } y = \frac{\omega^2\sqrt[3]{(a-1)(b-1)^2}}{(a-1)(b-1)};$$

or $$x = \frac{\omega^2\sqrt[3]{(a-1)^2(b-1)}}{(a-1)(b-1)}, \text{ and } y = \frac{\omega\sqrt[3]{(a-1)(b-1)^2}}{(a-1)(b-1)};$$

where $\omega = \frac{1}{2}(-1 + \sqrt{-3})$, and $\omega^2 = \frac{1}{2}(-1 - \sqrt{-3})$.

67. Solve

$$\left.\begin{aligned} x + y &= a \\ x^3 + y^3 &= bxy \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Divide (2) by (1), $$x^2 - xy + y^2 = \frac{bxy}{a}. \qquad (3)$$

Square (1), $$x^2 + 2xy + y^2 = a^2. \qquad (4)$$

Subtract (3) from (4), $$3xy = a^2 - \frac{bxy}{a}.$$

Multiply by a, $$3axy = a^3 - bxy.$$

Transpose and combine, $$(3a + b)xy = a^3.$$

$$\therefore\ xy = \frac{a^3}{3a+b}.$$

Multiply by 4, $$4xy = \frac{4a^3}{3a+b}. \qquad (5)$$

Subtract (5) from (4), $$x^2 - 2xy + y^2 = \frac{a^2(b-a)}{3a+b}.$$

Extract the square root, $$x - y = \pm a\sqrt{\frac{b-a}{3a+b}}. \qquad (6)$$

Add (1) and (6), $$2x = a \pm a\sqrt{\frac{b-a}{3a+b}}.$$

$$\therefore\ x = \frac{1}{2}a\left(1 \pm \sqrt{\frac{b-a}{3a+b}}\right).$$

Subtract (6) from (1), $$2y = a \mp a\sqrt{\frac{b-a}{3a+b}}.$$

$$\therefore\ y = \frac{1}{2}a\left(1 \mp \sqrt{\frac{b-a}{3a-b}}\right).$$

Therefore, $x = \frac{1}{2}a\left(1 + \sqrt{\frac{b-a}{3a+b}}\right)$, and $y = \frac{1}{2}a\left(1 - \sqrt{\frac{b-a}{3a+b}}\right)$;

or $x = \frac{1}{2}a\left(1 - \sqrt{\frac{b-a}{3a+b}}\right)$, and $y = \frac{1}{2}a\left(1 + \sqrt{\frac{b-a}{3a+b}}\right)$.

68. Solve $$\left.\begin{aligned} x^3 + y^3 &= a(x+y) \\ x^4 + y^4 &= b(x+y)^2 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

From (1), $$x^3 + y^3 - a(x+y) = 0.$$

$$(x+y)(x^2 - xy + y^2) - a(x+y) = 0.$$

Factor, $$(x+y)(x^2 - xy + y^2 - a) = 0.$$

$$\therefore\ x + y = 0, \text{ or } x^2 - xy + y^2 - a = 0.$$

(i) $$x + y = 0.$$

$$\therefore\ x = -y. \qquad (3)$$

Substitute the value of x in (2), $$y^4 + y^4 = b(-y+y)^2.$$

$$2y^4 = 0.$$

$$y^4 = 0.$$

$$\therefore\ y = 0.$$

Substitute the value of y in (3), $$x = 0.$$

(ii) $$x^2 - xy + y^2 - a = 0.$$

$$x^2 + y^2 = xy + a. \qquad (4)$$

Square, $$x^4 + 2x^2y^2 + y^4 = x^2y^2 + 2axy + a^2. \qquad (5)$$

From (2), $x^4 + y^4 = bx^2 + 2bxy + by^2.$ (6)

Subtract (6) from (5),

$$2x^2y^2 = x^2y^2 + 2axy - 2bxy - b(x^2 + y^2) + a^2. \quad (7)$$

Substitute in (7) the value of $x^2 + y^2$ from (4),

$$2x^2y^2 = x^2y^2 + 2axy - 2bxy - bxy - ab + a^2.$$

$$x^2y^2 - (2a - 3b)xy = a^2 - ab.$$

$$4x^2y^2 - (\quad) + (2a - 3b)^2 = 8a^2 - 16ab + 9b^2.$$

$$2xy - (2a - 3b) = \pm\sqrt{8a^2 - 16ab + 9b^2}.$$

$$2xy = 2a - 3b \pm \sqrt{8a^2 - 16ab + 9b^2}.$$

$$\therefore\ xy = a - \tfrac{3}{2}b \pm \tfrac{1}{2}\sqrt{8a^2 - 16ab + 9b^2}. \quad (8)$$

From (4), $x^2 - xy + y^2 = a.$ (9)

Subtract (8) from (9),

$$x^2 - 2xy + y^2 = \tfrac{3}{2}b \mp \tfrac{1}{2}\sqrt{8a^2 - 16ab + 9b^2}.$$

$$x - y = \pm\sqrt{\tfrac{3}{2}b \mp \tfrac{1}{2}\sqrt{8a^2 - 16ab + 9b^2}}. \quad (10)$$

Multiply (8) by 3, $3xy = 3a - \tfrac{9}{2}b \pm \tfrac{3}{2}\sqrt{8a^2 - 16ab + 9b^2}.$ (11)

Add (9) and (11),

$$x^2 + 2xy + y^2 = 4a - \tfrac{9}{2}b \pm \tfrac{3}{2}\sqrt{8a^2 - 16ab + 9b^2}.$$

$$x + y = \pm\sqrt{4a - \tfrac{9}{2}b \pm \tfrac{3}{2}\sqrt{8a^2 - 16ab + 9b^2}}. \quad (12)$$

Put r for $\tfrac{1}{2}\sqrt{8a^2 - 16ab + 9b^2}$.

Then $x + y = \pm\sqrt{4a - \tfrac{9}{2}b \pm 3r},$ (13)

and $x - y = \pm\sqrt{\tfrac{3}{2}b \mp r}.$ (14)

Add (13) and (14), $2x = \pm\sqrt{4a - \tfrac{9}{2}b \pm 3r} \pm \sqrt{\tfrac{3}{2}b \mp r}.$

$$\therefore\ x = \pm\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b \pm 3r} \pm \tfrac{1}{2}\sqrt{\tfrac{3}{2}b \mp r}.$$

Subtract (14) from (13),

$$2y = \pm\sqrt{4a - \tfrac{9}{2}b \pm 3r} \mp \sqrt{\tfrac{3}{2}b \mp \tfrac{1}{2}r}.$$

$$\therefore\ y = \pm\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b \pm 3r} \mp \tfrac{1}{2}\sqrt{\tfrac{3}{2}b \mp \tfrac{1}{2}r}.$$

Therefore,

$x = 0$, and $y = 0$;

$x = \tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b + 3r} + \tfrac{1}{2}\sqrt{\tfrac{3}{2}b - r}$, and $y = \tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b + 3r} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}b - r}$;

$x = \tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b + 3r} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}b - r}$, and $y = \tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b + 3r} + \tfrac{1}{2}\sqrt{\tfrac{3}{2}b - r}$;

$x = -\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b + 3r} + \tfrac{1}{2}\sqrt{\tfrac{3}{2}b - r}$, and $y = -\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b + 3r} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}b - r}$;

$x = -\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b + 3r} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}b - r}$, and $y = -\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b + 3r} + \tfrac{1}{2}\sqrt{\tfrac{3}{2}b - r}$;

$x = \tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b - 3r} + \tfrac{1}{2}\sqrt{\tfrac{3}{2}b + r}$, and $y = \tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b - 3r} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}b + r}$;

$x = \tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b - 3r} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}b + r}$, and $y = \tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b - 3r} + \tfrac{1}{2}\sqrt{\tfrac{3}{2}b + r}$;

$x = -\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b - 3r} + \tfrac{1}{2}\sqrt{\tfrac{3}{2}b + r}$, and $y = -\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b - 3r} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}b + r}$;

$x = -\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b - 3r} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}b + r}$, and $y = -\tfrac{1}{2}\sqrt{4a - \tfrac{9}{2}b - 3r} + \tfrac{1}{2}\sqrt{\tfrac{3}{2}b + r}$;

where $r = \tfrac{1}{2}\sqrt{8a^2 - 16ab + 9b^2}$.

69. Solve
$$\left.\begin{array}{l} x^2 - y^2 = a \\ x^3 + y^3 = b(x - y) \end{array}\right\} \qquad \begin{array}{r} (1) \\ (2) \end{array}$$

From (1),
$$x^2 = y^2 + a. \qquad (3)$$
$$\therefore\ x = \pm\sqrt{y^2 + a}.$$

Substitute the value of x in (2),
$$\pm(y^2 + a)\sqrt{y^2 + a} + y^3 = \pm b\sqrt{y^2 + a} - by.$$
$$\pm(y^2 + a - b)\sqrt{y^2 + a} = -y^3 - by.$$

Square, $[y^4 + 2(a - b)y^2 + (a^2 - 2ab + b^2)](y^2 + a) = y^6 + 2by^4 + b^2y^2.$

$$y^6 + (3a - 2b)y^4 + (3a^2 - 4ab + b^2)y^2 + a^3 - 2a^2b + ab^2 = y^6 + 2by^4 + b^2y^2.$$

Transpose and combine,
$$(3a - 4b)y^4 + (3a^2 - 4ab)y^2 = -a^3 + 2a^2b - ab^2.$$

Divide by $3a - 4b$, $y^4 + ay^2 = \dfrac{-a(a - b)^2}{3a - 4b}.$

$$4y^4 + (\ ) + a^2 = \frac{a(a - 2b)^2}{4b - 3a}.$$
$$2y^2 + a = \pm(a - 2b)\sqrt{\frac{a}{4b - 3a}}.$$
$$2y^2 = -a \pm (a - 2b)\sqrt{\frac{a}{4b - 3a}}.$$
$$y^2 = -\frac{1}{2}a \pm \frac{1}{2}(a - 2b)\sqrt{\frac{a}{4b - 3a}}.$$
$$\therefore\ y = \pm\frac{1}{2}\sqrt{-2a \pm 2(a - 2b)\sqrt{\frac{a}{4b - 3a}}}.$$

Substitute the value of y^2 in (3),
$$x^2 = \frac{1}{2}a \pm \frac{1}{2}(a - 2b)\sqrt{\frac{a}{4b - 3a}}.$$
$$\therefore\ x = \pm\frac{1}{2}\sqrt{2a \pm 2(a - 2b)\sqrt{\frac{a}{4b - 3a}}}.$$

Therefore, $x = \frac{1}{2}\sqrt{2a + r}$, and $y = \frac{1}{2}\sqrt{-2a + r}$;
$x = -\frac{1}{2}\sqrt{2a + r}$, and $y = -\frac{1}{2}\sqrt{-2a + r}$;
$x = \frac{1}{2}\sqrt{2a - r}$, and $y = \frac{1}{2}\sqrt{-2a - r}$;
or $x = -\frac{1}{2}\sqrt{2a - r}$, and $y = -\frac{1}{2}\sqrt{-2a - r}$;

where $r = 2(a - 2b)\sqrt{\dfrac{a}{4b - 3a}}.$

70. Solve
$$\left.\begin{aligned} x + y &= a \\ x^4 + y^4 &= b \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Raise (1) to the 4th power,
$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = a^4. \qquad (3)$$

Subtract (2) from (3),
$$4x^3y + 6x^2y^2 + 4xy^3 = a^4 - b.$$
$$2xy(2x^2 + 3xy + 2y^2) = a^4 - b. \qquad (4)$$

Square (1), $x^2 + 2xy + y^2 = a^2. \qquad (5)$
$$2x^2 + 4xy + 2y^2 = 2a^2.$$
$$2x^2 + 2y^2 = 2a^2 - 4xy.$$

Substitute the value of $2x^2 + 2y^2$ in (4),
$$2xy(2a^2 - xy) = a^4 - b.$$
$$4a^2xy - 2x^2y^2 = a^4 - b.$$
$$2x^2y^2 - 4a^2xy = b - a^4.$$
$$4x^2y^2 - 8a^2xy + 4a^4 = 2b + 2a^4.$$
$$2xy - 2a^2 = \pm\sqrt{2b + 2a^4}.$$
$$2xy = 2a^2 \pm \sqrt{2b + 2a^4}.$$
$$xy = a^2 \pm \tfrac{1}{2}\sqrt{2b + 2a^4}.$$
$$4xy = 4a^2 \pm 2\sqrt{2b + 2a^4}. \qquad (6)$$

Subtract (6) from (5),
$$x^2 - 2xy + y^2 = -3a^2 \mp 2\sqrt{2b + 2a^4}.$$
$$x - y = \pm\sqrt{-3a^2 \mp 2\sqrt{2b + 2a^4}}. \qquad (7)$$

Add (1) and (7), $2x = a \pm \sqrt{-3a^2 \pm 2\sqrt{2b + 2a^4}}.$
$$\therefore\ x = \tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{-3a^2 \pm 2\sqrt{2b + 2a^4}}.$$

Subtract (7) from (1), $2y = a \mp \sqrt{-3a^2 \pm 2\sqrt{2b + 2a^4}}.$
$$\therefore\ y = \tfrac{1}{2}a \mp \tfrac{1}{2}\sqrt{-3a^2 \pm 2\sqrt{2b + 2a^4}}.$$

Therefore, $x = \frac{1}{2}a + \frac{1}{2}\sqrt{-3a^2 + 2r}$, and $y = \frac{1}{2}a - \frac{1}{2}\sqrt{-3a^2 + 2r}$;
$x = \frac{1}{2}a + \frac{1}{2}\sqrt{-3a^2 - 2r}$, and $y = \frac{1}{2}a - \frac{1}{2}\sqrt{-3a^2 - 2r}$;
$x = \frac{1}{2}a - \frac{1}{2}\sqrt{-3a^2 + 2r}$, and $y = \frac{1}{2}a + \frac{1}{2}\sqrt{-3a^2 + 2r}$;
or $x = \frac{1}{2}a - \frac{1}{2}\sqrt{-3a^2 - 2r}$, and $y = \frac{1}{2}a + \frac{1}{2}\sqrt{-3a^2 - 2r}$;
where $r = \sqrt{2b + 2a^4}$.

71. Solve
$$\left.\begin{aligned} x^5 - y^5 &= a \\ x - y &= b \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Divide (1) by (2), $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \dfrac{a}{b}. \qquad (3)$

Raise (2) to the 4th power,
$$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = b^4. \qquad (4)$$

Subtract (4) from (3),

$$5x^3y - 5x^2y^2 + 5xy^3 = \frac{a-b^5}{b}.$$

$$x^3y - x^2y^2 + xy^3 = \frac{a-b^5}{5b}.$$

$$xy(x^2 - xy + y^2) = \frac{a-b^5}{5b}. \qquad (5)$$

Square (2),

$$x^2 - 2xy + y^2 = b^2. \qquad (6)$$

$$x^2 + y^2 = b^2 + 2xy.$$

Substitute the value of $x^2 + y^2$ in (5),

$$xy(b^2 + xy) = \frac{a-b^5}{5b}.$$

$$b^2xy + x^2y^2 = \frac{a-b^5}{5b}.$$

$$x^2y^2 + b^2xy = \frac{a-b^5}{5b}.$$

$$4x^2y^2 + (\;) + b^4 = \frac{4a+b^5}{5b}.$$

$$2xy + b^2 = \pm\sqrt{\frac{4a+b^5}{5b}}.$$

$$2xy = -b^2 \pm\sqrt{\frac{4a+b^5}{5b}}.$$

Put r for $\sqrt{\frac{4a+b^5}{5b}}$,

$$2xy = -b^2 \pm r.$$

$$4xy = -2b^2 \pm 2r. \qquad (7)$$

Add (6) and (7),

$$x^2 + 2xy + y^2 = -b^2 \pm 2r.$$

Extract the root,

$$x + y = \pm\sqrt{-b^2 \pm 2r}. \qquad (8)$$

Add (2) and (8),

$$2x = b \pm \sqrt{-b^2 \pm 2r}.$$

$$\therefore\ x = \tfrac{1}{2}b \pm \tfrac{1}{2}\sqrt{-b^2 \pm 2r}.$$

Subtract (2) from (8),

$$2y = -b \pm \sqrt{-b^2 \pm 2r}.$$

$$\therefore\ y = -\tfrac{1}{2}b \pm \tfrac{1}{2}\sqrt{-b^2 \pm 2r}.$$

Therefore, $x = \frac{1}{2}b + \frac{1}{2}\sqrt{-b^2+2r}$, and $y = -\frac{1}{2}b + \frac{1}{2}\sqrt{-b^2+2r}$;
$x = \frac{1}{2}b - \frac{1}{2}\sqrt{-b^2+2r}$, and $y = -\frac{1}{2}b - \frac{1}{2}\sqrt{-b^2+2r}$;
$x = \frac{1}{2}b + \frac{1}{2}\sqrt{-b^2-2r}$, and $y = -\frac{1}{2}b + \frac{1}{2}\sqrt{-b^2-2r}$;
or $x = \frac{1}{2}b - \frac{1}{2}\sqrt{-b^2-2r}$, and $y = -\frac{1}{2}b - \frac{1}{2}\sqrt{-b^2-2r}$;
where $r = \sqrt{\frac{4a+b^5}{5b}}$.

72. Solve
$$\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= a \\ x^2 - xy + y^2 &= 1 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Divide (1) by (2), $x^2 + xy + y^2 = a. \qquad (3)$

Subtract (2) from (3), $2xy = a - 1.$

$$xy = \frac{a-1}{2}. \qquad (4)$$

Add (3) and (4), $x^2 + 2xy + y^2 = \dfrac{3a-1}{2}.$

$$x + y = \pm \tfrac{1}{2}\sqrt{6a-2}. \qquad (5)$$

Subtract (4) from (2),

$$x^2 - 2xy + y^2 = \frac{3-a}{2}.$$

$$x - y = \pm \tfrac{1}{2}\sqrt{6-2a}. \qquad (6)$$

Add (5) and (6), $2x = \pm \tfrac{1}{2}\sqrt{6a-2}.\pm \tfrac{1}{2}\sqrt{6-2a}.$

$$\therefore\ x = \tfrac{1}{4}(\pm\sqrt{6a-2} \pm \sqrt{6-2a}).$$

Subtract (6) from (5), $2y = \pm \tfrac{1}{2}\sqrt{6a-2} \mp \tfrac{1}{2}\sqrt{6-2a}.$

$$\therefore\ y = \tfrac{1}{4}(\pm\sqrt{6a-2} \mp \sqrt{6-2a}).$$

Therefore,

$x = \tfrac{1}{4}(\sqrt{6a-2} + \sqrt{6-2a})$, and $y = \tfrac{1}{4}(\sqrt{6a-2} - \sqrt{6-2a})$;

$x = \tfrac{1}{4}(\sqrt{6a-2} - \sqrt{6-2a})$, and $y = \tfrac{1}{4}(\sqrt{6a-2} + \sqrt{6-2a})$;

$x = \tfrac{1}{4}(-\sqrt{6a-2} + \sqrt{6-2a})$, and $y = \tfrac{1}{4}(-\sqrt{6a-2} - \sqrt{6-2a})$;

or $x = \tfrac{1}{4}(-\sqrt{6a-2} - \sqrt{6-2a})$, and $y = \tfrac{1}{4}(-\sqrt{6a-2} + \sqrt{6-2a})$.

73. Solve
$$\left.\begin{aligned} \frac{x+y}{1+xy} &= \frac{2a}{1+a^2} \\ \frac{x-y}{1-xy} &= \frac{2b}{1+b^2} \end{aligned}\right\} \qquad \begin{aligned}(1)\\[2ex](2)\end{aligned}$$

From (1), $(1+a^2)x + (1+a^2)y = 2a + 2xy.$

$$(1+a^2)x - 2axy = 2a - (1+a^2)y.$$

$$(1+a^2-2ay)x = 2a - (1+a^2)y.$$

$$\therefore\ x = \frac{2a - (1+a^2)y}{(1+a^2) - 2ay}. \qquad (3)$$

From (2), $(1+b^2)x - (1+b^2)y = 2b - 2bxy.$

$$(1+b^2)x + 2bxy = 2b + (1+b^2)y.$$

$$(1+b^2+2by)x = 2b + (1+b^2)y.$$

$$\therefore\ x = \frac{2b + (1+b^2)y}{(1+b^2) + 2by}. \qquad (4)$$

Equate the values of x in (3) and (4),

$$\frac{2a-(1+a^2)y}{(1+a^2)-2ay}=\frac{2b+(1+b^2)y}{(1+b^2)+2by}.$$

Clear of fractions,

$$\begin{aligned}&2a(1+b^2)-(1+a^2)(1+b^2)y+4aby-2b(1+a^2)y^2\\ &\qquad=2b(1+a^2)-4aby+(1+a^2)(1+b^2)y-2a(1+b^2)y^2.\\ &[2a(1+b^2)-2b(1+a^2)]y^2-[2(1+a^2)(1+b^2)-8ab]y\\ &\qquad=2b(1+a^2)-2a(1+b^2).\end{aligned}$$

Divide by 2, and expand coefficients,

$$\begin{aligned}&(a+ab^2-b-a^2b)y^2-(1+a^2+b^2+a^2b^2-4ab)y\\ &\qquad=b+a^2b-a-ab^2.\\ &(a-b)(1-ab)y^2-[(a-b)^2+(1-ab)^2]y=(ab-1)(a-b).\end{aligned}$$

Complete the square,

$$\begin{aligned}&4(a-b)^2(1-ab)^2y^2-(\quad)+[(a-b)^2+(1-ab)^2]^2\\ &\quad=-4(a-b)^2(1-ab)^2+(a-b)^4+2(a-b)^2(1-ab)^2+(1-ab)^4\\ &\quad=(a-b)^4-2(a-b)^2(1-ab)^2+(1-ab)^4.\end{aligned}$$

Extract the root,

$$2(a-b)(1-ab)y-[(a-b)^2+(1-ab)^2]=\pm[(a-b)^2-(1-ab)^2].$$

$$2(a-b)(1-ab)y=2(a-b)^2, \text{ or } 2(1-ab)^2.$$

$$\therefore\ y=\frac{a-b}{1-ab}, \text{ or } \frac{1-ab}{a-b}.$$

Substitute the value of y in (3),

$$\begin{aligned}x&=\frac{2a-\dfrac{(1+a^2)(a-b)}{1-ab}}{(1+a^2)-\dfrac{2a(a-b)}{1-ab}}, \text{ or } \frac{2a-\dfrac{(1+a^2)(1-ab)}{a-b}}{(1+a^2)-\dfrac{2a(1-ab)}{a-b}}\\ &=\frac{2a-2a^2b-a+b-a^3+a^2b}{1+a^2-ab-a^3b-2a^2+2ab}, \text{ or } \frac{2a^2-2ab-1+ab-a^2+a^3b}{a+a^3-b-a^2b-2a+2a^2b}\\ &=\frac{a+b-a^3-a^2b}{1+ab-a^2-a^3b}, \text{ or } \frac{a^2+a^3b-ab-1}{a^3+a^2b-a-b}\\ &=\frac{(a+b)(1-a^2)}{(1+ab)(1-a^2)}, \text{ or } \frac{(1+ab)(a^2-1)}{(a+b)(a^2-1)}\\ &=\frac{a+b}{1+ab}, \text{ or } \frac{1+ab}{a+b}.\end{aligned}$$

Therefore, $x=\dfrac{a+b}{1+ab}$, and $y=\dfrac{a-b}{1-ab}$;

or $x=\dfrac{1+ab}{a+b}$, and $y=\dfrac{1-ab}{a-b}$.

74. Solve

$$\left.\begin{aligned}\frac{x+y}{1-xy}&=\frac{2a}{1-a^2}\\ \frac{x-y}{1+xy}&=\frac{2b}{1-b^2}\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

From (1), $(1-a^2)x+(1-a^2)y=2a-2axy.$

$$(1-a^2)x+2axy=2a-(1-a^2)y.$$

$$(1-a^2+2ay)x=2a-(1-a^2)y.$$

$$\therefore\ x=\frac{2a-(1-a^2)y}{(1-a^2)+2ay}. \qquad (3)$$

From (2), $(1-b^2)x-(1-b^2)y=2b+2bxy.$

$$(1-b^2)x-2bxy=2b+(1-b^2)y.$$

$$(1-b^2-2by)x=2b+(1-b^2)y.$$

$$\therefore\ x=\frac{2b+(1-b^2)y}{(1-b^2)-2by}. \qquad (4)$$

Equate the values of x in (3) and (4),

$$\frac{2a-(1-a^2)y}{(1-a^2)+2ay}=\frac{2b+(1-b^2)y}{(1-b^2)-2by}.$$

Clear of fractions,

$$2a(1-b^2)-(1-a^2)(1-b^2)y-4aby+2b(1-a^2)y^2$$
$$=2b(1-a^2)+4aby+(1-a^2)(1-b^2)y+2a(1-b^2)y^2.$$

$$[2a(1-b^2)-2b(1-a^2)]y^2+[2(1-a^2)(1-b^2)+8ab]y$$
$$=2a(1-b^2)-2b(1-a^2).$$

Divide by 2, and expand coefficients,

$$(a-ab^2-b+a^2b)y^2+(1-a^2-b^2+a^2b^2+4ab)y$$
$$=a-ab^2-b+a^2b.$$

$$(a-b)(1+ab)y^2+[(1+ab)^2-(a-b)^2]y=(a-b)(1+ab).$$

$$(a-b)(1+ab)y^2+[(1+ab)^2-(a-b)^2]y-(a-b)(1+ab)=0.$$

Factor, $[(1+ab)y-(a-b)][(a-b)y+(1+ab)]=0.$

$$\therefore\ y=\frac{a-b}{1+ab},\ \text{or}\ -\frac{1+ab}{a-b}.$$

Substitute the value of y in (3),

$$x=\frac{2a-\dfrac{(1-a^2)(a-b)}{1+ab}}{(1-a^2)+\dfrac{2a(a-b)}{1+ab}},\ \text{or}\ \frac{2a+\dfrac{(1-a^2)(1+ab)}{a-b}}{(1-a^2)-\dfrac{2a(1+ab)}{a-b}}$$

$$=\frac{2a+2a^2b-a+b+a^3-a^2b}{1-a^3b-a^2+ab+2a^2-2ab},\ \text{or}\ \frac{2a^2-2ab+1+ab-a^2-a^3b}{a-a^3-b+a^2b-2a-2a^2b}$$

$$=\frac{a+b+a^3+a^2b}{1+a^2-ab-a^3b},\ \text{or}\ \frac{1-ab+a^2-a^3b}{-a-b-a^3-a^2b}$$

$$= \frac{(a+b)(1+a^2)}{(1-ab)(1+a^2)}, \text{ or } \frac{(1-ab)(1+a^2)}{-(a+b)(1+a^2)}$$

$$= \frac{a+b}{1-ab}, \text{ or } -\frac{1-ab}{a+b}.$$

Therefore, $x = \frac{a+b}{1-ab}$, and $y = \frac{a-b}{1+ab}$;

or $x = -\frac{1-ab}{a+b}$, and $y = -\frac{1+ab}{a-b}$.

75. Solve

$$\left.\begin{aligned} ax^2 + by^2 &= cx^3 \\ cx^2 - dy^2 &= ax \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (1) by d, $adx^2 + bdy^2 = cdx^3$. (3)

Multiply (2) by b, $bcx^2 - bdy^2 = abx$. (4)

Add (3) and (4), $adx^2 + bcx^2 = cdx^3 + abx$.

$$cdx^3 - adx^2 - bcx^2 + abx = 0.$$

$$x(cdx^2 - adx - bcx + ab) = 0.$$

$$\therefore x = 0,$$

or $cdx^2 - adx - bcx + ab = 0.$

$$cdx^2 - (ad + bc)x = -ab.$$

$$4c^2d^2x^2 - (\ ) + (ad+bc)^2 = a^2d^2 - 2abcd + b^2c^2.$$

$$2cdx - (ad+bc) = \pm(ad - bc).$$

$$2cdx = 2ad, \text{ or } 2bc.$$

$$\therefore x = \frac{a}{c}, \text{ or } \frac{b}{d}.$$

$$\therefore x = 0, \frac{a}{c}, \text{ or } \frac{b}{d}.$$

Substitute the value of x in (1),

$$0 + by^2 = 0, \quad \frac{a^3}{c^2} + by^2 = \frac{a^3}{c^2}, \text{ or } \quad \frac{ab^2}{d^2} + by^2 = \frac{b^3c}{d^3}.$$

$$by^2 = 0, \quad by^2 = 0, \quad \text{or } ab^2d + bd^3y^2 = b^3c.$$

$$y^2 = 0, \quad y^2 = 0, \quad \text{or} \quad y^2 = \frac{b^3c - ab^2d}{bd^3}.$$

$$y = 0, \quad y = 0, \quad \text{or} \quad y = \pm\frac{1}{d^2}\sqrt{bd(bc - ad)}.$$

Therefore, $x = 0$, and $y = 0$;

$x = \frac{a}{c}$, and $y = 0$;

$x = \frac{b}{d}$, and $y = \frac{1}{d^2}\sqrt{bd(bc-ad)}$;

or $x = \frac{b}{d}$, and $y = -\frac{1}{d^2}\sqrt{bd(bc-ad)}$.

76. Solve
$$\left.\begin{aligned} x^2 + xy + y^2 &= 2a \\ x^2 - xy + y^2 &= 2b \end{aligned}\right\} \qquad \begin{aligned} &(1)\\ &(2)\end{aligned}$$

Subtract (2) from (1), $$2xy = 2a - 2b.$$
$$xy = a - b. \tag{3}$$

Add (1) and (3), $$x^2 + 2xy + y^2 = 3a - b.$$
$$x + y = \pm\sqrt{3a - b}. \tag{4}$$

Subtract (3) from (2),
$$x^2 - 2xy + y^2 = 3b - a.$$
$$x - y = \pm\sqrt{3b - a}. \tag{5}$$

Add (4) and (5), $$2x = \pm\sqrt{3a - b} \pm \sqrt{3b - a}.$$
$$\therefore\ x = \pm\tfrac{1}{2}\sqrt{3a - b} \pm \tfrac{1}{2}\sqrt{3b - a}.$$

Subtract (5) from (4), $$2y = \pm\sqrt{3a - b} \mp \sqrt{3b - a}.$$
$$\therefore\ y = \pm\tfrac{1}{2}\sqrt{3a - b} \mp \tfrac{1}{2}\sqrt{3b - a}.$$

Therefore,

$x = \tfrac{1}{2}\sqrt{3a - b} + \tfrac{1}{2}\sqrt{3b - a}$, and $y = \tfrac{1}{2}\sqrt{3a - b} - \tfrac{1}{2}\sqrt{3b - a}$;

$x = \tfrac{1}{2}\sqrt{3a - b} - \tfrac{1}{2}\sqrt{3b - a}$, and $y = \tfrac{1}{2}\sqrt{3a - b} + \tfrac{1}{2}\sqrt{3b - a}$;

$x = -\tfrac{1}{2}\sqrt{3a - b} + \tfrac{1}{2}\sqrt{3b - a}$, and $y = -\tfrac{1}{2}\sqrt{3a - b} - \tfrac{1}{2}\sqrt{3b - a}$;

or $x = -\tfrac{1}{2}\sqrt{3a - b} - \tfrac{1}{2}\sqrt{3b - a}$, and $y = -\tfrac{1}{2}\sqrt{3a - b} + \tfrac{1}{2}\sqrt{3b - a}$.

77. Solve
$$\left.\begin{aligned} x^2 - xy + y^2 &= 2a \\ x^4 - x^2y^2 + y^4 &= 2b \end{aligned}\right\} \qquad \begin{aligned} &(1)\\ &(2)\end{aligned}$$

From (1), $$x^2 + y^2 = xy + 2a.$$

Square, $$x^4 + 2x^2y^2 + y^4 = x^2y^2 + 4axy + 4a^2. \tag{3}$$

From (2), $$x^4 + y^4 = x^2y^2 + 2b.$$

Substitute the value of $x^4 + y^4$ in (3),
$$x^2y^2 + 2b + 2x^2y^2 = x^2y^2 + 4axy + 4a^2.$$
$$2x^2y^2 - 4axy = 4a^2 - 2b.$$
$$x^2y^2 - 2axy = 2a^2 - b.$$
$$x^2y^2 - 2axy + a^2 = 3a^2 - b.$$
$$xy - a = \pm\sqrt{3a^2 - b}.$$
$$xy = a \pm \sqrt{3a^2 - b}. \tag{4}$$
$$3xy = 3a \pm 3\sqrt{3a^2 - b}. \tag{5}$$

Add (1) and (5), $$x^2 + 2xy + y^2 = 5a \pm 3\sqrt{3a^2 - b}.$$
$$x + y = \pm\sqrt{5a \pm 3\sqrt{3a^2 - b}}. \tag{6}$$

Subtract (4) from (1),
$$x^2 - 2xy + y^2 = a \mp \sqrt{3a^2 - b}.$$
$$x - y = \pm\sqrt{a \mp \sqrt{3a^2 - b}}. \tag{7}$$

Add (6) and (7),

$$2x = \pm\sqrt{5a \pm 3\sqrt{3a^2 - b}} \pm \sqrt{a \mp \sqrt{3a^2 - b}}.$$

$$\therefore\ x = \pm\tfrac{1}{2}\sqrt{5a \pm 3\sqrt{3a^2 - b}} \pm \tfrac{1}{2}\sqrt{a \mp \sqrt{3a^2 - b}}.$$

Subtract (7) from (6),

$$2y = \pm\sqrt{5a \pm 3\sqrt{3a^2 - b}} \mp \sqrt{a \mp \sqrt{3a^2 - b}}.$$

$$\therefore\ y = \pm\tfrac{1}{2}\sqrt{5a \pm 3\sqrt{3a^2 - b}} \mp \tfrac{1}{2}\sqrt{a \mp \sqrt{3a^2 - b}}.$$

Therefore,

$$x = \tfrac{1}{2}\sqrt{5a + 3r} + \tfrac{1}{2}\sqrt{a - r}, \text{ and } y = \tfrac{1}{2}\sqrt{5a + 3r} - \tfrac{1}{2}\sqrt{a - r};$$
$$x = \tfrac{1}{2}\sqrt{5a + 3r} - \tfrac{1}{2}\sqrt{a - r}, \text{ and } y = \tfrac{1}{2}\sqrt{5a + 3r} + \tfrac{1}{2}\sqrt{a - r};$$
$$x = -\tfrac{1}{2}\sqrt{5a + 3r} + \tfrac{1}{2}\sqrt{a - r}, \text{ and } y = -\tfrac{1}{2}\sqrt{5a + 3r} - \tfrac{1}{2}\sqrt{a - r};$$
$$x = -\tfrac{1}{2}\sqrt{5a + 3r} - \tfrac{1}{2}\sqrt{a - r}, \text{ and } y = -\tfrac{1}{2}\sqrt{5a + 3r} + \tfrac{1}{2}\sqrt{a - r};$$
$$x = \tfrac{1}{2}\sqrt{5a - 3r} + \tfrac{1}{2}\sqrt{a + r}, \text{ and } y = \tfrac{1}{2}\sqrt{5a - 3r} - \tfrac{1}{2}\sqrt{a + r};$$
$$x = \tfrac{1}{2}\sqrt{5a - 3r} - \tfrac{1}{2}\sqrt{a + r}, \text{ and } y = \tfrac{1}{2}\sqrt{5a - 3r} + \tfrac{1}{2}\sqrt{a + r};$$
$$x = -\tfrac{1}{2}\sqrt{5a - 3r} + \tfrac{1}{2}\sqrt{a + r}, \text{ and } y = -\tfrac{1}{2}\sqrt{5a - 3r} - \tfrac{1}{2}\sqrt{a + r};$$

or $x = -\tfrac{1}{2}\sqrt{5a - 3r} - \tfrac{1}{2}\sqrt{a + r}$, and $y = -\tfrac{1}{2}\sqrt{5a - 3r} + \tfrac{1}{2}\sqrt{a + r}$;

where $r = \sqrt{3a^2 - b}$.

78. Solve

$$\left.\begin{aligned} a(x - y) &= b(x + y) \\ xy &= a(x - y) \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

From (1),

$$ax - ay = bx + by.$$
$$ax - bx = ay + by.$$
$$(a - b)x = (a + b)y.$$
$$\therefore\ x = \frac{(a + b)y}{a - b}. \qquad (3)$$

Substitute the value of x in (2),

$$\frac{(a + b)y^2}{a - b} = \frac{a(a + b)y}{a - b} - ay.$$
$$(a + b)y^2 = (a^2 + ab - a^2 + ab)y.$$
$$(a + b)y^2 = 2aby.$$
$$(a + b)y^2 - 2aby = 0.$$
$$y[(a + b)y - 2ab] = 0.$$
$$\therefore\ y = 0, \text{ or } \frac{2ab}{a + b}.$$

Substitute the value of y in (3),

$$x = 0, \text{ or } \frac{2ab}{a - b}.$$

Therefore, $x = 0$, and $y = 0$;

or $x = \dfrac{2ab}{a - b}$, and $y = \dfrac{2ab}{a + b}$.

Exercise 122. Page 306

1. Plot the graph of the function

$$x^2 + 5x + 4.$$

Put $x^2 + 5x + 4 = y$.

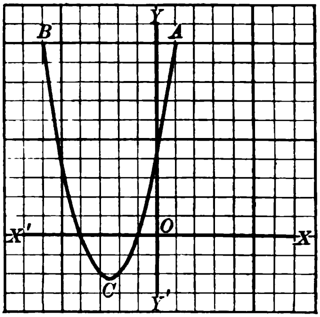
$$x^2 + 5x = y - 4.$$

$$4x^2 + (\ ) + 25 = 4y + 9.$$

$$2x + 5 = \pm\sqrt{4y + 9}.$$

$$2x = -5 \pm \sqrt{4y + 9}.$$

$$\therefore x = \tfrac{1}{2}(-5 \pm \sqrt{4y + 9}).$$



If $y = -2\tfrac{1}{4}$, $x = -2.5$;
$y = -2$, $x = -2$ or -3;
$y = -1$, $x = -1.38$ or -3.62;
$y = 0$, $x = -1$ or -4;
$y = +1$, $x = -0.70$ or -4.30;
$y = +2$, $x = -0.44$ or -4.56;
$y = +3$, $x = -0.21$ or -4.79.

If $y = +4$, $x = 0$ or -5;
$y = +5$, $x = +0.19$ or -5.19;
$y = +6$, $x = +0.37$ or -5.37;
$y = +7$, $x = +0.54$ or -5.54;
$y = +8$, $x = +0.70$ or -5.70;
$y = +9$, $x = +0.85$ or -5.85;
$y = +10$, $x = +1$ or -6.

Plot the points found, $(-2.5, -2\tfrac{1}{4})$, $(-2, -2)$, $(-3, -2)$, and so on. Through these points with a free hand draw the smooth curve *ACB*.

The curve *ACB* is the graph of the function $x^2 + 5x + 4$.

The graph is a parabola.

2. Plot the graph of the function

$$x^2 + x - 2.$$

Put $x^2 + x - 2 = y$.

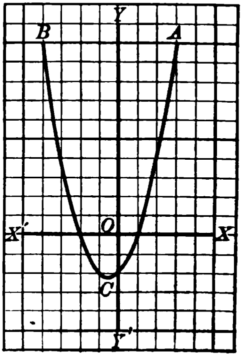
$$x^2 + x = y + 2.$$

$$4x^2 + (\ ) + 1 = 4y + 9.$$

$$2x + 1 = \pm\sqrt{4y + 9}.$$

$$2x = -1 \pm \sqrt{4y + 9}.$$

$$\therefore x = \tfrac{1}{2}(-1 \pm \sqrt{4y + 9}).$$



If $y=-2\frac{1}{4}$, $x=-0.5$;
$y=-2$, $x=0$ or -1;
$y=-1$, $x=+0.62$ or -1.62;
$y=0$, $x=+1$ or -2;
$y=+1$, $x=+1.30$ or -2.30;
$y=+2$, $x=+1.56$ or -2.56;
$y=+3$, $x=+1.79$ or -2.79.

If $y=+4$, $x=+2$ or -3;
$y=+5$, $x=+2.19$ or -3.19;
$y=+6$, $x=+2.37$ or -3.37;
$y=+7$, $x=+2.54$ or -3.54;
$y=+8$, $x=+2.70$ or -3.70;
$y=+9$, $x=+2.85$ or -3.85;
$y=+10$, $x=+3$ or -4.

Plot the points found, $(-0.5, -2\frac{1}{4})$, $(0, -2)$, $(-1, -2)$, and so on. Through these points with a free hand draw the smooth curve ACB.

The curve ACB is the graph of the function x^2+x-2.

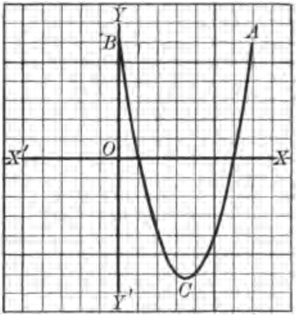
The graph is a parabola.

3. Plot the graph of the function

$$x^2-7x+6.$$

Put $x^2-7x+6=y$.

$$x^2-7x=y-6.$$
$$4x^2-(\quad)+49=4y+25.$$
$$2x-7=\pm\sqrt{4y+25}.$$
$$2x=7\pm\sqrt{4y+25}.$$
$$\therefore\ x=\tfrac{1}{2}(7\pm\sqrt{4y+25}).$$



If $y=-6\frac{1}{4}$, $x=+3.5$;
$y=-6$, $x=+4$ or $+3$;
$y=-5$, $x=+4.62$ or $+2.38$;
$y=-4$, $x=+5$ or $+2$;
$y=-3$, $x=+5.30$ or $+1.70$;
$y=-2$, $x=+5.56$ or $+1.44$;
$y=-1$, $x=+5.79$ or $+1.21$.

If $y=0$, $x=+6$ or $+1$;
$y=+1$, $x=+6.19$ or $+0.81$;
$y=+2$, $x=+6.37$ or $+0.63$;
$y=+3$, $x=+6.54$ or $+0.46$;
$y=+4$, $x=+6.70$ or $+0.30$;
$y=+5$, $x=+6.85$ or $+0.15$;
$y=+6$, $x=+7$ or 0.

Plot the points found, $(3.5, -6\frac{1}{4})$, $(4, -6)$, $(3, -6)$, and so on. Through these points with a free hand draw the smooth curve ACB.

The curve ACB is the graph of the function x^2-7x+6.

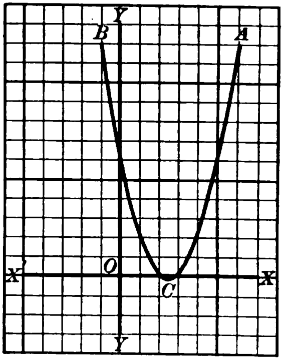
The graph is a parabola.

4. Plot the graph of the function $x^2 - 5x + 6$.

Put $x^2 - 5x + 6 = y$.

$$x^2 - 5x = y - 6.$$
$$4x^2 - (\ ) + 25 = 4y + 1.$$
$$2x - 5 = \pm\sqrt{4y + 1}.$$
$$2x = 5 \pm \sqrt{4y + 1}.$$
$$\therefore\ x = \tfrac{1}{2}(5 \pm \sqrt{4y + 1}).$$

If $y = -\frac{1}{4}$, $x = +2.5$;
$y = 0$, $x = +3$ or $+2$;
$y = +1$, $x = +3.62$ or $+1.38$;
$y = +2$, $x = +4$ or $+1$;
$y = +3$, $x = +4.30$ or $+0.70$;
$y = +4$, $x = +4.56$ or $+0.44$;
$y = +5$, $x = +4.79$ or $+0.21$;
$y = +6$, $x = +5$ or 0;
$y = +7$, $x = +5.19$ or -0.19;
$y = +8$, $x = +5.37$ or -0.37;
$y = +9$, $x = +5.54$ or -0.54;
$y = +10$, $x = +5.70$ or -0.70;
$y = +11$, $x = +5.85$ or -0.85;
$y = +12$, $x = +6$ or -1.



Plot the points found, $(2.5, -\frac{1}{4})$, $(3, 0)$, $(2, 0)$, and so on. Through these points with a free hand draw the smooth curve ACB.

The curve ACB is the graph of the function $x^2 - 5x + 6$.

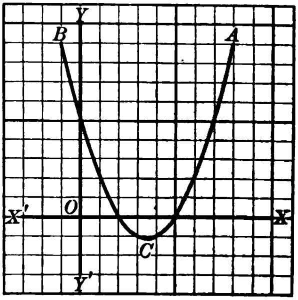
The graph is a parabola.

5. Plot the graph of the function

$$2x^2 - 7x + 5.$$

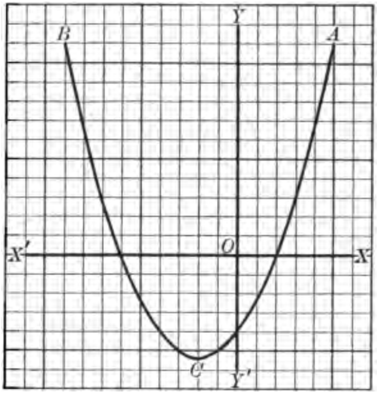
Put $2x^2 - 7x + 5 = y$.

$$2x^2 - 7x = y - 5.$$
$$16x^2 - (\ ) + 49 = 8y + 9.$$
$$4x - 7 = \pm\sqrt{8y + 9}.$$
$$4x = 7 \pm \sqrt{8y + 9}.$$
$$\therefore\ x = \tfrac{1}{4}(7 \pm \sqrt{8y + 9}).$$



If $y = -1\frac{1}{8}$, $x = +1.75$;
$y = -1$, $x = +2$ or $+1.5$;
$y = 0$, $x = +2.5$ or $+1$;
$y = +1$, $x = +2.78$ or $+0.72$;
$y = +2$, $x = +3$ or $+0.5$;
$y = +3$, $x = +3.19$ or $+0.31$.
If $y = +4$, $x = +3.35$ or $+0.15$;
$y = +5$, $x = +3.5$ or 0;
$y = +6$, $x = +3.64$ or -0.14;
$y = +7$, $x = +3.77$ or -0.27;
$y = +8$, $x = +3.89$ or -0.39;
$y = +9$, $x = +4$ or -0.5.

Plot the points found, $(1.75, -1\frac{1}{8})$, $(2, -1)$, $(1.5, -1)$, and so on, to insure greater accuracy and beauty, using two spaces of the coördinate paper for one unit of x, and one space for one unit of y. Through these points with a free hand draw the smooth curve ACB.



The curve ACB is the graph of the function $2x^2 - 7x + 5$.

The graph is a parabola.

6. Plot the graph of the function $3x^2 + 4x - 4$.

Put $3x^2 + 4x - 4 = y$.

$$3x^2 + 4x = y + 4.$$
$$9x^2 + (\ ) + 4 = 3y + 16.$$
$$3x + 2 = \pm\sqrt{3y + 16}.$$
$$3x = -2 \pm \sqrt{3y + 16}.$$
$$\therefore\ x = \tfrac{1}{3}(-2 \pm \sqrt{3y + 16}).$$

If $y = -5\frac{1}{3}$, $x = -0.67$;
$y = -5$, $x = -0.33$ or -1;
$y = -4$, $x = 0$ or -1.33;
$y = -3$, $x = +0.22$ or -1.55;
$y = -2$, $x = +0.39$ or -1.72;
$y = -1$, $x = +0.54$ or -1.87;
$y = 0$, $x = +0.67$ or -2;
$y = +1$, $x = +0.79$ or -2.12;
$y = +2$, $x = +0.90$ or -2.23.
If $y = +3$, $x = +1$ or -2.33;
$y = +4$, $x = +1.10$ or -2.43;
$y = +5$, $x = +1.19$ or -2.52;
$y = +6$, $x = +1.28$ or -2.61;
$y = +7$, $x = +1.36$ or -2.69;
$y = +8$, $x = +1.44$ or -2.77;
$y = +9$, $x = +1.52$ or -2.85;
$y = +10$, $x = +1.59$ or -2.93;
$y = +11$, $x = +1.67$ or -3.

Plot the points found, $(-0.67, -5\frac{1}{3})$, $(-0.33, -5)$, $(-1, -5)$, and so on, to insure greater accuracy, using three spaces of the coördinate paper for one unit of x, and one space for one unit of y. Through these points with a free hand draw the smooth curve ACB.

The curve ACB is the graph of the function $3x^2 + 4x - 4$.

The graph is a parabola.

7. Find, by the method of graphs, the roots of $x^2 - x - 6 = 0$.

Let $x^2 = y$, and put y for x^2.

Then the given equation becomes $y - x - 6 = 0$.

Since the graph of $y - x - 6 = 0$ is a straight line, it is necessary to locate only two points to determine the graph.

If $y = 0$, $x = -6$; and if $x = 0$, $y = +6$.

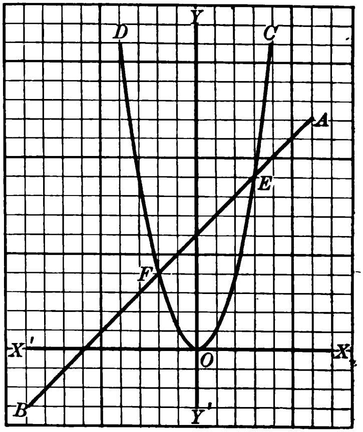
Therefore, the points $(-6, 0)$ and $(0, 6)$ determine the graph of $y - x - 6 = 0$.

The straight line AB, drawn through the points $(-6, 0)$ and $(0, 6)$, is the graph of the equation $y - x - 6 = 0$.

To plot the graph of the equation $x^2 = y$.

$$x^2 = y.$$
$$\therefore x = \pm\sqrt{y}.$$

If $y = 0$, $x = 0$;
$y = +\frac{1}{2}$, $x = \pm 0.71$;
$y = +1$, $x = \pm 1$;
$y = +2$, $x = \pm 1.41$;
$y = +3$, $x = \pm 1.73$;
$y = +4$, $x = \pm 2$;
$y = +5$, $x = \pm 2.24$;
$y = +6$, $x = \pm 2.45$;
$y = +7$, $x = \pm 2.65$;
$y = +8$, $x = \pm 2.83$;
$y = +9$, $x = \pm 3$;
$y = +10$, $x = \pm 3.16$;
$y = +11$, $x = \pm 3.32$;
$y = +12$, $x = \pm 3.46$;
$y = +13$, $x = \pm 3.61$;
$y = +14$, $x = \pm 3.74$;
$y = +15$, $x = \pm 3.87$;
$y = +16$, $x = \pm 4$.



Plot the points found, $(0, 0)$, $(0.71, 0.5)$, $(-0.71, 0.5)$, and so on. Through these points with a free hand draw the smooth curve COD.

The parabola COD is the graph of the equation $x^2 = y$.

The straight line AB intersects the parabola COD at the points E and F.

The abscissas of E and F are $+3$ and -2, respectively.

Therefore (§ 358), the roots of the given equation are 3 and -2.

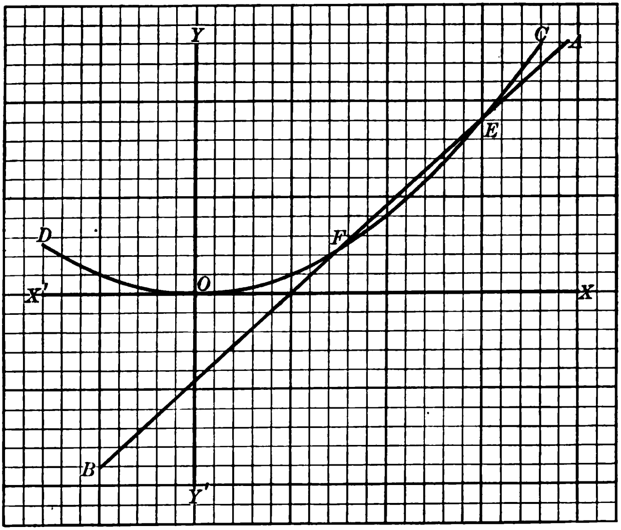
8. **Find, by the method of graphs, the roots of $2x^2 - 9x + 9 = 0$.**

Let $x^2 = y$, and put y for x^2.

Then the given equation becomes $2y - 9x + 9 = 0$.

To increase the chance for accuracy in drawing the graphs use five spaces of the coördinate paper for one unit of x, and one space for one unit of y.

Two points on the line $2y - 9x + 9 = 0$ are $(3, 9)$ and $(-1, -9)$.



The graph of the equation $2y - 9x + 9 = 0$ is the straight line AB, and the graph of the equation $x^2 = y$ is the parabola COD, as in Example 7.

AB intersects COD in the points E and F, the abscissas of which are 3 and $1\frac{1}{2}$, respectively.

Therefore (§ 358), the roots of the given equation are 3 and $1\frac{1}{2}$.

In plotting points of the parabola COD we use the table of Example 7, and in addition the following:

$$y = x^2.$$

If $x = \pm 0.2,\ y = 0.04$;
$x = \pm 0.4,\ y = 0.16$;
$x = \pm 0.6,\ y = 0.36$;
$x = \pm 0.8,\ y = 0.64$;
$x = \pm 1.2,\ y = 1.44$;
$x = \pm 1.4,\ y = 1.96$;
$x = \pm 1.6,\ y = 2.56$;
$x = \pm 1.8,\ y = 3.24$.

If $x = \pm 2.2,\ y = 4.84$;
$x = \pm 2.4,\ y = 5.76$;
$x = \pm 2.6,\ y = 6.76$;
$x = \pm 2.8,\ y = 7.84$;
$x = \pm 3.2,\ y = 10.24$;
$x = \pm 3.4,\ y = 11.56$;
$x = \pm 3.6,\ y = 12.96$;
$x = \pm 3.8,\ y = 14.44$.

9. Find, by the method of graphs, the roots of $5x^2 - 20 = 0$.

Let $x^2 = y$, and put y for x^2.

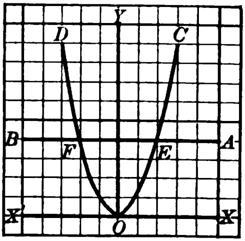
Then the given equation becomes

$$5y - 20 = 0, \text{ or } y = 4.$$

The graph of the equation $5y - 20 = 0$ is the line AB parallel to XX' and at a distance of 4 units above it.

As in Example 7, the graph of the equation $x^2 = y$ is the parabola COD.

AB intersects COD in the points E and F, the abscissas of which are $+2$ and -2, respectively.



Therefore (§ 358), the roots of the given equation are 2 and -2.

10. Find, by the method of graphs, the roots of $7x^2 + 14x - 21 = 0$.

Divide by 7, $x^2 + 2x - 3 = 0$.

Let $x^2 = y$, and put y for x^2.

Then the given equation becomes

$$y + 2x - 3 = 0.$$

Two points on the straight line

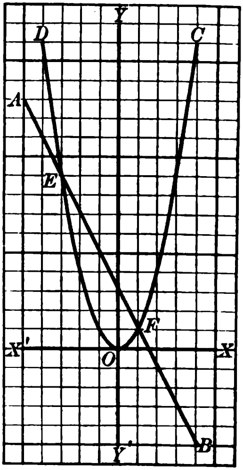
$$y + 2x - 3 = 0$$

are $(4, -5)$ and $(-3, 9)$.

The graph of the equation $y + 2x - 3 = 0$ is the straight line AB, and the graph of the equation $x^2 = y$ is the parabola COD, as in Example 7.

AB intersects COD in the points E and F, the abscissas of which are -3 and $+1$, respectively.

Therefore (§ 358), the roots of the given equation are -3 and 1.

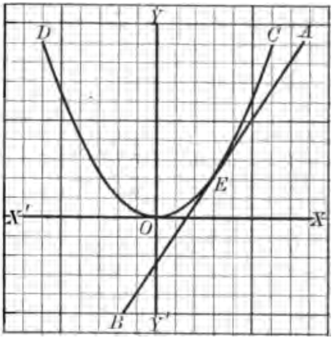


11. Find, by the method of graphs, the roots of $4x^2 - 12x + 9 = 0$.

Let $x^2 = y$, and put y for x^2.

Then the given equation becomes $4y - 12x + 9 = 0$.

Two points on the straight line $4y - 12x + 9 = 0$ are $(-\frac{1}{2}, -3\frac{3}{4})$ and $(2\frac{1}{2}, 5\frac{1}{4})$.



In plotting the graphs use two spaces of the coördinate paper for one unit of x, and one space for one unit of y.

The graph of the equation

$$4y - 12x + 9 = 0$$

is the straight line AB, and the graph of the equation $x^2 = y$ is the parabola COD, as in Example 7.

AB is tangent to COD at the point E, the abscissa of which is $+1\frac{1}{2}$.

Therefore, the roots of the given equation are $1\frac{1}{2}$ and $1\frac{1}{2}$.

12. Find, by the method of graphs, the roots of $25x^2+60x+36=0$.

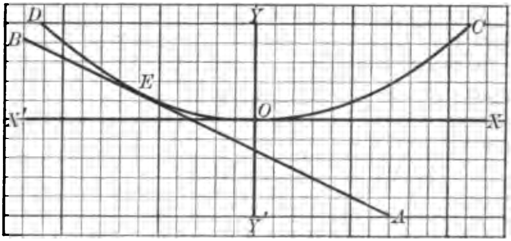
Let $x^2 = y$, and put y for x^2.

Then the given equation becomes $25y + 60x + 36 = 0$.

In plotting the graphs use five spaces of the coördinate paper for one unit of x, and one space for one unit of y.

Two points on the straight line $25y + 60x + 36 = 0$ are $(1, -3\frac{21}{25})$ and $(-2, 3\frac{9}{25})$.

The graph of the equation $25y + 60x + 36 = 0$ is the straight line AB, and the graph of the equation $x^2 = y$ is the parabola COD, as in Example 7.



AB is tangent to COD at the point E, the abscissa of which is $-1\frac{1}{5}$.

Therefore, the roots of the given equation are $-1\frac{1}{5}$ and $-1\frac{1}{5}$.

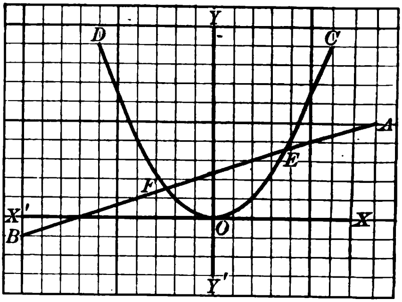
13. **Find approximately the roots of $3x^2 - 2x - 7 = 0$.**

Let $x^2 = y$, and put y for x^2.

Then the given equation becomes $3y - 2x - 7 = 0$.

In plotting the graphs use two spaces of the coördinate paper for one unit of x, and one space for one unit of y.

Two points on the straight line $3y - 2x - 7 = 0$ are $(2\frac{1}{2}, 4)$ and $(-5, -1)$.



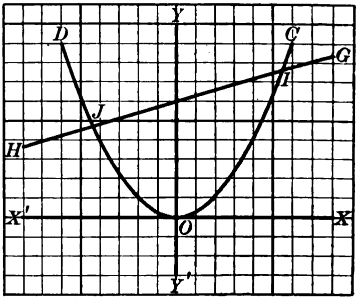
The graph of the equation $3y - 2x - 7 = 0$ is the straight line AB, and the graph of the equation $x^2 = y$ is the parabola COD, as in Example 7.

AB intersects COD in the points E and F, the abscissas of which are approximately $+1\frac{7}{8}$ and $-1\frac{1}{4}$, respectively.

Therefore, the roots of the given equation are approximately $1\frac{7}{8}$ and $-1\frac{1}{4}$.

On page 688 the graphs AB and COD are drawn to larger scale, five spaces of the coördinate paper for one unit of x, and two spaces for one unit of y.

By that figure the roots of the given equation are seen to be 1.90 and $-$ 1.23, approximately.



14. Find approximately the roots of $5x^2 - 3x - 30 = 0$.

Let $x^2 = y$, and put y for x^2.

Then the given equation becomes $5y - 3x - 30 = 0$.

In plotting the graphs use two spaces of the coördinate paper for one unit of x, and one space for one unit of y.

Two points on the straight line $5y - 3x - 30 = 0$ are $(0, 6)$ and $(-5, 3)$.

The graph of the equation $5y - 3x - 30 = 0$ is the straight line GH, and the graph of the equation $x^2 = y$ is the parabola COD, as in Example 7.

GH intersects *COD* in the points *I* and *J*, the abscissas of which are approximately $+ 2\frac{3}{4}$ and $- 2\frac{1}{6}$, respectively.

Therefore, the roots of the given equation are approximately $2\frac{3}{4}$ and $- 2\frac{1}{6}$.

On page 688 the graphs *GH* and *COD* are drawn to larger scale, five spaces of the coördinate paper for one unit of x, and two spaces for one unit of y.

By that figure the roots of the given equation are seen to be 2.77 and $- 2.17$, approximately.

15. Find approximately the roots of $5x^2 - 7x - 1 = 0$.

Let $x^2 = y$, and put y for x^2.

Then the given equation becomes $5y - 7x - 1 = 0$.

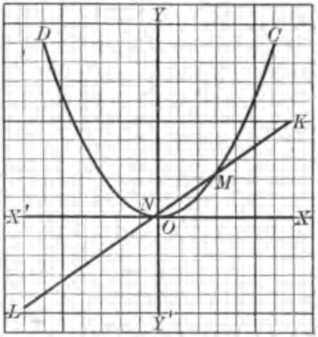
In plotting the graphs use two spaces of the coördinate paper for one unit of x, and one space for one unit of y.

Two points on the straight line $5y - 7x - 1 = 0$ are $(7, 10)$ and $(-3, -4)$.

The graph of the equation

$$5y - 7x - 1 = 0$$

is the straight line *KL*, and the graph of the equation $x^2 = y$ is the parabola *COD*, as in Example 7.



KL intersects *COD* in the points *M* and *N*, the abscissas of which are approximately $+ 1\frac{1}{2}$ and $- \frac{1}{6}$, respectively.

Therefore, the roots of the given equation are approximately $1\frac{1}{2}$ and $- \frac{1}{6}$.

On page 688 the graphs *KL* and *COD* are drawn to larger scale, five spaces of the coördinate paper for one unit of x, and two spaces for one unit of y.

By that figure the roots of the given equation are seen to be 1.53 and $- 0.13$, approximately.

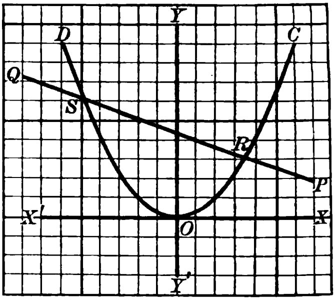
16. Find approximately the roots of $7x^2 + 5x - 31 = 0$.

Let $x^2 = y$, and put y for x^2.

Then the given equation becomes $7y + 5x - 31 = 0$.

In plotting the graphs use two spaces of the coördinate paper for one unit of x, and one space for one unit of y.

Two points on the straight line $7y + 5x - 31 = 0$ are $(9, -2)$ and $(-5, 8)$.



The graph of the equation

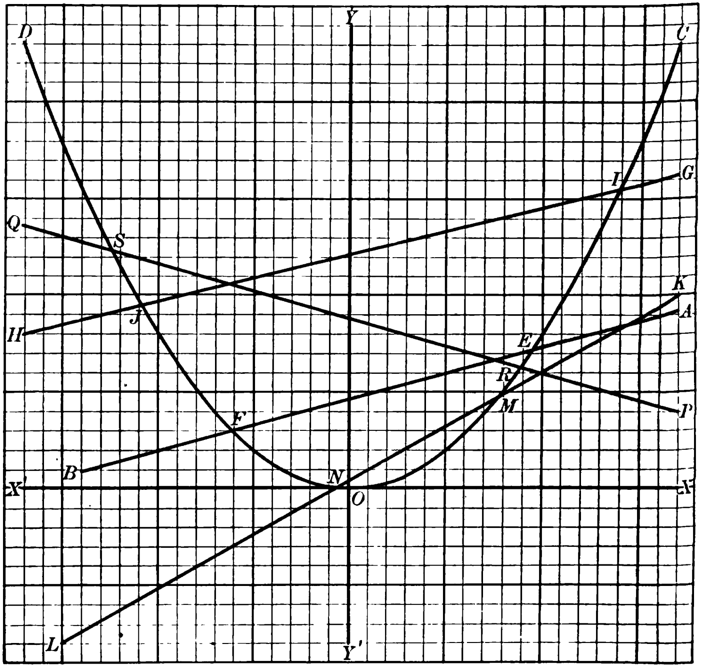
$$7y + 5x - 31 = 0$$

is the straight line *PQ*, and the graph of the equation $x^2 = y$ is the parabola *COD*, as in Example 7.

PQ intersects *COD* in the points *R* and *S*, the abscissas of which are approximately $+1\frac{3}{4}$ and $-2\frac{1}{2}$, respectively.

Therefore, the roots of the given equation are approximately $1\frac{3}{4}$ and $-2\frac{1}{2}$.

At the bottom of this page the graphs *PQ* and *COD* are drawn to larger scale, five spaces of the coördinate paper for one unit of x, and two spaces for one unit of y.



By that figure the roots of the given equation are seen to be 1.78 and − 2.49, approximately.

17. What is the nature of the roots of the equation $3x^2 + 2x + 4 = 0$? Construct the graph of the equation.

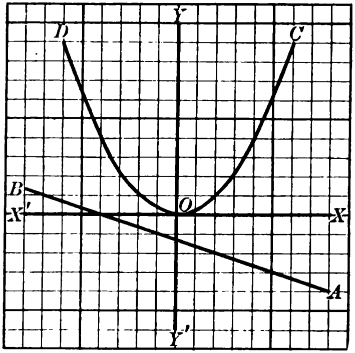
Let $x^2 = y$, and put y for x^2.

Then the given equation becomes $3y + 2x + 4 = 0$.

In plotting the graphs use two spaces of the coördinate paper for one unit of x, and one space for one unit of y.

Two points on the straight line $3y + 2x + 4 = 0$ are $(4, -4)$ and $(-3\frac{1}{2}, 1)$.

The graph of the equation $3y + 2x + 4 = 0$ is the straight line AB, and the graph of the equation $x^2 = y$ is the parabola COD, as in Example 7.



AB does not cut or touch COD and will not, however far the graphs are produced.

Therefore, the roots of the given equation are imaginary, as may be seen by solving the equation.

$$3x^2 + 2x + 4 = 0.$$
$$3x^2 + 2x = -4.$$
$$9x^2 + (\ ) + 1 = -11.$$
$$3x + 1 = \pm\sqrt{-11}.$$
$$3x = -1 \pm \sqrt{-11}.$$
$$\therefore x = \tfrac{1}{3}(-1 \pm \sqrt{-11}).$$

Therefore, the roots of the given equation are $\frac{1}{3}(-1 \pm \sqrt{-11})$.

18. How does the graph of a quadratic equation indicate the fact that the roots of the equation are real and unequal? real and equal? imaginary?

If the roots of a quadratic equation are real and unequal, the graph of the equation twice crosses the axis of x.

If the roots of a quadratic equation are real and equal, the graph of the equation touches the axis of x but does not cut it; in other words, the axis of x is tangent to the graph.

If the roots of a quadratic equation are imaginary, the graph of the equation does not cut or touch the axis of x.

If the method of § 358 is used, if the straight line function of the given equation cuts the parabola $x^2 = y$ in two points, the roots of the equation are real and unequal; if the straight line function is tangent to the parabola, the roots of the equation are real and equal; if the straight line function does not cut or touch the parabola, the roots are imaginary.

Exercise 123. Page 308

1. What does the right-hand figure on page 307 show about the relation of AB, CD, and EF? How do the coefficients of x and y in equations (2), (3), and (4) show this? Are the graphs of $ax + by + c = 0$ and $ax + by + d = 0$ parallel?

The right-hand figure on page 307 shows that the straight lines AB, CD, and EF are parallel.

The coefficients of x and y in equations (2), (3), and (4) show that the lines AB, CD, and EF are parallel by the fact that in the three equations the coefficients of x are equal and the coefficients of y are equal, while the third terms are all different.

The graphs of $ax + by + c = 0$ and $ax + by + d = 0$ are parallel, since the coefficients of x are equal, the coefficients of y are equal, and the third terms are unequal.

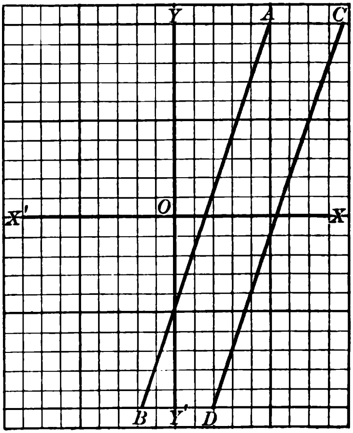
2. Write the equations of two parallel lines and construct their graphs.

The equations of two parallel lines are $3x - y - 5 = 0$ and $3x - y - 16 = 0$.

Two points of the graph of $3x - y - 5 = 0$ are $(5, 10)$ and $(-1, -8)$.

Two points of the graph of $3x - y - 16 = 0$ are $(2, -10)$ and $(8, 8)$.

The graph of $3x - y - 5 = 0$ is the straight line AB, and the graph of $3x - y - 16 = 0$ is the straight line CD.



3. If a straight line and a circle touch each other, how many values has x? how many has y? How many values have x and y when the line cuts the circle? What is the nature of the roots of two equations when the graphs do not intersect?

If a straight line and a circle touch each other, x has one value and y has one value; that is, there is a double root, or the two roots of the equations are real and equal.

If a straight line cuts a circle, x has two values and y has two values; that is, the two roots of the equations are real and unequal.

If the graphs of two equations do not intersect and the equations are simultaneous, the roots of the equations are imaginary.

If the graphs of two equations do not intersect and the equations are not simultaneous, the roots of the equations are infinity.

4. Solve exactly or approximately by the method of graphs,

$$\left.\begin{aligned} x^2+y^2-169&=0\\ 3x-2y+9&=0\end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

From (1), $x^2 = 169 - y^2$, and $y^2 = 169 - x^2$.

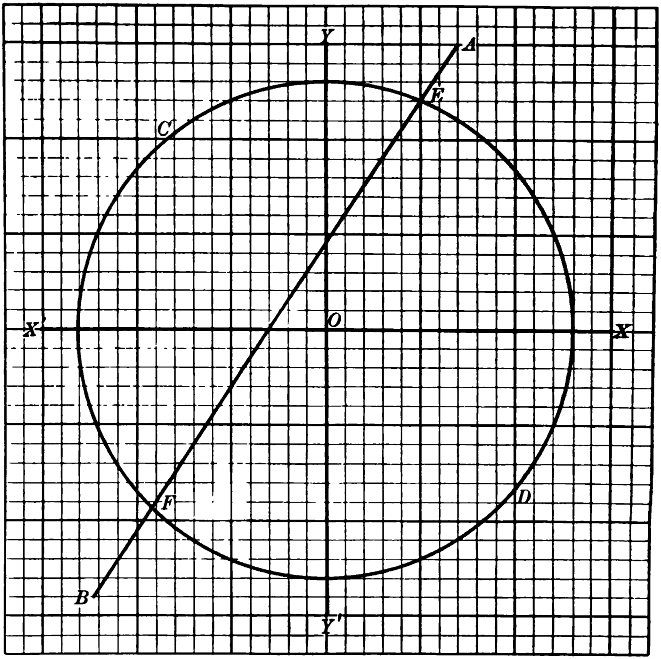
$\therefore\ x = \pm\sqrt{169 - y^2}$, and $y = \pm\sqrt{169 - x^2}$.

If $y = 0,\ x = \pm 13;$	If $x = 0,\ y = \pm 13;$
$y = \pm 1,\ x = \pm 12.96;$	$x = \pm 1,\ y = \pm 12.96;$
$y = \pm 2,\ x = \pm 12.85;$	$x = \pm 2,\ y = \pm 12.85;$
$y = \pm 3,\ x = \pm 12.65;$	$x = \pm 3,\ y = \pm 12.65;$
$y = \pm 4,\ x = \pm 12.37;$	$x = \pm 4,\ y = \pm 12.37;$
$y = \pm 5,\ x = \pm 12;$	$x = \pm 5,\ y = \pm 12;$
$y = \pm 6,\ x = \pm 11.53;$	$x = \pm 6,\ y = \pm 11.53;$
$y = \pm 7,\ x = \pm 10.95;$	$x = \pm 7,\ y = \pm 10.95;$
$y = \pm 8,\ x = \pm 10.25;$	$x = \pm 8,\ y = \pm 10.25;$
$y = \pm 9,\ x = \pm 9.38;$	$x = \pm 9,\ y = \pm 9.38;$
$y = \pm 10,\ x = \pm 8.31;$	$x = \pm 10,\ y = \pm 8.31;$
$y = \pm 11,\ x = \pm 6.93;$	$x = \pm 11,\ y = \pm 6.93;$
$y = \pm 12,\ x = \pm 5;$	$x = \pm 12,\ y = \pm 5;$
$y = \pm 13,\ x = 0.$	$x = \pm 13,\ y = 0.$

The graph of equation (1) is found to be the circle CD; a circle with O for a centre and a radius equal to $\sqrt{169}$, or 13.

In equation (2), if $x = 5$, $y = 12$, and if $x = -9$, $y = -9$.

The graph of equation (2) is the straight line AB.



AB cuts CD in the point $E\,(5, 12)$ exactly, and the point $F\,(-9\frac{1}{5}, -9\frac{1}{4})$, approximately.

Therefore, the solutions of the given equations are $x = 5$, $y = 12$ (exactly), and $x = -9\frac{1}{5}$, $y = -9\frac{1}{4}$ (approximately).

5. Solve exactly or approximately by the method of graphs,

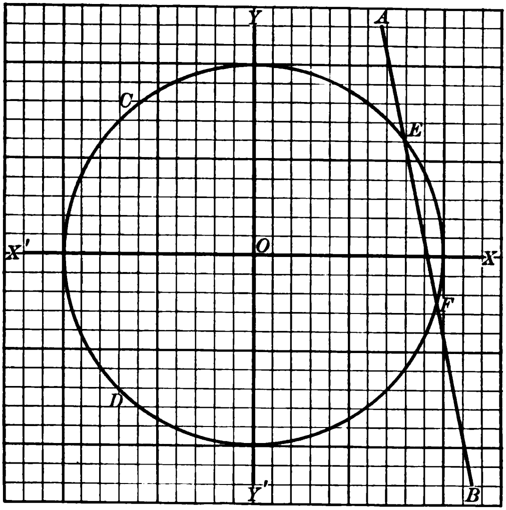
$$\left.\begin{aligned} x^2 + y^2 &= 100 \\ 5x + y &= 46 \end{aligned}\right\} \qquad \begin{aligned} (1) \\ (2) \end{aligned}$$

From (1), $x^2 = 100 - y^2$, and $y^2 = 100 - x^2$.

$$\therefore\ x = \pm\sqrt{100 - y^2}, \text{ and } y = \pm\sqrt{100 - x^2}.$$

$y = 0,\ x = \pm 10;$
$y = \pm 1,\ x = \pm 9.95;$
$y = \pm 2,\ x = \pm 9.80;$
$y = \pm 3,\ x = \pm 9.54;$
$y = \pm 4,\ x = \pm 9.17;$
$y = \pm 5,\ x = \pm 8.66;$
$y = \pm 6,\ x = \pm 8;$
$y = \pm 7,\ x = \pm 7.14;$
$y = \pm 8,\ x = \pm 6;$
$y = \pm 9,\ x = \pm 4.36;$
$y = \pm 10,\ x = 0.$

If $x = 0,\ y = \pm 10;$
$x = \pm 1,\ y = \pm 9.95;$
$x = \pm 2,\ y = \pm 9.80;$
$x = \pm 3,\ y = \pm 9.54;$
$x = \pm 4,\ y = \pm 9.17;$
$x = \pm 5,\ y = \pm 8.66;$
$x = \pm 6,\ y = \pm 8;$
$x = \pm 7,\ y = \pm 7.14;$
$x = \pm 8,\ y = \pm 6;$
$x = \pm 9,\ y = \pm 4.36;$
$x = \pm 10,\ y = 0.$



The graph of equation (1) is found to be the circle *CD*; a circle with *O* for a centre and a radius equal to 10.

In equation (2), if $x = 7$, $y = 11$, and if $x = 11$, $y = -9$.

The graph of equation (2) is the straight line *AB*.

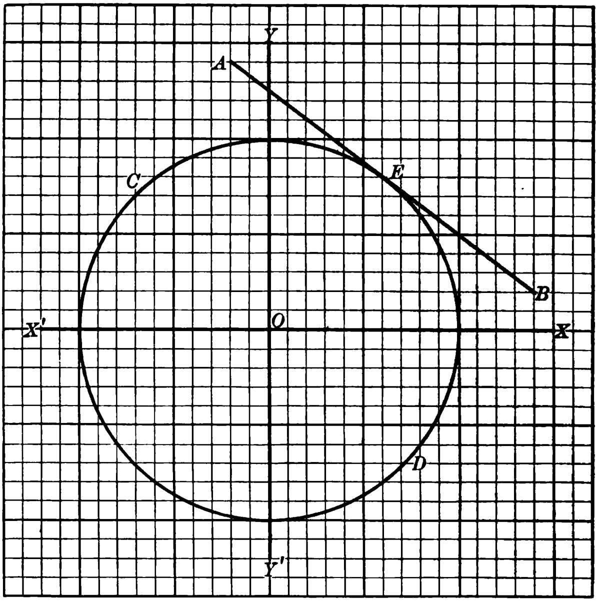
AB cuts *CD* in the point *E* (8, 6) exactly and the point *F* $(9\frac{2}{3}, -2\frac{1}{2})$ approximately.

Therefore, the solutions of the given equations are $x = 8$, $y = 6$ (exactly), and $x = 9\frac{2}{3}$, $y = -2\frac{1}{2}$ (approximately).

6. Solve exactly or approximately by the method of graphs,

$$\left.\begin{aligned} x^2 + y^2 &= 100 \\ 3x + 4y &= \ \ 50 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

The graph of equation (1) is the circle *CD*, as in Example 5. In equation (2), if $x = 2$, $y = 11$, and if $x = 10$, $y = 5$.



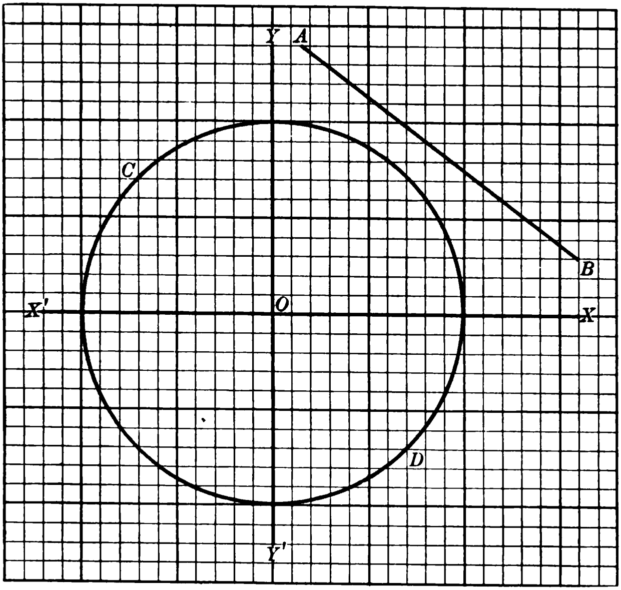
The graph of equation (2) is the straight line *AB*.
AB is tangent to the circle at the point *E* (6, 8).
Hence, the two pairs of solutions of the given equations are the same.

Therefore, the solutions of the given equations are $x = 6$, $y = 8$, and $x = 6$, $y = 8$.

7. Solve exactly or approximately by the method of graphs,

$$\left.\begin{aligned} x^2+y^2&=100 \\ 3x+4y&=\ \ 60 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

The graph of equation (1) is the circle *CD*, as in Example 5.
In equation (2), if $x = 4$, $y = 12$, and if $x = 16$, $y = 3$.



The graph of equation (2) is the straight line *AB*.

AB does not cut or touch *CD* and will not, however far *AB* is produced in either direction.

Therefore, the roots of the given equations are imaginary numbers and cannot be found by the method of graphs.

8. Solve the equations of Example 7 as simultaneous equations, and explain why their graphs do not intersect.

We have
$$\left.\begin{aligned} x^2 + y^2 &= 100 \\ 3x + 4y &= 60 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

From (2),
$$x = \frac{60 - 4y}{3}. \qquad (3)$$

Substitute the value of x in (1),
$$\frac{3600 - 480y + 16y^2}{9} + y^2 = 100.$$
$$3600 - 480y + 16y^2 + 9y^2 = 900.$$
$$25y^2 - 480y = -2700.$$
$$25y^2 - 480y + 2304 = -396.$$
$$5y - 48 = \pm 6\sqrt{-11}.$$
$$5y = 48 \pm 6\sqrt{-11}.$$
$$\therefore\ y = \tfrac{6}{5}(8 \pm \sqrt{-11}).$$

Substitute the value of y in (3), $x = \tfrac{4}{5}(9 \mp 2\sqrt{-11})$.

Therefore, $x = \tfrac{4}{5}(9 + 2\sqrt{-11})$, and $y = \tfrac{6}{5}(8 - \sqrt{-11})$;
or $x = \tfrac{4}{5}(9 - 2\sqrt{-11})$, and $y = \tfrac{6}{5}(8 + \sqrt{-11})$.

If the graphs intersected, the roots would be real, and since the roots are not real, the graphs do not intersect.

9. The figure on page 308 shows the graph of the ellipse $4x^2 + 9y^2 = 288$, and the graph of the parabola $3y^2 = 8x$. What roots satisfy these equations?

By the figure it is seen that the parabola $3y^2 = 8x$ cuts the ellipse $4x^2 + 9y^2 = 288$ in the points (6, 4) and (6, -4).

Therefore, the solutions of these equations are $x = 6$, $y = 4$, and $x = 6$, $y = -4$.

By solving the equations as simultaneous equations the additional solutions $x = -12$, $y = 4\sqrt{-2}$, and $x = -12$, $y = -4\sqrt{-2}$ are found.

10. The equation of the circle $ax^2 + ay^2 = c$ differs in what respect from the equation of the ellipse $ax^2 + by^2 = c$? What is the shape of the ellipse when a and b differ greatly in value? when a and b are nearly equal? when a and b are equal?

The equation of the circle $ax^2 + ay^2 = c$ differs from the equation of the ellipse $ax^2 + by^2 = c$ in the one respect that in the equation of the circle the coefficients of x^2 and y^2 are alike, and in the equation of the ellipse the coefficients of x^2 and y^2 are different.

When a and b differ greatly in value the ellipse is greatly elongated. When a is greater than b the ellipse cuts the axis of y at a greater distance from the origin than it cuts the axis of x; when a is less than b the ellipse cuts the axis of x at a greater distance from the origin than it cuts the axis of y.

When a and b are nearly equal the ellipse takes a nearly round form and is somewhat like a flattened circle.

When a and b are equal the ellipse becomes a circle.

11. The figure on page 309 shows the graph of the circle $x^2 + y^2 = 26$ and the graph of the hyperbola $xy = 5$. What are the coördinates of their points of intersection? What roots satisfy the equations?

The circle $x^2 + y^2 = 26$ and the hyperbola $xy = 5$ intersect in the points B, C, D, and A, which, as is shown by the figure, are the points $(1, 5)$, $(-1, -5)$, $(5, 1)$, and $(-5, -1)$, respectively.

Therefore, the solutions of the equations are $x = 1$, $y = 5$; $x = -1$, $y = -5$; $x = 5$, $y = 1$; and $x = -5$, $y = -1$.

12. Solve $x^2 + y^2 = 26$ and $xy = 5$ as simultaneous quadratics, and notice that the results are the answers to Example 11.

We have
$$\left.\begin{aligned} x^2 + y^2 &= 26 \\ xy &= 5 \end{aligned}\right\} \quad \begin{matrix}(1)\\(2)\end{matrix}$$

From (2),
$$x = \frac{5}{y}. \qquad (3)$$

Substitute the value of x in (1),
$$\frac{25}{y^2} + y^2 = 26.$$
$$25 + y^4 = 26y^2.$$
$$y^4 - 26y^2 + 25 = 0.$$
$$(y^2 - 1)(y^2 - 25) = 0.$$
$$(y - 1)(y + 1)(y - 5)(y + 5) = 0.$$
$$\therefore\ y = 1,\ -1,\ 5,\ \text{or}\ -5.$$

Substitute the value of y in (3),
$$x = 5,\ -5,\ 1,\ \text{or}\ -1.$$

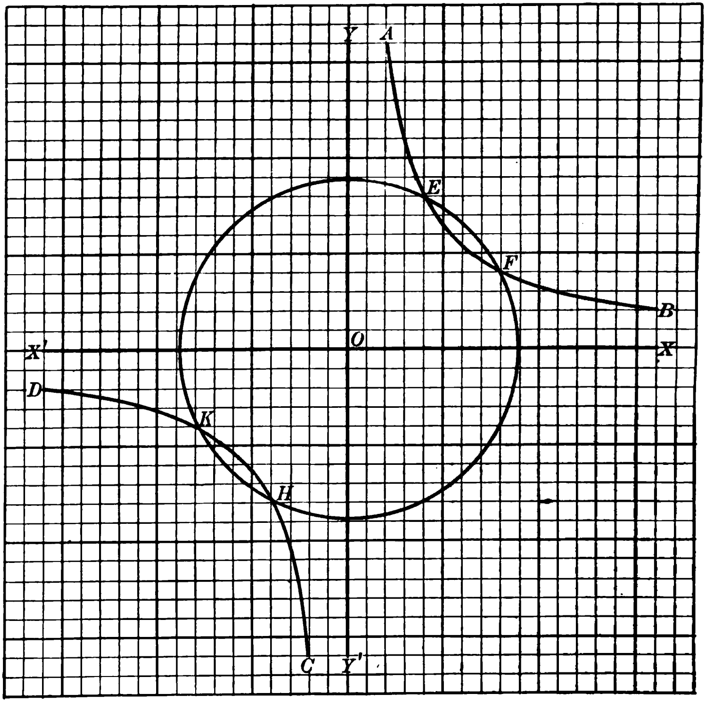
Therefore, $x = 1$, and $y = 5$;
$x = -1$, and $y = -5$;
$x = 5$, and $y = 1$;
or $x = -5$, and $y = -1$.

13. Solve by graphs,

$$\left.\begin{aligned} x^2 + y^2 &= 80 \\ xy &= 32 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

From (1), $x^2 = 80 - y^2$, and $y^2 = 80 - x^2$.

$\therefore\ x = \pm\sqrt{80 - y^2}$, and $y = \pm\sqrt{80 - x^2}$.



If $y = 0$, $x = \pm 8.94$;
$y = \pm 1$, $x = \pm 8.89$;
$y = \pm 2$, $x = \pm 8.72$;
$y = \pm 3$, $x = \pm 8.43$;
$y = \pm 4$, $x = \pm 8$;
$y = \pm 5$, $x = \pm 7.42$;
$y = \pm 6$, $x = \pm 6.63$;
$y = \pm 7$, $x = \pm 5.57$;
$y = \pm 8$, $x = \pm 4$.

If $x = 0$, $y = \pm 8.94$;
$x = \pm 1$, $y = \pm 8.89$;
$x = \pm 2$, $y = \pm 8.72$;
$x = \pm 3$, $y = \pm 8.43$;
$x = \pm 4$, $y = \pm 8$;
$x = \pm 5$, $y = \pm 7.42$;
$x = \pm 6$, $y = \pm 6.63$;
$x = \pm 7$, $y = \pm 5.57$;
$x = \pm 8$, $y = \pm 4$.

The graph of the equation $x^2 + y^2 = 80$ is the circle EFH, a circle with O for a centre and a radius equal to $\sqrt{80}$, or 8.944.

From (2), $y = \frac{32}{x}$, and $x = \frac{32}{y}$.

If $x = +\ 2,\ y = +16;$
$x = +\ 3,\ y = +10\frac{2}{3};$
$x = +\ 4,\ y = +\ 8;$
$x = +\ 5,\ y = +\ 6\frac{2}{5};$
$x = +\ 6,\ y = +\ 5\frac{1}{3};$
$x = +\ 7,\ y = +\ 4\frac{4}{7};$
$x = +\ 8,\ y = +\ 4;$
$x = +\ 9,\ y = +\ 3\frac{5}{9};$
$x = +10,\ y = +\ 3\frac{1}{5};$
$x = +11,\ y = +\ 2\frac{10}{11}.$

If $x = +12,\ y = +2\frac{2}{3};$
$x = +13,\ y = +2\frac{6}{13};$
$x = +14,\ y = +2\frac{2}{7};$
$x = +15,\ y = +2\frac{2}{15};$
$x = +16,\ y = +2;$
$y = +15,\ x = +2\frac{2}{15};$
$y = +14,\ x = +2\frac{2}{7};$
$y = +13,\ x = +2\frac{6}{13};$
$y = +12,\ x = +2\frac{2}{3};$
$y = +\ 9,\ x = +3\frac{5}{9}.$

For the negative of the assumed values of x and y in the left-hand columns in the above table there correspond values of y and x equal in absolute value to the values in the right-hand columns but with negative signs in place of the positive signs shown in the table.

The graph of the equation $xy = 32$ is the hyperbola of which the branches are AB and CD.

The points of intersection of the circle and hyperbola are E, F, H, K, which are the points (4, 8), (8, 4), (– 4, – 8), (– 8, – 4), respectively.

Therefore, the solutions of the given equations are $x = 4,\ y = 8$; $x = -4,\ y = -8$; $x = 8,\ y = 4$; and $x = -8,\ y = -4$.

14. Solve by graphs,

$$\left.\begin{aligned} x^2 + y^2 &= 34 \\ xy &= 15 \end{aligned}\right\} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

From (1), $x^2 = 34 - y^2$, and $y^2 = 34 - x^2$.

$$\therefore\ x = \pm\sqrt{34 - y^2}, \text{ and } y = \pm\sqrt{34 - x^2}.$$

If $y = 0,\ x = \pm 5.83;$
$y = \pm 1,\ x = \pm 5.74;$
$y = \pm 2,\ x = \pm 5.48;$
$y = \pm 3,\ x = \pm 5;$
$y = \pm 4,\ x = \pm 4.24;$
$y = \pm 5,\ x = \pm 3.$

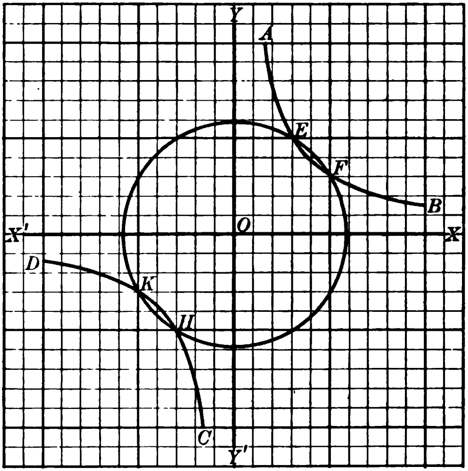
If $x = 0,\ y = \pm 5.83;$
$x = \pm 1,\ y = \pm 5.74;$
$x = \pm 2,\ y = \pm 5.48;$
$x = \pm 3,\ y = \pm 5;$
$x = \pm 4,\ y = \pm 4.24;$
$x = \pm 5,\ y = \pm 3.$

The graph of the equation $x^2 + y^2 = 34$ is the circle EFH, a circle with O for a centre and a radius equal to $\sqrt{34}$, or 5.83.

From (2), $x = \frac{15}{y}$, and $y = \frac{15}{x}$.

If $y = +2, x = +7\frac{1}{2}$;
$y = +3, x = +5$;
$y = +4, x = +3\frac{3}{4}$;
$y = +5, x = +3$;
$y = +6, x = +2\frac{1}{2}$;
$y = +7, x = +2\frac{1}{7}$;
$y = +8, x = +1\frac{7}{8}$.

If $y = +9, x = +1\frac{2}{3}$;
$y = +10, x = +1\frac{1}{2}$;
$x = +10, y = +1\frac{1}{2}$;
$x = +9, y = +1\frac{2}{3}$;
$x = +8, y = +1\frac{7}{8}$;
$x = +7, y = +2\frac{1}{7}$;
$x = +6, y = +2\frac{1}{2}$.



Similarly for the negative of the above values, as in Example 13.

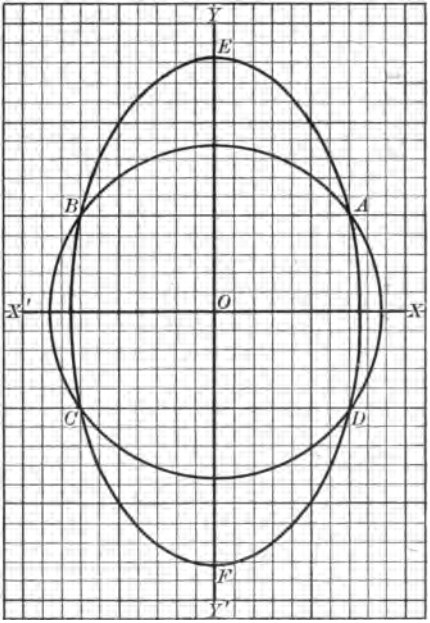
The graph of the equation $xy = 15$ is the hyperbola of which the branches are AB and CD.

The points of intersection of the circle and hyperbola are E, F, H, K, which are the points (3, 5), (5, 3), (– 3, – 5), (– 5, – 3), respectively.

Therefore, the solutions of the given equations are $x = 3$, $y = 5$; $x = -3$, $y = -5$; $x = 5$, $y = 3$; and $x = -5$, $y = -3$.

15. Solve by graphs,
$$\left.\begin{aligned} x^2 + y^2 &= 74 \\ 3x^2 + y^2 &= 172 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

From (1), $x^2 = 74 - y^2$, and $y^2 = 74 - x^2$.
$$\therefore\ x = \pm\sqrt{74 - y^2}, \text{ and } y = \pm\sqrt{74 - x^2}.$$



If $y = 0$, $x = \pm 8.60$;
$y = \pm 1$, $x = \pm 8.54$;
$y = \pm 2$, $x = \pm 8.37$;
$y = \pm 3$, $x = \pm 8.06$;
$y = \pm 4$, $x = \pm 7.62$;
$y = \pm 5$, $x = \pm 7$;
$y = \pm 6$, $x = \pm 6.16$;
$y = \pm 7$, $x = \pm 5$;
$y = \pm 8$, $x = \pm 3.16$.

If $x = 0$, $y = \pm 8.60$;
$x = \pm 1$, $y = \pm 8.54$;
$x = \pm 2$, $y = \pm 8.37$;
$x = \pm 3$, $y = \pm 8.06$;
$x = \pm 4$, $y = \pm 7.62$;
$x = \pm 5$, $y = \pm 7$;
$x = \pm 6$, $y = \pm 6.16$;
$x = \pm 7$, $y = \pm 5$;
$x = \pm 8$, $y = \pm 3.16$.

The graph of the equation $x^2 + y^2 = 74$ is the circle ABC, a circle with O for a centre and a radius equal to $\sqrt{74}$, or 8.602.

From (2), $$y^2 = 172 - 3x^2, \text{ and } x^2 = \frac{172 - y^2}{3}.$$
$$\therefore\ y = \pm\sqrt{172 - 3x^2}, \text{ and } x = \pm\tfrac{1}{3}\sqrt{516 - 3y^2}.$$

If $x = 0$, $y = \pm 13.11$;
$x = \pm 1$, $y = \pm 13$;
$x = \pm 2$, $y = \pm 12.65$;
$x = \pm 3$, $y = \pm 12.04$;
$x = \pm 4$, $y = \pm 11.14$;
$x = \pm 5$, $y = \pm 9.85$.
$x = \pm 6$, $y = \pm 8$;
$x = \pm 7$, $y = \pm 5$.

If $y = 0$, $x = \pm 7.57$;
$y = \pm 1$, $x = \pm 7.55$;
$y = \pm 2$, $x = \pm 7.48$;
$y = \pm 3$, $x = \pm 7.37$;
$y = \pm 4$, $x = \pm 7.21$;
$y = \pm 6$, $x = \pm 6.73$;
$y = \pm 7$, $x = \pm 6.40$;
$y = \pm 9$, $x = \pm 5.51$.

The graph of the equation $3x^2 + y^2 = 172$ is the ellipse EF.

The points of intersection of the circle and ellipse are A, B, C, D, which are the points (7, 5), (– 7, 5), (– 7, – 5), (7, – 5), respectively.

Therefore, the solutions of the given equations are $x = 7$, $y = 5$; $x = 7$, $y = -5$; $x = -7$, $y = 5$; and $x = -7$, $y = -5$.

16. Solve by graphs, $$\left.\begin{aligned} 5x^2 + y^2 &= 321 \\ 5y^2 - 196x &= 0 \end{aligned}\right\}\qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

From (1), $$y^2 = 321 - 5x^2, \text{ and } x^2 = \frac{321 - y^2}{5}.$$
$$\therefore\ y = \pm\sqrt{321 - 5x^2}, \text{ and } x = \pm\tfrac{1}{5}\sqrt{1605 - 5y^2}.$$

If $x = 0$, $y = \pm 17.92$;
$x = \pm 1$, $y = \pm 17.78$;
$x = \pm 2$, $y = \pm 17.35$;
$x = \pm 3$, $y = \pm 16.61$;
$x = \pm 4$, $y = \pm 15.52$;
$x = \pm 5$, $y = \pm 14$;
$x = \pm 6$, $y = \pm 11.87$;
$x = \pm 7$, $y = \pm 8.72$;
$x = \pm 8$, $y = \pm 1$.

If $y = 0$, $x = \pm 8.01$;
$y = \pm 1$, $x = \pm 8$;
$y = \pm 2$, $x = \pm 7.96$;
$y = \pm 3$, $x = \pm 7.90$;
$y = \pm 4$, $x = \pm 7.81$;
$y = \pm 5$, $x = \pm 7.69$;
$y = \pm 6$, $x = \pm 7.55$;
$y = \pm 7$, $x = \pm 7.38$;
$y = \pm 8$, $x = \pm 7.17$.

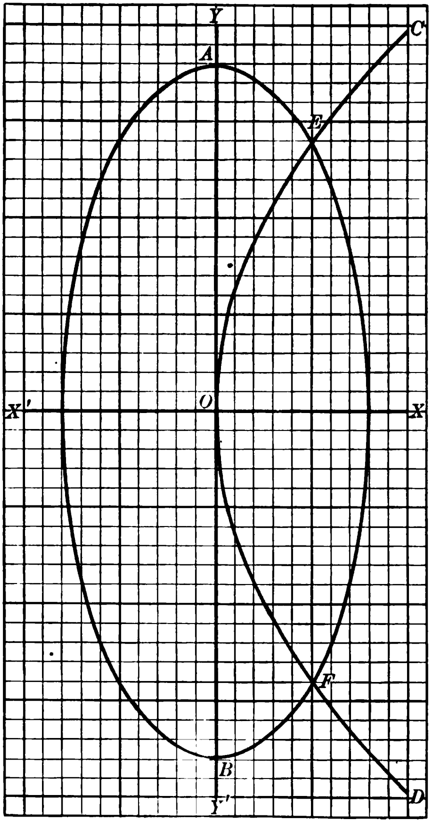
The graph of the equation $5x^2 + y^2 = 321$ is the ellipse AEB.

From (2), $$y^2 = \frac{196x}{5}, \text{ and } 196x = 5y^2.$$

$$\therefore\ y = \pm \tfrac{14}{5}\sqrt{5x}, \text{ and } x = \tfrac{5}{196}y^2.$$

If $x = 0$, $y = 0$;
$x = +1$, $y = \pm 6.26$;
$x = +2$, $y = \pm 8.85$;
$x = +3$, $y = \pm 10.84$;
$x = +4$, $y = \pm 12.52$;
$x = +5$, $y = \pm 14$;
$x = +6$, $y = \pm 15.34$;
$x = +7$, $y = \pm 16.57$;
$x = +8$, $y = \pm 17.71$;
$x = +9$, $y = \pm 18.78$;
$x = +10$, $y = \pm 19.80$.

If $y = 0$, $x = 0$;
$y = \pm 1$, $x = +0.026$;
$y = \pm 2$, $x = +0.10$;
$y = \pm 3$, $x = +0.23$;
$y = \pm 4$, $x = +0.41$;
$y = \pm 5$, $x = +0.64$;
$y = \pm 7$, $x = +1.25$;
$y = \pm 8$, $x = +1.63$;
$y = \pm 10$, $x = +2.55$.



The graph of the equation $5y^2 - 196x = 0$ is the parabola COD.

The ellipse AEB and the parabola COD intersect in the points E and F, which are the points (5, 14) and (5, -14), respectively.

Therefore, the solutions of the given equations are $x = 5$, $y = 14$, and $x = 5$, $y = -14$.

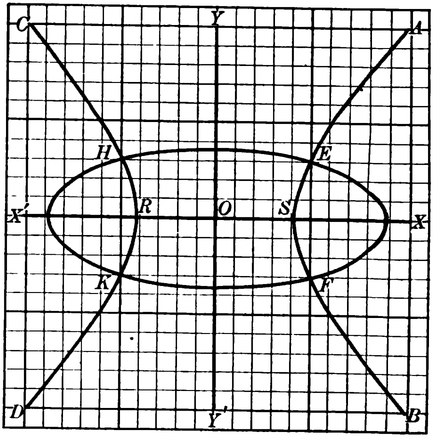
17. Solve by graphs,
$$\left.\begin{aligned} x^2 + 6y^2 &= 79 \\ 5x^2 - 4y^2 &= 89 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

From (1), $x^2 = 79 - 6y^2$, and $y^2 = \dfrac{79 - x^2}{6}$.

$$\therefore\ x = \pm\sqrt{79 - 6y^2}, \text{ and } y = \pm\tfrac{1}{6}\sqrt{474 - 6x^2}.$$

If $y = 0$, $x = \pm 8.89$;
$y = \pm 1$, $x = \pm 8.54$;
$y = \pm 2$, $x = \pm 7.42$;
$y = \pm 3$, $x = \pm 5$.

If $x = 0$, $y = \pm 3.63$;
$x = \pm 1$, $y = \pm 3.61$;
$x = \pm 2$, $y = \pm 3.54$;
$x = \pm 3$, $y = \pm 3.42$;
$x = \pm 4$, $y = \pm 3.24$;
$x = \pm 6$, $y = \pm 2.68$;
$x = \pm 7$, $y = \pm 2.24$;
$x = \pm 8$, $y = \pm 1.58$.



The graph of the equation $x^2 + 6y^2 = 79$ is the ellipse *EFH*.

From (2),
$$5x^2 = 89 + 4y^2.$$
$$x^2 = \frac{89 + 4y^2}{5}.$$
$$\therefore\ x = \pm\tfrac{1}{5}\sqrt{445 + 20y^2}.$$

If $y = 0$, $x = \pm 4.22$;
$y = \pm 1$, $x = \pm 4.31$;
$y = \pm 2$, $x = \pm 4.58$;
$y = \pm 3$, $x = \pm 5$;
$y = \pm 4$, $x = \pm 5.53$;
$y = \pm 5$, $x = \pm 6.15$.

If $y = \pm 6$, $x = \pm 6.83$;
$y = \pm 7$, $x = \pm 7.55$;
$y = \pm 8$, $x = \pm 8.31$;
$y = \pm 9$, $x = \pm 9.09$;
$y = \pm 10$, $x = \pm 9.89$.

The graph of the equation $5x^2 - 4y^2 = 89$ is the hyperbola of which the branches are *ASB* and *CRD*.

The ellipse and the hyperbola intersect in the points *E*, *F*, *H*, *K*, which are the points (5, 3), (5, − 3), (− 5, 3), (− 5, − 3), respectively.

Therefore, the solutions of the given equations are $x = 5$, $y = 3$; $x = 5$, $y = -3$; $x = -5$, $y = 3$; and $x = -5$, $y = -3$.

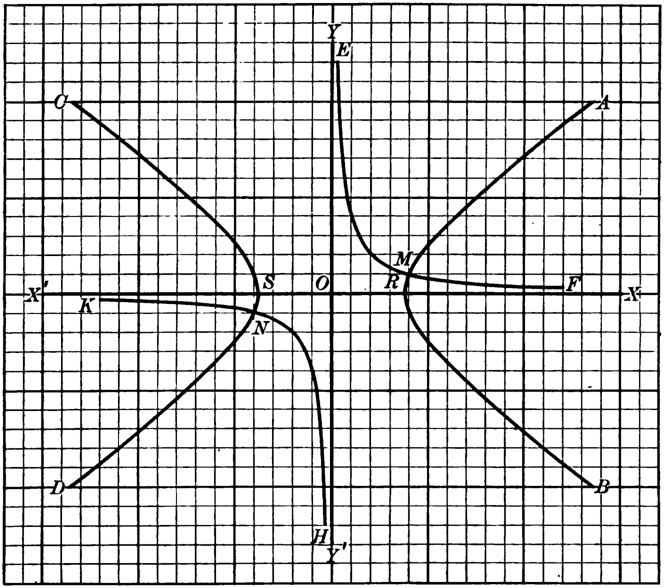
18. Solve by graphs, $$\left.\begin{aligned} 3x^2 - 5y^2 &= 43 \\ xy &= 4 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

From (1), $$x^2 = \frac{43 + 5y^2}{3}.$$

$$\therefore\ x = \pm \tfrac{1}{3}\sqrt{129 + 15y^2}.$$

If $y = 0$, $x = \pm 3.79$;
$y = \pm 1$, $x = \pm 4$;
$y = \pm 2$, $x = \pm 4.58$;
$y = \pm 3$, $x = \pm 5.42$;
$y = \pm 4$, $x = \pm 6.40$;
$y = \pm 5$, $x = \pm 7.48$.

If $y = \pm 6$, $x = \pm 8.62$;
$y = \pm 7$, $x = \pm 9.80$;
$y = \pm 8$, $x = \pm 11$;
$y = \pm 9$, $x = \pm 12.22$;
$y = \pm 10$, $x = \pm 13.45$.



The graph of the equation $3x^2 - 5y^2 = 43$ is the hyperbola of which the branches are *ARB* and *CSD*.

From (2), $$x = \frac{4}{y}, \text{ and } y = \frac{4}{x}.$$

[illegible] $y = -6$, $x = -\frac{2}{3}$;
[illegible] $y = -7$, $x = -\frac{4}{7}$;
[illegible] $y = -8$, $x = -\frac{1}{2}$;
[illegible] $y = -9$, $x = -\frac{4}{9}$;
[illegible] $y = -10$, $x = -\frac{2}{5}$;
[illegible] $y = -11$, $x = -\frac{4}{11}$;
[illegible] $y = -12$, $x = -\frac{1}{3}$.

Similarly for the negative of the above values, as in Example 13.

The graph of the equation $xy = 4$ is the hyperbola of which the branches are [illegible].

The two hyperbolas intersect in the points M and N, which are the points (4, 1) and (−4, −1), respectively.

Therefore, the solutions of the given equations are $x = 4$, $y = 1$, and $x = -4$, $y = -1$.

19. Explain the meaning of the imaginary roots of the equations in Example 18.

Solve the equations in Example 18.

We have $$3x^2 - 5y^2 = 43 \qquad (1)$$
$$xy = 4 \qquad (2)$$

From (2), $$y = \frac{4}{x}. \qquad (3)$$

Substitute the value of y in (1), $$3x^2 - \frac{80}{x^2} = 43.$$
$$3x^4 - 80 = 43x^2.$$
$$3x^4 - 43x^2 = 80.$$
$$36x^4 - [illegible] + 1849 = 2809.$$
$$6x^2 - 43 = \pm 53.$$
$$6x^2 = 96, \text{ or } -10.$$
$$x^2 = 16, \text{ or } -\tfrac{5}{3}.$$
$$\therefore x = \pm 4, \text{ or } \pm \tfrac{1}{3}\sqrt{-15}.$$

Substitute the value of x in (3), $$y = \pm 1, \text{ or } \mp \tfrac{4}{5}\sqrt{-15}.$$

Therefore, $x = 4$ and $y = 1$, $x = -4$ and $y = -1$, $x = \frac{1}{3}\sqrt{-15}$ and $y = -\frac{4}{5}\sqrt{-15}$, or $x = -\frac{1}{3}\sqrt{-15}$ and $y = \frac{4}{5}\sqrt{-15}$.

The meaning of the imaginary solutions, the two last pairs of solutions, is that, while from the form of the two given equations we expect four pairs of solutions, the graphs do really intersect in only two points, the two first pairs of solutions. In other words, the meaning of the imaginary solutions is that the graphs do not intersect.

20. Determine, by inspection, the shape of the graph of $2x-11y=7$.

Since the given equation is of the first degree, its graph is a straight line.

21. Determine, by inspection, the shape of the graph of $5x^2+8y^2=6$.

The given equation is of the second degree, with the terms in x, y, and xy missing. The signs of the x^2 and y^2 terms are alike and the coefficients are different.

Therefore, the graph of the equation is an ellipse.

22. Determine, by inspection, the shape of the graph of $y^2=8x$.

The given equation is of the second degree, with the terms in x^2, y, and xy missing.

Therefore, the graph of the equation is a parabola.

23. Determine, by inspection, the shape of the graph of $x^2+y^2=18$.

The given equation is of the second degree; the x^2 and y^2 terms have like signs and equal coefficients.

Therefore, the graph of the equation is a circle.

24. Determine, by inspection, the shape of the graph of $x=3y$.

The given equation is of the first degree.

Therefore, the graph of the equation is a straight line.

25. Determine, by inspection, the shape of the graph of $2x^2=7y$.

The given equation is of the second degree, with the terms in y^2, x, and xy missing.

Therefore, the graph of the equation is a parabola.

26. Determine, by inspection, the shape of the graph of $4x^2+4y^2=27$.

The given equation is of the second degree; the x^2 and y^2 terms have like signs and equal coefficients.

Therefore, the graph of the equation is a circle.

27. Determine, by inspection, the shape of the graph of $xy=12$.

The given equation is of the second degree, with all the terms missing with the exception of the xy term.

Therefore, the graph of the equation is an hyperbola.

28. Determine, by inspection, the shape of the graph of $2x^2-5y^2=12$.

The given equation is of the second degree; the x^2 and y^2 terms have unlike signs.

Therefore, the graph of the equation is an hyperbola.

Exercise 124. Page 311

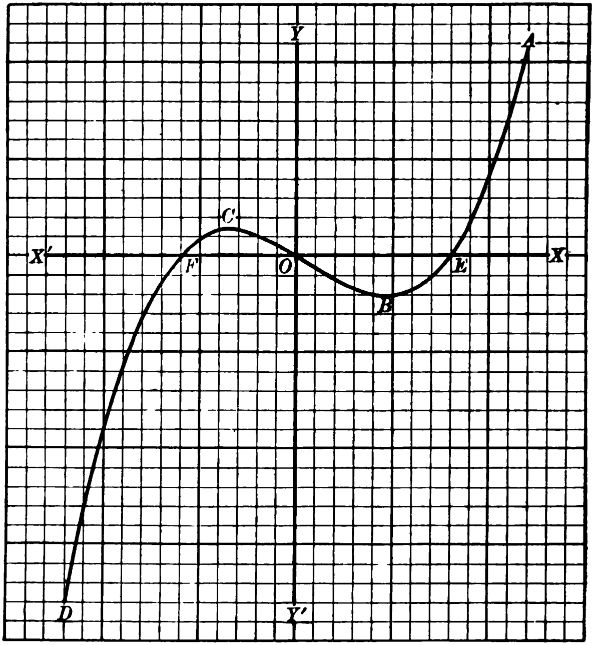
1. Find by a graph the roots of $x^3 - x^2 - 12x = 0$.

Put $x^3 - x^2 - 12x = y$.

If $x = 0$, $y = 0$;
$x = +\frac{1}{2}$, $y = -6\frac{1}{8}$;
$x = +1$, $y = -12$;
$x = +1\frac{1}{2}$, $y = -16\frac{7}{8}$;
$x = +2$, $y = -20$;
$x = +2\frac{1}{2}$, $y = -20\frac{5}{8}$;
$x = +3$, $y = -18$;
$x = +3\frac{1}{2}$, $y = -11\frac{3}{8}$;
$x = +4$, $y = -0$.

If $x = +4\frac{1}{2}$, $y = +16\frac{7}{8}$;
$x = +5$, $y = +40$;
$x = +5\frac{1}{2}$, $y = +70\frac{1}{8}$;
$x = +6$, $y = +108$;
$x = -\frac{1}{2}$, $y = +5\frac{5}{8}$;
$x = -1$, $y = +10$;
$x = -1\frac{1}{2}$, $y = +12\frac{3}{8}$;
$x = -2$, $y = +12$.

If $x = -2\frac{1}{2}$, $y = +8\frac{1}{8}$;
$x = -3$, $y = 0$;
$x = -3\frac{1}{2}$, $y = -13\frac{1}{8}$;
$x = -4$, $y = -32$;
$x = -4\frac{1}{2}$, $y = -57\frac{3}{8}$;
$x = -5$, $y = -90$;
$x = -5\frac{1}{2}$, $y = -130\frac{5}{8}$;
$x = -6$, $y = -180$.



In plotting the graph use two spaces of the coördinate paper for one unit of x, and one space for ten units of y.

The graph of the equation $x^3 - x^2 - 12x = 0$ is the curve $ABCD$.

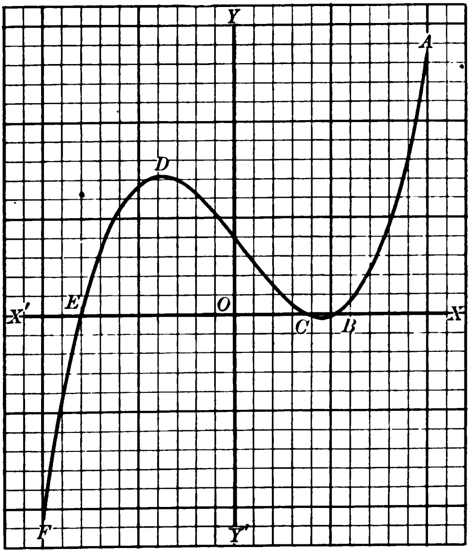
The graph cuts the axis of x at the points E, O, and F, the abscissas of which are 4, 0, and -3, respectively.

Therefore, the roots of the given equation are 0, 4, and -3.

2. Find by a graph the roots of $2x^3 - x^2 - 26x + 40 = 0$.

Put $2x^3 - x^2 - 26x + 40 = y$.

If $x = 0$,	$y = +40$;	If $x = +3$,	$y = +7$;	If $x = -2$,	$y = +72$;
$x = +\frac{1}{2}$,	$y = +27$;	$x = +3\frac{1}{2}$,	$y = +22\frac{1}{2}$;	$x = -2\frac{1}{2}$,	$y = +67\frac{1}{2}$;
$x = +1$,	$y = +15$;	$x = +4$,	$y = +48$;	$x = -3$,	$y = +55$;
$x = +1\frac{1}{2}$,	$y = +5\frac{1}{2}$;	$x = +4\frac{1}{2}$,	$y = +85$;	$x = -3\frac{1}{2}$,	$y = +33$;
$x = +2$,	$y = 0$;	$x = +5$,	$y = +135$;	$x = -4$,	$y = 0$;
$x = +2\frac{1}{4}$,	$y = -\frac{25}{32}$;	$x = -\frac{1}{2}$,	$y = +52\frac{1}{2}$;	$x = -4\frac{1}{2}$,	$y = -45\frac{1}{2}$;
$x = +2\frac{1}{2}$,	$y = 0$;	$x = -1$,	$y = +63$;	$x = -5$,	$y = -105$.
$x = +2\frac{3}{4}$,	$y = +2\frac{17}{32}$.	$x = -1\frac{1}{2}$,	$y = +70$.		



In plotting the graph use two spaces of the coördinate paper for one unit of x, and one space for ten units of y.

The graph of the equation $2x^3 - x^2 - 26x + 40 = 0$ is the curve *ABDF*.

The graph cuts the axis of x at the points B, C, and E, the abscissas of which are $2\frac{1}{2}$, 2, and -4, respectively.

Therefore, the roots of the given equation are 2, $2\frac{1}{2}$, and -4.

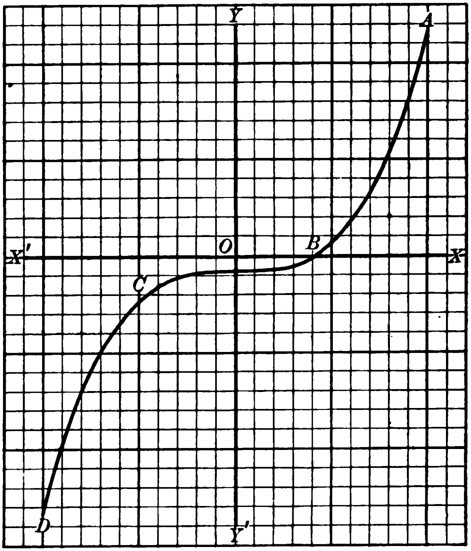
3. Find by a graph the number of real roots of $x^3 - 8 = 0$.

Put $x^3 - 8 = y$.

If $x = 0$, $y = -8$;
$x = +\frac{1}{2}$, $y = -7\frac{7}{8}$;
$x = +1$, $y = -7$;
$x = +1\frac{1}{2}$, $y = -4\frac{5}{8}$;
$x = +2$, $y = 0$;
$x = +2\frac{1}{2}$, $y = +7\frac{5}{8}$;
$x = +3$, $y = +19$.

If $x = +3\frac{1}{2}$, $y = +34\frac{7}{8}$;
$x = +4$, $y = +56$;
$x = +4\frac{1}{2}$, $y = +83\frac{1}{8}$;
$x = +5$, $y = +117$;
$x = -\frac{1}{2}$, $y = -8\frac{1}{8}$;
$x = -1$, $y = -9$;
$x = -1\frac{1}{2}$, $y = -11\frac{3}{8}$.

If $x = -2$, $y = -16$;
$x = -2\frac{1}{2}$, $y = -23\frac{5}{8}$;
$x = -3$, $y = -35$;
$x = -3\frac{1}{2}$, $y = -50\frac{7}{8}$;
$x = -4$, $y = -72$;
$x = -4\frac{1}{2}$, $y = -99\frac{1}{8}$;
$x = -5$, $y = -133$.



In plotting the graph use two spaces of the coördinate paper for one unit of x, and one space for ten units of y.

The graph of the equation $x^3 - 8 = 0$ is the curve *ABCD*.

The graph cuts the axis of x at the point B, the abscissa of which is 2.

Therefore, one root of the given equation is 2.

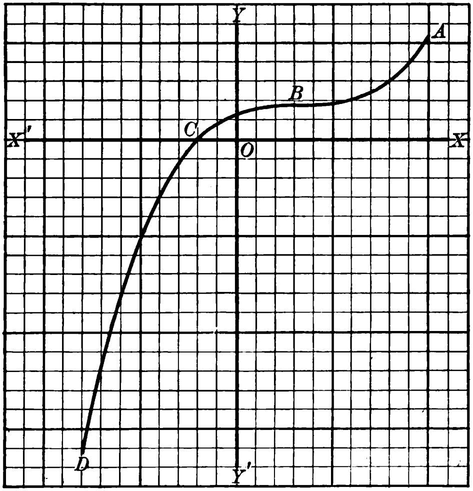
It is evident from the degree of the given equation that it has three roots. It is also evident that if x is taken greater than $+5$ or less than -5, the absolute value of y becomes greater and greater and the graph recedes farther and farther from XX'.

It is reasonable, therefore, to conclude that since the graph cuts the axis of x in only one point, the given equation has one real root and two imaginary roots. The only possible chance for error in making this conclusion is that the graph *might* cross XX' *twice* between two adjacent assumed values of x.

Therefore, the equation has one real root.

4. Find by a graph the number of real roots of $x^3-5x^2+8x+14=0$.

Put $x^3-5x^2+8x+14=y$.



If $x=0,\ y=+14;$
$x=+\frac{1}{2},\ y=+16\frac{7}{8};$
$x=+1,\ y=+18;$
$x=+1\frac{1}{2},\ y=+18\frac{1}{8};$
$x=+2,\ y=+18;$
$x=+2\frac{1}{2},\ y=+18\frac{3}{8};$
$x=+3,\ y=+20.$

If $x=+3\frac{1}{2},\ y=+23\frac{5}{8};$
$x=+4,\ y=+30;$
$x=+4\frac{1}{2},\ y=+39\frac{7}{8};$
$x=+5,\ y=+54;$
$x=-\frac{1}{2},\ y=+8\frac{5}{8};$
$x=-1,\ y=0.$

If $x=-1\frac{1}{2},\ y=-12\frac{5}{8};$
$x=-2,\ y=-30;$
$x=-2\frac{1}{2},\ y=-52\frac{7}{8};$
$x=-3,\ y=-82;$
$x=-3\frac{1}{2},\ y=-118\frac{1}{8};$
$x=-4,\ y=-162.$

In plotting the graph use two spaces of the coördinate paper for one unit of x, and one space for ten units of y.

The graph of the equation $x^3-5x^2+8x+14=0$ is the curve $ABCD$.

The graph cuts the axis of x at the point C, the abscissa of which is -1.

Therefore, one root of the given equation is -1.

It is evident from the degree of the given equation that it has three roots. It is also evident that as x is taken greater than $+5$ or less than -4, the absolute value of y becomes greater and greater, and the graph recedes farther and farther from XX'.

It is reasonable, therefore, to conclude that since the graph cuts the axis of x in only one point, the given equation has one real root and two imaginary roots. The only possible chance for error in making this conclusion is that the graph *might* cross XX' *twice* between two adjacent assumed values of x.

Therefore, the equation has one real root.

Exercise 125. Page 312

1. The sum of the squares of two numbers is 130, and twice the product of the numbers is 126. Find the numbers.

Let $x =$ one number,

and $y =$ the other number.

$$\therefore x^2+y^2=130, \qquad (1)$$

and
$$2xy=126. \qquad (2)$$

Add (1) and (2), $x^2+2xy+y^2=256.$

Extract the root, $x+y=\pm 16. \qquad (3)$

Subtract (2) from (1), $x^2-2xy+y^2=4.$

Extract the root, $x-y=\pm 2. \qquad (4)$

Add (3) and (4), $2x=\pm 18, \text{ or } \pm 14.$

$\therefore x=\pm 9, \text{ or } \pm 7.$

Subtract (4) from (3), $2y=\pm 14, \text{ or } \pm 18.$

$\therefore y=\pm 7, \text{ or } \pm 9.$

Therefore, the numbers required are 9 and 7, or -9 and -7.

2. The sum of two numbers added to the sum of their squares is 686, and the difference of the numbers added to the difference of their squares is 74. Find the numbers.

Let $x =$ the larger number,

and $y =$ the smaller number.

$$\therefore\ x + y + x^2 + y^2 = 686, \qquad (1)$$

and $$x - y + x^2 - y^2 = 74. \qquad (2)$$

Add (1) and (2), $$2x + 2x^2 = 760.$$

$$x^2 + x = 380.$$

$$x^2 + x - 380 = 0.$$

$$(x - 19)(x + 20) = 0.$$

$$\therefore\ x = 19, \text{ or } -20.$$

Substitute the value of x in (1),

$$19 + y + 361 + y^2 = 686,$$

or $$-20 + y + 400 + y^2 = 686.$$

$$y^2 + y - 306 = 0.$$

$$(y - 17)(y + 18) = 0.$$

$$\therefore\ y = 17, \text{ or } -18.$$

Therefore, the required numbers are 19 and 17, 19 and -18, -20 and 17, or -20 and -18.

3. If a number of two digits is divided by the product of the digits, the quotient is 3. If the digits are reversed and the resulting number is divided by the sum of the digits, the quotient is 7. Find the number.

Let $x =$ the digit in the tens' place,

and $y =$ the digit in the units' place.

Then $10x + y =$ the number,

and $10y + x =$ the number with the digits reversed.

$$\therefore\ \frac{10x + y}{xy} = 3, \qquad (1)$$

and $$\frac{10y + x}{x + y} = 7. \qquad (2)$$

From (2), $$10y + x = 7x + 7y.$$

$$3y = 6x.$$

$$\therefore\ y = 2x. \qquad (3)$$

Substitute the value of y in (1), $$\frac{10x + 2x}{2x^2} = 3.$$

$$10x + 2x = 6x^2.$$

$$6x^2 - 12x = 0.$$

$$x^2 - 2x = 0.$$

$$x(x - 2) = 0.$$

$$\therefore\ x = 0, \text{ or } 2.$$

Substitute the value of x in (3), $$y = 0, \text{ or } 4.$$

Then $$10x + y = 24.$$

Therefore, the required number is 24.

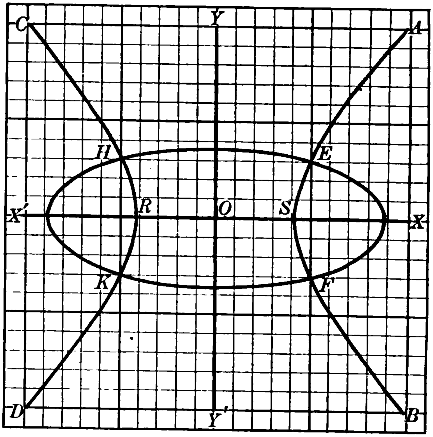
17. Solve by graphs, $$\left.\begin{aligned} x^2 + 6y^2 &= 79 \\ 5x^2 - 4y^2 &= 89 \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

From (1), $x^2 = 79 - 6y^2$, and $y^2 = \dfrac{79 - x^2}{6}$.

$$\therefore\ x = \pm\sqrt{79 - 6y^2}, \text{ and } y = \pm\tfrac{1}{6}\sqrt{474 - 6x^2}.$$

If $y = 0$, $x = \pm 8.89$;
$y = \pm 1$, $x = \pm 8.54$;
$y = \pm 2$, $x = \pm 7.42$;
$y = \pm 3$, $x = \pm 5$.

If $x = 0$, $y = \pm 3.63$;
$x = \pm 1$, $y = \pm 3.61$;
$x = \pm 2$, $y = \pm 3.54$;
$x = \pm 3$, $y = \pm 3.42$;
$x = \pm 4$, $y = \pm 3.24$;
$x = \pm 6$, $y = \pm 2.68$;
$x = \pm 7$, $y = \pm 2.24$;
$x = \pm 8$, $y = \pm 1.58$.



The graph of the equation $x^2 + 6y^2 = 79$ is the ellipse *EFH*.

From (2), $$5x^2 = 89 + 4y^2.$$
$$x^2 = \frac{89 + 4y^2}{5}.$$
$$\therefore\ x = \pm\tfrac{1}{5}\sqrt{445 + 20y^2}.$$

If $y = 0$, $x = \pm 4.22$;
$y = \pm 1$, $x = \pm 4.31$;
$y = \pm 2$, $x = \pm 4.58$;
$y = \pm 3$, $x = \pm 5$;
$y = \pm 4$, $x = \pm 5.53$;
$y = \pm 5$, $x = \pm 6.15$.

If $y = \pm 6$, $x = \pm 6.83$;
$y = \pm 7$, $x = \pm 7.55$;
$y = \pm 8$, $x = \pm 8.31$;
$y = \pm 9$, $x = \pm 9.09$;
$y = \pm 10$, $x = \pm 9.89$.

The graph of the equation $5x^2 - 4y^2 = 89$ is the hyperbola of which the branches are *ASB* and *CRD*.

The ellipse and the hyperbola intersect in the points *E*, *F*, *H*, *K*, which are the points (5, 3), (5, − 3), (− 5, 3), (− 5, − 3), respectively.

Therefore, the solutions of the given equations are $x = 5$, $y = 3$; $x = 5$, $y = -3$; $x = -5$, $y = 3$; and $x = -5$, $y = -3$.

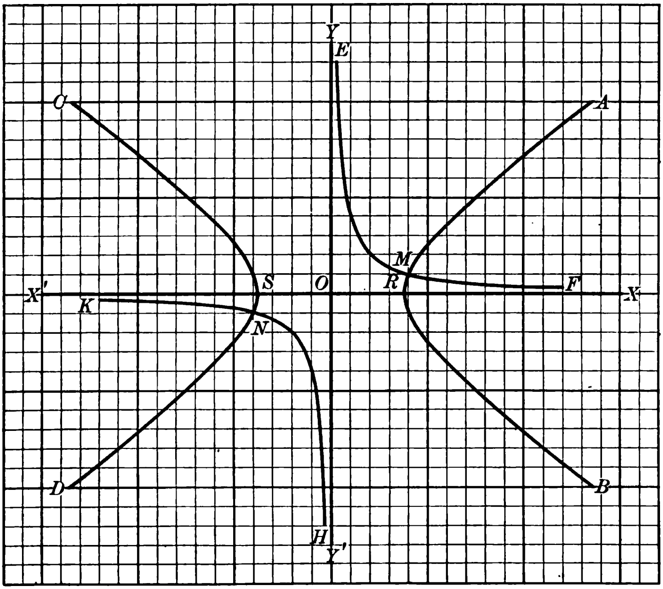
18. Solve by graphs, $\left.\begin{aligned}3x^2-5y^2&=43\\ xy&=\ \ 4\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$

From (1), $$x^2=\frac{43+5y^2}{3}.$$

$$\therefore\ x=\pm\tfrac{1}{3}\sqrt{129+15y^2}.$$

If $y=0$, $x=\pm 3.79$;
$y=\pm 1$, $x=\pm 4$;
$y=\pm 2$, $x=\pm 4.58$;
$y=\pm 3$, $x=\pm 5.42$;
$y=\pm 4$, $x=\pm 6.40$;
$y=\pm 5$, $x=\pm 7.48$.

If $y=\pm 6$, $x=\pm 8.62$;
$y=\pm 7$, $x=\pm 9.80$;
$y=\pm 8$, $x=\pm 11$;
$y=\pm 9$, $x=\pm 12.22$;
$y=\pm 10$, $x=\pm 13.45$.



The graph of the equation $3x^2-5y^2=43$ is the hyperbola of which the branches are *ARB* and *CSD*.

From (2), $$x=\frac{4}{y},\ \text{and}\ y=\frac{4}{x}.$$

$$\therefore \frac{10x+y}{xy} = 5 + \frac{2}{xy}, \qquad (1)$$

and
$$\frac{10y+x}{xy} = 2 + \frac{5}{xy}. \qquad (2)$$

Multiply (1) by xy, $\quad 10x + y = 5xy + 2. \qquad (3)$

Multiply (2) by xy, $\quad 10y + x = 2xy + 5. \qquad (4)$

Multiply (3) by 2, $\quad 20x + 2y = 10xy + 4. \qquad (5)$

Multiply (4) by 5, $\quad 5x + 50y = 10xy + 25. \qquad (6)$

Subtract (6) from (5), $\quad 15x - 48y = -21.$

$$5x - 16y = -7.$$

$$5x = 16y - 7.$$

$$\therefore x = \frac{16y-7}{5}. \qquad (7)$$

Substitute the value of x in (4),

$$10y + \frac{16y-7}{5} = \frac{32y^2 - 14y}{5} + 5.$$

$$50y + 16y - 7 = 32y^2 - 14y + 25.$$

$$32y^2 - 80y + 32 = 0.$$

$$2y^2 - 5y + 2 = 0.$$

$$(y-2)(2y-1) = 0.$$

$$\therefore y = 2, \text{ or } \tfrac{1}{2}.$$

Substitute the value of y in (7), $\quad x = 5, \text{ or } \tfrac{1}{5}.$

Then $\quad 10x + y = 52.$

Therefore, the required number is 52.

8. A walks a quarter of a mile an hour faster than B, and, in consequence, requires a quarter of an hour less time to walk 15 miles. Find the rate at which each walks.

Let $\quad x =$ the number of miles A walks per hour,

and $\quad y =$ the number of miles B walks per hour.

$$\therefore x - y = \tfrac{1}{4}, \qquad (1)$$

and
$$\frac{15}{y} - \frac{15}{x} = \frac{1}{4}. \qquad (2)$$

From (2), $\quad 60x - 60y = xy. \qquad (3)$

Substitute in (3) the value of $x - y$ from (1),

$$15 = xy.$$

$$4xy = 60. \qquad (4)$$

Square (1), $\quad x^2 - 2xy + y^2 = \tfrac{1}{16}. \qquad (5)$

Add (4) and (5), $\quad x^2 + 2xy + y^2 = \tfrac{961}{16}.$

$$x + y = \pm \tfrac{31}{4}. \qquad (6)$$

Add (1) and (6), $2x = 8$, or $-\frac{15}{2}$.

$\therefore x = 4$, or $-3\frac{3}{4}$.

Subtract (1) from (6), $2y = \frac{15}{2}$, or -8.

$\therefore y = 3\frac{3}{4}$, or -4.

Therefore, A walks 4 miles an hour, and B walks $3\frac{3}{4}$ miles an hour.

9. Two square gardens have together a surface of 3469 square yards. A rectangular garden whose dimensions are respectively equal to the sides of the two squares contains $24\frac{1}{2}$ square yards less than half the area of the two squares together. Find the sides of the two squares.

Let $x =$ the number of yards in a side of one square,

and $y =$ the number of yards in a side of the other square.

$\therefore x^2 + y^2 = 3469,$ (1)

and $xy = \frac{1}{2}(3469) - 24\frac{1}{2}.$ (2)

Multiply (2) by 2, $2xy = 3420.$ (3)

Add (1) and (3), $x^2 + 2xy + y^2 = 6889.$

$x + y = \pm 83.$ (4)

Subtract (3) from (1), $x^2 - 2xy + y^2 = 49.$

$x - y = \pm 7.$ (5)

Add (4) and (5), $2x = 90$, or -76.

$\therefore x = 45$, or -38.

Subtract (5) from (4), $2y = 76$, or -90.

$\therefore y = 38$, or -45.

Therefore, the sides of the two squares are 45 yards and 38 yards.

10. The sum of the squares of two numbers is 370. If the smaller number was 1 greater and the larger number 3 greater, the sum of their squares would be 500. Find the numbers.

Let $x =$ the larger number,

and $y =$ the smaller number.

$\therefore x^2 + y^2 = 370,$ (1)

and $(x + 3)^2 + (y + 1)^2 = 500.$ (2)

From (2), $x^2 + 6x + 9 + y^2 + 2y + 1 = 500.$

$x^2 + y^2 + 6x + 2y = 490.$ (3)

Subtract (1) from (3), $6x + 2y = 120.$

$3x + y = 60.$

$\therefore y = 60 - 3x.$ (4)

Substitute the value of y in (1),

$$x^2 + 3600 - 360x + 9x^2 = 370.$$

$$10x^2 - 360x = -3230.$$

$$x^2 - 36x = -323.$$

$$x^2 - 36x + 324 = 1.$$

$$x - 18 = \pm 1.$$

$$\therefore\ x = 19, \text{ or } 17.$$

Substitute the value of x in (4), $y = 3$, or 9.

Therefore, the required numbers are 19 and 3, or 17 and 9.

11. The diagonal of a rectangle is 89 inches. If each side of the rectangle was 3 inches shorter, the diagonal would be 85 inches. Find the length of the sides.

Let $x =$ the number of inches in the length,
and $y =$ the number of inches in the breadth.
Then $\sqrt{x^2 + y^2} =$ the number of inches in the diagonal,
and $\sqrt{(x-3)^2 + (y-3)^2} =$ the number of inches in the diagonal if each side was 3 inches shorter.

$$\therefore\ \sqrt{x^2 + y^2} = 89, \tag{1}$$

and

$$\sqrt{(x-3)^2 + (y-3)^2} = 85. \tag{2}$$

Square (1), $$x^2 + y^2 = 7921. \tag{3}$$

Square (2), $$x^2 - 6x + 9 + y^2 - 6y + 9 = 7225.$$

$$x^2 + y^2 - 6x - 6y = 7207. \tag{4}$$

Subtract (4) from (3), $$6x + 6y = 714.$$

$$x + y = 119. \tag{5}$$

Square (5), $$x^2 + 2xy + y^2 = 14161. \tag{6}$$

Subtract (3) from (6), $$2xy = 6240.$$

$$4xy = 12480. \tag{7}$$

Subtract (7) from (6), $$x^2 - 2xy + y^2 = 1681.$$

$$x - y = \pm 41. \tag{8}$$

Add (5) and (8), $$2x = 160, \text{ or } 78.$$

$$\therefore\ x = 80, \text{ or } 39.$$

Substitute the value of x in (5), $y = 39$, or 80.

Therefore, the length of the rectangle is 80 inches, and the breadth 39 inches.

12. The diagonal of a rectangle is 65 feet. If the shorter side of the rectangle was 17 feet shorter and the longer side 7 feet longer, the length of the diagonal would be unchanged. Find the length of the sides.

Let $x =$ the number of feet in the length,
and $y =$ the number of feet in the breadth.
Then $\sqrt{x^2 + y^2} =$ the number of feet in the diagonal,
and $\sqrt{(x+7)^2 + (y-17)^2} =$ the number of feet in the diagonal if the length was 7 feet longer, and the breadth was 17 feet shorter.

$$\therefore \sqrt{x^2 + y^2} = 65, \quad (1)$$

and
$$\sqrt{(x+7)^2 + (y-17)^2} = 65. \quad (2)$$

Square (1),
$$x^2 + y^2 = 4225. \quad (3)$$

Square (2), $x^2 + 14x + 49 + y^2 - 34y + 289 = 4225.$

$$x^2 + y^2 + 14x - 34y = 3887. \quad (4)$$

Subtract (3) from (4),
$$14x - 34y = -338.$$
$$14x = 34y - 338.$$
$$\therefore x = \frac{17y - 169}{7}. \quad (5)$$

Substitute the value of x in (3),

$$\frac{289y^2 - 5746y + 28561}{49} + y^2 = 4225.$$
$$289y^2 - 5746y + 28561 + 49y^2 = 207025.$$
$$338y^2 - 5746y = 178464.$$
$$y^2 - 17y = 528.$$
$$4y^2 - (\ ) + 289 = 2401.$$
$$2y - 17 = \pm 49.$$
$$2y = 66, \text{ or } -32.$$
$$\therefore y = 33, \text{ or } -16.$$

Substitute the value of y in (5), $x = 56, \text{ or } -63.$

Therefore, the length of the rectangle is 56 feet, and the breadth 33 feet.

13. A farmer received $245 for some apples. If he had sold 10 barrels more and had received for each barrel a quarter of a dollar more, he would have received $300. How many barrels of apples did he sell, and what was the price per barrel?

Let $x =$ the number of barrels of apples the farmer sold,
and $y =$ the number of dollars he received for each barrel.

$$\therefore\ xy = 245, \tag{1}$$

and
$$(x+10)(y+\tfrac{1}{4}) = 300. \tag{2}$$

Expand (2),
$$xy + \frac{x}{4} + 10y + \frac{5}{2} = 300.$$

$$4xy + x + 40y + 10 = 1200. \tag{3}$$

Substitute in (3) the value of xy from (1),

$$980 + x + 40y + 10 = 1200.$$

$$x = 210 - 40y. \tag{4}$$

Substitute in (1) the value of x from (4),

$$210y - 40y^2 = 245.$$

$$40y^2 - 210y = -245.$$

$$8y^2 - 42y = -49.$$

$$64y^2 - (\ ) + 441 = 49.$$

$$8y - 21 = \pm 7.$$

$$8y = 28, \text{ or } 14.$$

$$y = 3\tfrac{1}{2}, \text{ or } 1\tfrac{3}{4}.$$

Substitute the value of y in (4), $x = 70$, or 140.

Therefore, the farmer sold 70 barrels at $3.50 each, or 140 barrels at $1.75 each.

14. If the speed of a railway train was increased 5 miles an hour, $37\frac{1}{2}$ minutes would be saved in making a certain run. If the speed was decreased 5 miles an hour, 50 minutes would be lost in making the same run. Find the length of the run and the rate of the train.

Let x = the number of miles in the run,

and y = the number of miles per hour in the rate of the train.

Then $\frac{x}{y}$ = the number of hours required to make the run.

$37\frac{1}{2}$ minutes $= \frac{5}{8}$ hour; 50 minutes $= \frac{5}{6}$ hour.

$$\therefore\ \frac{x}{y+5} = \frac{x}{y} - \frac{5}{8}, \tag{1}$$

and
$$\frac{x}{y-5} = \frac{x}{y} + \frac{5}{6}. \tag{2}$$

From (1),
$$8xy = 8xy + 40x - 5y^2 - 25y.$$

$$40x = 5y^2 + 25y.$$

$$8x = y^2 + 5y. \tag{3}$$

From (2),
$$6xy = 6xy - 30x + 5y^2 - 25y.$$

$$30x = 5y^2 - 25y.$$

$$6x = y^2 - 5y. \quad (4)$$

Subtract (4) from (3), $2x = 10y.$

$$\therefore\ x = 5y. \quad (5)$$

Substitute the value of x in (3), $40y = y^2 + 5y.$

$$y^2 - 35y = 0.$$

$$y(y - 35) = 0.$$

$$\therefore\ y = 0, \text{ or } 35.$$

Substitute the value of y in (5), $x = 0, \text{ or } 175.$

Therefore, the length of the run is 175 miles, and the rate of the train is 35 miles an hour.

15. Find the fraction that is increased by $\frac{1}{40}$ when 2 is added to the numerator and 3 to the denominator, and is diminished by $\frac{1}{10}$ when 2 is subtracted from the numerator and 3 from the denominator.

Let $x =$ the numerator,

and $y =$ the denominator.

Then $\frac{x}{y} =$ the fraction.

$$\therefore\ \frac{x+2}{y+3} = \frac{x}{y} + \frac{1}{40}, \quad (1)$$

and

$$\frac{x-2}{y-3} = \frac{x}{y} - \frac{1}{10}. \quad (2)$$

From (1), $40xy + 80y = 40xy + 120x + y^2 + 3y.$

$$y^2 - 77y + 120x = 0. \quad (3)$$

From (2), $10xy - 20y = 10xy - 30x - y^2 + 3y.$

$$y^2 - 23y + 30x = 0. \quad (4)$$

Multiply (4) by 4, $4y^2 - 92y + 120x = 0. \quad (5)$

Subtract (3) from (5), $3y^2 - 15y = 0.$

$$y^2 - 5y = 0.$$

$$y(y - 5) = 0.$$

$$\therefore\ y = 0, \text{ or } 5.$$

Substitute the value of y in (4),

$$0 - 0 + 30x = 0, \text{ or } 25 - 115 + 30x = 0.$$

$$30x = 0, \text{ or } 90.$$

$$\therefore\ x = 0, \text{ or } 3.$$

Then $\frac{x}{y} = \frac{3}{5}.$

Therefore, the required fraction is $\frac{3}{5}$.

16. Find the number of two digits which is 4 less than the sum of the squares of its digits, and is 5 greater than twice the product of the digits.

Let $x =$ the digit in the tens' place,
and $y =$ the digit in the units' place.
Then $10x + y =$ the number.

$$\therefore 10x + y + 4 = x^2 + y^2, \qquad (1)$$

and

$$10x + y - 5 = 2xy. \qquad (2)$$

Subtract (2) from (1), $9 = x^2 - 2xy + y^2.$

$$x^2 - 2xy + y^2 = 9.$$

$$x - y = \pm 3. \qquad (3)$$

$$\therefore x = y \pm 3. \qquad (4)$$

Substitute the value of x in (2),

$$10y \pm 30 + y - 5 = 2y^2 \pm 6y.$$

$$2y^2 - 5y = 25, \text{ or } 2y^2 - 17y = -35.$$

Complete the square,

$$16y^2 - (\ ) + 25 = 225, \text{ or } 16y^2 - (\ ) + 289 = 9.$$

$$4y - 5 = \pm 15, \text{ or } 4y - 17 = \pm 3.$$

$$4y = 20,\ -10,\ 20, \text{ or } -14.$$

$$\therefore y = 5,\ -2\tfrac{1}{2},\ 5, \text{ or } -3\tfrac{1}{2}.$$

Substitute the value of y in (4), $x = 8,\ \tfrac{1}{2},\ 2, \text{ or } -6\tfrac{1}{2}.$

Then $10x + y = 85, \text{ or } 25.$

Therefore, the number required is 85, or 25.

17. Two rectangular fields were supposed to contain each 4 acres; but on accurate measurement the first field was found to contain 140 square yards more than 4 acres, and the second 160 square yards less than 4 acres. The second field was 10 yards longer and 10 yards narrower than the first. Find the dimensions of the fields.

Let $x =$ the number of yards in the length of the first field,
and $y =$ the number of yards in the breadth of the first field.
Then $x + 10 =$ the number of yards in the length of the second field,
and $y - 10 =$ the number of yards in the breadth of the second field.
1 acre = 4840 square yards.

$$\therefore xy = 4 \times 4840 + 140, \qquad (1)$$

and

$$(x + 10)(y - 10) = 4 \times 4840 - 160. \qquad (2)$$

From (1), $xy = 19500. \qquad (3)$

From (2), $xy - 10x + 10y - 100 = 19200.$

$$xy - 10x + 10y = 19300. \qquad (4)$$

Subtract (4) from (3), $10x - 10y = 200.$

$$x - y = 20. \quad (5)$$

Square (5), $x^2 - 2xy + y^2 = 400. \quad (6)$

Multiply (3) by 4, $4xy = 78000. \quad (7)$

Add (6) and (7), $x^2 + 2xy + y^2 = 78400.$

$$x + y = \pm 280. \quad (8)$$

Add (5) and (8), $2x = 300, \text{ or } -260.$

$$\therefore x = 150, \text{ or } -130.$$

Substitute the value of x in (5), $y = 130, \text{ or } -150.$

Then $x + 10 = 160,$

and $y - 10 = 120.$

Therefore, the length of the first field is 150 yards, and the breadth 130 yards; the length of the second field is 160 yards, and the breadth 120 yards.

18. The altitude of a trapezoid is 18 feet. Its area is equal to that of a rectangle with sides equal to the parallel bases of the trapezoid. Three times the smaller base added to the larger base is four times the altitude of the trapezoid. Find the two bases.

Let x = the number of feet in the length of the larger base,

and y = the number of feet in the length of the smaller base.

The area of a trapezoid is equal to the product of the altitude by half the sum of the bases.

$$\therefore 9(x + y) = xy, \quad (1)$$

and $x + 3y = 72. \quad (2)$

From (1), $9x + 9y = xy. \quad (3)$

From (2), $x = 72 - 3y. \quad (4)$

Substitute the value of x in (3),

$$648 - 27y + 9y = 72y - 3y^2.$$
$$3y^2 - 90y + 648 = 0.$$
$$y^2 - 30y + 216 = 0.$$
$$(y - 12)(y - 18) = 0.$$
$$\therefore y = 12, \text{ or } 18.$$

Substitute the value of y in (4), $x = 36, \text{ or } 18.$

Since a trapezoid has its two bases unequal, the second pair of values does not apply.

Therefore, the bases are 36 feet and 12 feet.

19. A certain sum at simple interest at a certain per cent amounts to $22,781 in one year. If the sum was $200 greater and the rate of interest $\frac{1}{4}$ per cent higher, the amount would be $23,045. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the sum,
and $y =$ the rate per cent.

$$\therefore x\left(\frac{100+y}{100}\right) = 22781, \qquad (1)$$

and
$$(x+200)\left(\frac{100+y+\frac{1}{4}}{100}\right) = 23045. \qquad (2)$$

From (1),
$$100x + xy = 2278100. \qquad (3)$$

From (2), $100x + xy + \frac{x}{4} + 20000 + 200y + 50 = 2304500.$

$$100x + xy + \frac{x}{4} + 200y = 2284450. \qquad (4)$$

Subtract (3) from (4),
$$\frac{x}{4} + 200y = 6350.$$
$$x + 800y = 25400.$$
$$\therefore x = 25400 - 800y. \qquad (5)$$

Substitute the value of x in (3),

$$2540000 - 80000y + 25400y - 800y^2 = 2278100.$$
$$800y^2 + 54600y = 261900.$$
$$8y^2 + 546y = 2619.$$
$$64y^2 + (\ ) + 273^2 = 95481.$$
$$8y + 273 = \pm 309.$$
$$8y = 36, \text{ or } -582.$$
$$\therefore y = 4\tfrac{1}{2}.$$

Substitute the value of y in (5),
$$x = 25400 - 3600.$$
$$\therefore x = 21800.$$

Therefore, the sum at interest is $21,800, and the rate of interest is $4\frac{1}{2}$ per cent.

20. The product of the sum and the difference of two numbers is a, and the quotient of the sum divided by the difference is b. Find the numbers.

Let $x =$ the larger number,
and $y =$ the smaller number.

$$\therefore\ (x+y)(x-y)=a, \tag{1}$$

and
$$\frac{x+y}{x-y}=b. \tag{2}$$

Multiply (1) by (2),
$$(x+y)^2=ab.$$
$$\therefore\ x+y=\pm\sqrt{ab}. \tag{3}$$

Divide (1) by (2),
$$(x-y)^2=\frac{a}{b}.$$
$$\therefore\ x-y=\pm\frac{1}{b}\sqrt{ab}. \tag{4}$$

Add (3) and (4),
$$2x=\pm\left(1+\frac{1}{b}\right)\sqrt{ab}.$$
$$\therefore\ x=\pm\frac{b+1}{2b}\sqrt{ab}.$$

Subtract (4) from (3),
$$2y=\pm\left(1-\frac{1}{b}\right)\sqrt{ab}.$$
$$\therefore\ y=\pm\frac{b-1}{2b}\sqrt{ab}.$$

Therefore, the numbers required are $\frac{b+1}{2b}\sqrt{ab}$ and $\frac{b-1}{2b}\sqrt{ab}$, or $-\frac{b+1}{2b}\sqrt{ab}$ and $-\frac{b-1}{2b}\sqrt{ab}$.

21. Find the specific gravity of gold and of copper, if a compound of 28 ounces of gold and 11 ounces of copper has a specific gravity of 14.4, and if the specific gravity of gold is 10.4 greater than that of copper.

Let x = the specific gravity of gold,
and y = the specific gravity of copper.
Let a = the number of cubic inches occupied by 1 ounce of water.
Then $\frac{a}{x}$ = the number of cubic inches occupied by 1 ounce of gold,
$\frac{a}{y}$ = the number of cubic inches occupied by 1 ounce of copper,
and $\frac{a}{14.4}$ = the number of cubic inches occupied by 1 ounce of the compound of gold and copper.

$$\therefore\ \frac{28a}{x}+\frac{11a}{y}=\frac{39a}{14.4}, \tag{1}$$

and
$$x-y=10.4. \tag{2}$$

From (1),
$$\frac{28}{x}+\frac{11}{y}=\frac{13}{4.8}.$$
$$134.4\,y+52.8\,x=13\,xy. \qquad (3)$$
From (2),
$$x=y+10.4. \qquad (4)$$
Substitute the value of x in (3),
$$134.4\,y+52.8(y+10.4)=13\,y(y+10.4).$$
$$134.4\,y+52.8\,y+549.12=13\,y^2+135.2\,y.$$
Transpose and combine,
$$13\,y^2-52\,y=549.12.$$
$$y^2-4\,y=42.24.$$
$$y^2-4\,y+4=46.24.$$
$$y-2=\pm 6.8.$$
$$\therefore\ y=8.8,\text{ or }-4.8.$$
Substitute the value of y in (4),
$$x=19.2.$$
Therefore, the specific gravity of gold is 19.2, and of copper is 8.8.

22. Find the specific gravity of two substances A and B, if a compound of a ounces of the first and b ounces of the second has a specific gravity of m, and a compound of c ounces of the first and d ounces of the second has a specific gravity of n.

Let x = the specific gravity of A,

and y = the specific gravity of B.

Let p = the number of cubic inches occupied by 1 ounce of water.

Then $\frac{p}{x}$ = the number of cubic inches occupied by 1 ounce of A,

$\frac{p}{y}$ = the number of cubic inches occupied by 1 ounce of B,

$\frac{p}{m}$ = the number of cubic inches occupied by 1 ounce of the first compound,

and $\frac{p}{n}$ = the number of cubic inches occupied by 1 ounce of the second compound.

$$\therefore\ \frac{ap}{x}+\frac{bp}{y}=\frac{(a+b)p}{m}, \qquad (1)$$
and
$$\frac{cp}{x}+\frac{dp}{y}=\frac{(c+d)p}{n}. \qquad (2)$$
From (1),
$$\frac{a}{x}+\frac{b}{y}=\frac{a+b}{m}. \qquad (3)$$
From (2),
$$\frac{c}{x}+\frac{d}{y}=\frac{c+d}{n}. \qquad (4)$$
Multiply (3) by d,
$$\frac{ad}{x}+\frac{bd}{y}=\frac{(a+b)d}{m}. \qquad (5)$$

Multiply (4) by b, $$\frac{bc}{x}+\frac{bd}{y}=\frac{(c+d)b}{n}. \tag{6}$$

Subtract (6) from (5), $$\frac{ad-bc}{x}=\frac{(a+b)dn-(c+d)bm}{mn}.$$

$$\therefore x=\frac{(ad-bc)mn}{(a+b)dn-(c+d)bm}.$$

Multiply (3) by c, $$\frac{ac}{x}+\frac{bc}{y}=\frac{(a+b)c}{m}. \tag{7}$$

Multiply (4) by a, $$\frac{ac}{x}+\frac{ad}{y}=\frac{(c+d)a}{n}. \tag{8}$$

Subtract (7) from (8), $$\frac{ad-bc}{y}=\frac{(c+d)am-(a+b)cn}{mn}.$$

$$\therefore y=\frac{(ad-bc)mn}{(c+d)am-(a+b)cn}.$$

Therefore, the specific gravity of A is $\frac{(ad-bc)mn}{(a+b)dn-(c+d)bm}$, and the specific gravity of B is $\frac{(ad-bc)mn}{(c+d)am-(a+b)cn}$.

23. A farmer sold a certain number of sheep for \$286. He received for each sheep \$2 more than he paid for it and gained thereby on the cost of the sheep half as many per cent as each sheep cost him dollars. Find the number of sheep.

Let $x =$ the number of sheep,

and $y =$ the number of dollars the farmer paid for each sheep.

Then $y + 2 =$ the number of dollars he received for each sheep,

$\frac{286}{y+2} =$ the number of sheep,

and $\frac{2}{y} \times 100 =$ the gain per cent.

$$\therefore \frac{286}{y+2}=x, \tag{1}$$

and $$\frac{2}{y}\times 100=\frac{y}{2}. \tag{2}$$

From (2), $$y^2=400.$$

$$\therefore y=\pm 20.$$

Substitute the value of y in (1), $$\frac{286}{\pm 20+2}=x.$$

$$\therefore x=13, \text{ or } -15\tfrac{8}{9}.$$

Therefore, the number of sheep was 13.

24. A and B together take a contract to set in type in 12 days a certain number of pages for $90, and agree with each other to divide the $90 in proportion to the amount each does. After working 8 days, however, they call in C to help them, and finish the work on time. They still divide the $90 in proportion to the amount each does, and B receives $4 less than if he and A had finished the work alone. A sets just half the pages. How many days would it take B alone and C alone to set all the pages?

Let $x =$ the number of days it would take B alone,

and $y =$ the number of days it would take C alone.

Then $\frac{1}{x} =$ the part B can do in 1 day,

$\frac{1}{y} =$ the part C can do in 1 day,

$\frac{12}{x} =$ the part B does in all,

$\frac{4}{y} =$ the part C does in all,

$\frac{12}{x} + \frac{4}{y} =$ the part A does in all,

and $\dfrac{\frac{12}{x} + \frac{4}{y}}{12} = \frac{1}{x} + \frac{1}{3y} =$ the part A does in 1 day.

$$\therefore \frac{12}{x} + \frac{4}{y} + \frac{12}{x} + \frac{4}{y} = 1. \tag{1}$$

If C had not come into the partnership, the $90 would have been divided between A and B in proportion to $\frac{12}{x} + \frac{4}{y}$ and $\frac{12}{x}$.

Then B would have received

$$\frac{\frac{12}{x}}{\frac{24}{x} + \frac{4}{y}} \text{ of } \$90 = \frac{3y}{6y + x} \text{ of } \$90 = \frac{270y}{6y + x}.$$

As it was, B received

$$\frac{\frac{12}{x}}{\frac{12}{x} + \frac{4}{y} + \frac{12}{x} + \frac{4}{y}} \text{ of } \$90 = \frac{\frac{3}{x}}{\frac{6}{x} + \frac{2}{y}} \text{ of } \$90 = \frac{3y}{6y + 2x} \text{ of } \$90 = \frac{135y}{3y + x}.$$

$$\therefore \frac{270\,y}{6\,y+x} - \frac{135\,y}{3\,y+x} = 4. \qquad (2)$$

From (1),
$$\frac{24}{x} + \frac{8}{y} = 1.$$
$$24\,y + 8\,x = xy. \qquad (3)$$

From(2), $810\,y^2 + 270\,xy - 810\,y^2 - 135\,xy = 72\,y^2 + 36\,xy + 4\,x^2.$
$$4\,x^2 - 99\,xy + 72\,y^2 = 0.$$

Factor,
$$(4\,x - 3\,y)(x - 24\,y) = 0.$$
$$\therefore x = \frac{3\,y}{4}, \text{ or } 24\,y. \qquad (4)$$

Substitute the value of x in (3),

$$24\,y + 6\,y = \frac{3\,y^2}{4}, \quad \text{or} \quad 24\,y + 192\,y = 24\,y^2.$$
$$120\,y = 3\,y^2, \quad \text{or} \quad 216\,y = 24\,y^2.$$
$$40\,y = y^2, \quad \text{or} \quad 9\,y = y^2.$$
$$y^2 - 40\,y = 0, \quad \text{or} \quad y^2 - 9\,y = 0.$$
$$y\,(y - 40) = 0, \quad \text{or} \quad y\,(y - 9) = 0.$$
$$\therefore y = 0, \text{ or } 40, \quad \text{or} \quad \therefore y = 0, \text{ or } 9.$$

Substitute the value of y in (4),

$$x = 0, \text{ or } 30; \text{ or } 0, \text{ or } 216.$$

The 0 values of x and y do not apply.

Therefore, it would take B alone 30 days, and C alone 40 days; or B alone 216 days, and C alone 9 days.

Exercise 126. Page 319

1. Compound the ratios 5 : 7 and 14 : 45.

$$\tfrac{5}{7} \times \tfrac{14}{45} = \tfrac{2}{9}.$$

Therefore, the compounded ratio is 2 : 9.

2. Compound the duplicate of 4 : 9 with the triplicate of 3 : 8.

$$\tfrac{4}{9} \times \tfrac{4}{9} \times \tfrac{3}{8} \times \tfrac{3}{8} \times \tfrac{3}{8} = \tfrac{1}{96}.$$

Therefore, the compounded ratio is 1 : 96.

3. Arrange the ratios 4 : 7, 2 : 3, 5 : 6, 9 : 14, 11 : 21 in order of magnitude.

$$\tfrac{4}{7},\ \tfrac{2}{3},\ \tfrac{5}{6},\ \tfrac{9}{14},\ \tfrac{11}{21} = \tfrac{24}{42},\ \tfrac{28}{42},\ \tfrac{35}{42},\ \tfrac{27}{42},\ \tfrac{22}{42}.$$

Therefore, the required order is 5 : 6, 2 : 3, 9 : 14, 4 : 7, 11 : 21.

4. Find the ratio compounded of 15 : 28, 7 : 33, 22 : 45.

$$\tfrac{15}{28} \times \tfrac{7}{33} \times \tfrac{22}{45} = \tfrac{1}{18}.$$

Therefore, the compounded ratio is 1 : 18.

5. Find the ratio compounded of 17 : 13, 39 : 44, 77 : 85.

$$\tfrac{17}{13} \times \tfrac{39}{44} \times \tfrac{77}{85} = \tfrac{21}{20}.$$

Therefore, the compounded ratio is 21 : 20.

6. Find the ratio compounded of $8a^2b : c$, $4ab : cd$, $c^2d : 8a^3$, $cd : 4b$.

$$\frac{8a^2b}{c} \times \frac{4ab}{cd} \times \frac{c^2d}{8a^3} \times \frac{cd}{4b} = \frac{bcd}{1}.$$

Therefore, the compounded ratio is $bcd : 1$.

7. Find the ratio compounded of $x^3 + a^3 : x^2 - 9a^2$ and $x + 3a : x + a$.

$$\frac{x^3 + a^3}{x^2 - 9a^2} \times \frac{x + 3a}{x + a} = \frac{x^2 - ax + a^2}{x - 3a}.$$

Therefore, the compounded ratio is $x^2 - ax + a^2 : x - 3a$.

8. Find the condition that the ratio $a : b$ shall be the duplicate of the ratio $a + c : b + c$.

If $a : b$ is the duplicate of $a + c : b + c$,

then $$\frac{a}{b} = \frac{a+c}{b+c} \times \frac{a+c}{b+c},$$

or $$\frac{a}{b} = \frac{a^2 + 2ac + c^2}{b^2 + 2bc + c^2}.$$

Clear of fractions, $ab^2 + 2abc + ac^2 = a^2b + 2abc + bc^2.$

Transpose and combine, $ac^2 - bc^2 = a^2b - ab^2.$

Factor, $(a - b)c^2 = ab(a - b).$

$\therefore c^2 = ab.$

Therefore, the required condition is $c^2 = ab$.

9. What numbers must be added to the terms of the ratio $a : b$ that it may become equal to the ratio $c : d$?

The required numbers are $c - a$ and $d - b$, respectively.

10. Find to six places of decimals the value of the ratio of the diagonal of a cube to an edge of the cube.

Let $a =$ the length of an edge of a cube.

Then the diagonal of a face

$$= \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}.$$

The diagonal of the cube

$$= \sqrt{(a\sqrt{2})^2 + a^2} = \sqrt{2a^2 + a^2} = \sqrt{3a^2} = a\sqrt{3}.$$

$\therefore$ diagonal : edge $= a\sqrt{3} : a = \sqrt{3} : 1 = \sqrt{3} = 1.732050.$

```
          3.00 00(1.732050
          1
     27)2 00
          1 89
      343)11 00
            10 29
        3462)71 00
               69 24
      346405)1 76 00 00
               1 73 20 25
       3464100)2 79 75 00
```

11. Two numbers are in the ratio 3 : 4, and if 12 is added to each, the sums are in the ratio 5 : 6. Find the numbers.

Let $3x =$ the first number.

Then $4x =$ the second number,

$3x + 12 : 4x + 12 =$ the ratio if 12 is added to each number,

$$3x + 12 : 4x + 12 = \frac{3x + 12}{4x + 12}, \text{ and } 5 : 6 = \frac{5}{6}.$$

$$\therefore \frac{3x + 12}{4x + 12} = \frac{5}{6}.$$

$$18x + 72 = 20x + 60.$$

$$2x = 12.$$

$$\therefore x = 6.$$

Then $3x = 18,$

and $4x = 24.$

Therefore, the required numbers are 18 and 24.

12. The ages of two brothers are in the ratio of 9 : 5, and in $2\frac{1}{2}$ years their ages will be in the ratio of 8 : 5. In how many years will their ages be in the ratio of 7 : 5?

Let $9x =$ the number of years in the age of the elder.
Then $5x =$ the number of years in the age of the younger.

Then
$$\frac{9x + 2\frac{1}{2}}{5x + 2\frac{1}{2}} = \frac{8}{5}.$$
$$\frac{18x + 5}{10x + 5} = \frac{8}{5}.$$
$$\frac{18x + 5}{2x + 1} = 8.$$
$$18x + 5 = 16x + 8.$$
$$2x = 3.$$
$$\therefore x = 1\tfrac{1}{2}.$$
Then
$$9x = 13\tfrac{1}{2},$$
and
$$5x = 7\tfrac{1}{2}.$$

Therefore, the elder is $13\frac{1}{2}$ years old, and the younger $7\frac{1}{2}$ years old.

Let $y =$ the required number of years.

Then
$$\frac{13\frac{1}{2} + y}{7\frac{1}{2} + y} = \frac{7}{5}.$$
$$\frac{27 + 2y}{15 + 2y} = \frac{7}{5}.$$
$$135 + 10y = 105 + 14y.$$
$$4y = 30.$$
$$\therefore y = 7\tfrac{1}{2}.$$

Therefore, the required number of years is $7\frac{1}{2}$.

13. An alloy is composed of *a* pounds of copper and *b* pounds of tin. If there had been *a* pounds of tin and *b* pounds of copper, the volume of the alloy would have been increased by 9 per cent. The weights of equal volumes of copper and tin are in the ratio 11 to 9. Find the ratio of *a* to *b*.

Let $11s =$ the specific gravity of copper.
Then $9s =$ the specific gravity of tin.
Let $c =$ the number of cubic inches occupied by 1 pound of water.

Then $\frac{c}{11s}$ = the number of cubic inches occupied by 1 pound of copper,

$\frac{c}{9s}$ = the number of cubic inches occupied by 1 pound of tin,

$\frac{ac}{11s}$ = the number of cubic inches occupied by a pounds of copper,

$\frac{bc}{9s}$ = the number of cubic inches occupied by b pounds of tin,

$\frac{bc}{11s}$ = the number of cubic inches occupied by b pounds of copper,

and $\frac{ac}{9s}$ = the number of cubic inches occupied by a pounds of tin.

$$\therefore \frac{109}{100}\left(\frac{ac}{11s}+\frac{bc}{9s}\right)=\frac{bc}{11s}+\frac{ac}{9s}.$$

Multiply by $\frac{s}{c}$, $$\frac{109}{100}\left(\frac{a}{11}+\frac{b}{9}\right)=\frac{b}{11}+\frac{a}{9}.$$

Multiply by 99, $$\tfrac{109}{100}(9a+11b)=9b+11a.$$

$$981a+1199b=900b+1100a.$$

$$299b=119a.$$

$$\therefore \frac{a}{b}=\frac{299}{119}.$$

$$\therefore a:b=299:119.$$

Therefore, the ratio of a to b is 299 : 119.

Exercise 127. Page 325

1. Find the third proportional to 12 and 24.

$$12:24=24:x.$$

$$\therefore x=\frac{24\times 24}{12}=48.$$

2. Find the third proportional to 27 and 3.

$$27:3=3:x.$$

$$\therefore x=\frac{3\times 3}{27}=\frac{1}{3}.$$

3. Find the third proportional to $\frac{b^2}{a^2-b^2}$ and $\frac{ab-b^2}{(a+b)^2}$.

$$\frac{b^2}{a^2-b^2} : \frac{ab-b^2}{(a+b)^2} = \frac{ab-b^2}{(a+b)^2} : x.$$

$$\frac{b^2}{(a+b)(a-b)} : \frac{b(a-b)}{(a+b)^2} = \frac{b(a-b)}{(a+b)^2} : x.$$

$$\therefore x = \frac{\frac{b(a-b)}{(a+b)^2} \times \frac{b(a-b)}{(a+b)^2}}{\frac{b^2}{(a+b)(a-b)}}$$

$$= \frac{b(a-b)}{(a+b)^2} \times \frac{b(a-b)}{(a+b)^2} \times \frac{(a+b)(a-b)}{b^2}$$

$$= \frac{(a-b)^3}{(a+b)^3}.$$

4. Find the mean proportional between 64 and 81.

$$x = \pm\sqrt{64 \times 81} = \pm(8 \times 9) = \pm 72.$$

5. Find the mean proportional between 25 and 196.

$$x = \pm\sqrt{25 \times 196} = \pm(5 \times 14) = \pm 70.$$

6. Find the mean proportional between $1\frac{1}{2}$ and $16\frac{2}{3}$.

$$x = \pm\sqrt{1\tfrac{1}{2} \times 16\tfrac{2}{3}} = \pm\sqrt{\tfrac{3}{2} \times \tfrac{50}{3}} = \pm\sqrt{25} = \pm 5.$$

7. Find the mean proportional between

$$\frac{b+c+a}{b+c-a} \text{ and } \frac{2bc+b^2+c^2-a^2}{4b^2c^2}.$$

$$\frac{2bc+b^2+c^2-a^2}{4b^2c^2} = \frac{(b^2+2bc+c^2)-a^2}{4b^2c^2}$$

$$= \frac{(b+c)^2-a^2}{4b^2c^2}$$

$$= \frac{(b+c+a)(b+c-a)}{4b^2c^2}.$$

$$x = \pm\sqrt{\frac{b+c+a}{b+c-a} \times \frac{(b+c+a)(b+c-a)}{4b^2c^2}}$$

$$= \pm\sqrt{\frac{(b+c+a)^2}{4b^2c^2}}$$

$$= \pm\frac{b+c+a}{2bc}.$$

8. Find the mean proportional between

$$\left(x-\frac{xy-y^2}{x+y}\right)\left(x-\frac{xy^2-y^3}{x^2+y^2}\right) \text{ and } \frac{x^2}{x^2-xy+y^2}.$$

$$\begin{aligned}\left(x-\frac{xy-y^2}{x+y}\right)&\left(x-\frac{xy^2-y^3}{x^2+y^2}\right)\\ &=\frac{x^2+xy-xy+y^2}{x+y}\times\frac{x^3+xy^2-xy^2+y^3}{x^2+y^2}\\ &=\frac{x^2+y^2}{x+y}\times\frac{x^3+y^3}{x^2+y^2}\\ &=\frac{x^2+y^2}{x+y}\times\frac{(x+y)(x^2-xy+y^2)}{x^2+y^2}\\ &=x^2-xy+y^2.\end{aligned}$$

Therefore, the mean proportional

$$\begin{aligned}&=\pm\sqrt{(x^2-xy+y^2)\times\frac{x^2}{x^2-xy+y^2}}\\ &=\pm\sqrt{x^2}\\ &=\pm x.\end{aligned}$$

9. Find the fourth proportional to 5, 18, and 20.

$$5:18=20:x.$$

$$\therefore x=\frac{18\times 20}{5}=72.$$

10. Find the fourth proportional to 9, 16, and 36.

$$9:16=36:x.$$

$$\therefore x=\frac{16\times 36}{9}=64.$$

11. Find the fourth proportional to 12, 17, and 21.

$$12:17=21:x.$$

$$\therefore x=\frac{17\times 21}{12}=\frac{119}{4}=29\tfrac{3}{4}.$$

12. Find the fourth proportional to 64, 48, and 56.

$$64:48=56:x.$$

$$\therefore x=\frac{48\times 56}{64}=42.$$

13. Find the fourth proportional to $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$.

$$\frac{1}{a}:\frac{1}{b}=\frac{1}{c}:x.$$

$$\therefore x=\frac{\frac{1}{b}\times\frac{1}{c}}{\frac{1}{a}}=\frac{1}{b}\times\frac{1}{c}\times a=\frac{a}{bc}.$$

14. Find the value of x in the proportion $8:23=12:x$.

$$x=\frac{23\times 12}{8}=\frac{69}{2}=34\tfrac{1}{2}.$$

15. Find the value of x in the proportion $9:18=x:45$.

$$x=\frac{9\times 45}{18}=\frac{45}{2}=22\tfrac{1}{2}.$$

16. Find the value of x in the proportion $7:x=24\frac{1}{2}:52$.

$$x=\frac{7\times 52}{24\frac{1}{2}}=\frac{7\times 52\times 2}{49}=\frac{104}{7}=14\tfrac{6}{7}.$$

17. Find the value of x in the proportion $x:16=7:8$.

$$x=\frac{16\times 7}{8}=14.$$

18. Show that if $a:b=c:d$,

$$3a+7b:3a-7b=3c+7d:3c-7d.$$

If $$a:b=c:d,$$

then $$\frac{a}{b}=\frac{c}{d},$$

and $$\frac{3a}{7b}=\frac{3c}{7d},$$

or $$3a:7b=3c:7d.$$

$$\therefore 3a+7b:3a-7b=3c+7d:3c-7d. \qquad (\S\ 384)$$

19. Show that if $a:b=c:d,$

$$2a^2+5b^2:2a^2-5b^2=2c^2+5d^2:2c^2-5d^2.$$

If $a:b=c:d,$

then $a^2:b^2=c^2:d^2,$ (§ 386)

or $\dfrac{a^2}{b^2}=\dfrac{c^2}{d^2},$

and $\dfrac{2a^2}{5b^2}=\dfrac{2c^2}{5d^2},$

or $2a^2:5b^2=2c^2:5d^2.$

$\therefore\ 2a^2+5b^2:2a^2-5b^2=2c^2+5d^2:2c^2-5d^2.$ (§ 384)

20. Show that if $a:b=b:c,$

$$ab+b^2:b^2=bc+c^2:c^2.$$

If $a:b=b:c,$

then $a+b:b=b+c:c,$ (§ 382)

and $ab+b^2:b^2=bc+c^2:c^2.$

21. Show that if $a:b=b:c,$

$$a-c:c=a(a+2b)-c(2b+c):(b+c)^2.$$

If $a:b=b:c,$

then $a+b:a=b+c:b,$ (§ 382)

and $(a+b)^2:a^2=(b+c)^2:b^2.$ (§ 386)

Now $b^2=ac.$ (§ 379)

$\therefore\ (a+b)^2:a^2=(b+c)^2:ac.$

$a^2:ac=(a+b)^2:(b+c)^2.$ (§ 385)

$a:c=(a+b)^2:(b+c)^2.$

$a-c:c=(a+b)^2-(b+c)^2:(b+c)^2.$ (§ 383)

$a-c:c=a^2+2ab+b^2-b^2-2bc-c^2:(b+c)^2.$

$a-c:c=a^2+2ab-2bc-c^2:(b+c)^2.$

$\therefore\ a-c:c=a(a+2b)-c(2b+c):(b+c)^2.$

22. Find x from the proportion

$$6x+a:4x+b=3x-b:2x-a.$$

By § 379, $(6x+a)(2x-a)=(4x+b)(3x-b).$

Expand, $12x^2-4ax-a^2=12x^2-bx-b^2.$

Transpose and combine, $(4a-b)x=b^2-a^2.$

$$\therefore\ x=\frac{b^2-a^2}{4a-b}.$$

23. Find x from the proportion $5x : 5x - 4 = 6 - 7x : 10 - 7x$.

By division, $4 : 5x = -4 : 6 - 7x$, (§ 383)

or $4 : 5x = 4 : 7x - 6$.

$\therefore\ 5x = 7x - 6$.

$2x = 6$.

$\therefore\ x = 3$.

24. Find x from the proportion

$$2x + 11 : 6x + 47 = 2x + 3 : 6x + 16.$$

By alternation, $2x + 11 : 2x + 3 = 6x + 47 : 6x + 16$. (§ 385)

By division, $8 : 2x + 3 = 31 : 6x + 16$. (§ 383)

$48x + 128 = 62x + 93$. (§ 379)

$14x = 35$.

$\therefore\ x = 2\frac{1}{2}$.

25. Find x from the proportion $x - 1 : x - 2 = 4x + 5 : 3x + 2$.

$(x - 1)(3x + 2) = (x - 2)(4x + 5)$. (§ 379)

Expand, $3x^2 - x - 2 = 4x^2 - 3x - 10$.

Transpose and combine, $x^2 - 2x - 8 = 0$.

Factor, $(x - 4)(x + 2) = 0$.

$\therefore\ x = 4$, or -2.

26. Find x from the proportion $x - 4 : x - 7 = 2x - 1 : x + 1$.

$(x - 4)(x + 1) = (x - 7)(2x - 1)$. (§ 379)

Expand, $x^2 - 3x - 4 = 2x^2 - 15x + 7$.

Transpose and combine, $x^2 - 12x + 11 = 0$.

Factor, $(x - 1)(x - 11) = 0$.

$\therefore\ x = 1$, or 11.

27. Find x from the proportion

$$x + a + b : x - 3a + b = 2x - 3a + 2b - c : x + b + c.$$

By § 383, $4a : x - 3a + b = x - 3a + b - 2c : x + b + c$.

$\therefore\ 4a(x + b + c) = (x - 3a + b)(x - 3a + b - 2c)$.

$4a(x + b + c) = (x - 3a + b)^2 - 2c(x - 3a + b)$.

$4ax + 4ab + 4ac = x^2 + 9a^2 + b^2 - 6ax + 2bx - 6ab - 2cx + 6ac - 2bc$.

Transpose and combine,

$x^2 - 10ax + 2bx - 2cx = 10ab - 2ac + 2bc - 9a^2 - b^2$.

$x^2 - 2(5a - b + c)x = 10ab - 2ac + 2bc - 9a^2 - b^2$.

$x^2 - (\ ) + (5a - b + c)^2 = 16a^2 + 8ac + c^2$.

$x - (5a - b + c) = \pm(4a + c)$.

$\therefore\ x = 9a - b + 2c$, or $a - b$.

28. Find x and y from the proportions

$$x + y - 4 : 2x + y + 1 = 1 : 2, \tag{1}$$

$$2x + y - 9 : x + 2y + 7 = 3 : 4. \tag{2}$$

From (1), $2x + 2y - 8 = 2x + y + 1.$

Transpose and combine, $y = 9. \quad (3)$

From (2), $8x + 4y - 36 = 3x + 6y + 21.$

Transpose and combine, $5x = 2y + 57. \quad (4)$

Substitute in (4) the value of y from (3), $5x = 18 + 57.$

$$5x = 75.$$

$$\therefore x = 15.$$

Therefore, $x = 15$, and $y = 9$.

29. Find x and y from the proportions

$$x + y : x - y = a : b - c, \tag{1}$$

$$x + c : y + b = a + b : a + c. \tag{2}$$

From (1) by § 384, $2x : 2y = a + b - c : a - b + c,$

or $x : y = a + b - c : a - b + c.$

$$\therefore (a - b + c)x = (a + b - c)y. \tag{3}$$

$$\therefore x = \frac{a + b - c}{a - b + c}y. \tag{4}$$

From (2), $(a + c)x + ac + c^2 = (a + b)y + ab + b^2. \quad (5)$

Subtract (3) from (5), $bx + ac + c^2 = cy + ab + b^2.$

$$bx - cy = ab + b^2 - ac - c^2. \tag{6}$$

Substitute in (6) the value of x from (4),

$$\frac{ab + b^2 - bc}{a - b + c}y - cy = ab + b^2 - ac - c^2.$$

$$(ab + b^2 - bc - ac + bc - c^2)y = (a - b + c)(ab + b^2 - ac - c^2).$$

$$(ab + b^2 - ac - c^2)y = (a - b + c)(ab + b^2 - ac - c^2).$$

$$\therefore y = a - b + c.$$

Substitute the value of y in (4), $x = a + b - c.$

Therefore, $x = a + b - c$, and $y = a - b + c$.

30. Two numbers are in the ratio $5 : 4$. If 8 is added to the first number and 4 to the second, the difference of the squares of the new numbers is to the difference of the squares of the numbers in the ratio $35 : 3$. Find the numbers.

Let $5x =$ the larger number.

Then $4x =$ the smaller number.

$$\therefore (5x+8)^2-(4x+4)^2:(5x)^2-(4x)^2=35:3.$$
$$25x^2+80x+64-16x^2-32x-16:25x^2-16x^2=35:3.$$
$$9x^2+48x+48:9x^2=35:3.$$
$$3x^2+16x+16:3x^2=35:3.$$
$$9x^2+48x+48=105x^2. \qquad (\S\ 379)$$
$$96x^2-48x-48=0.$$
$$2x^2-x-1=0.$$
$$(x-1)(2x+1)=0.$$
$$\therefore x=1, \text{ or } -\tfrac{1}{2}.$$

Then $5x=5, \text{ or } -2\tfrac{1}{2},$

and $4x=4, \text{ or } -2.$

Therefore, the required numbers are 5 and 4, or $-2\frac{1}{2}$ and -2.

31. What number must be added to each of the numbers 2, 7, 14, 29, in order that the sums may be in proportion?

Let $x=$ the number required.

$$\therefore 2+x:7+x=14+x:29+x.$$

By § 383, $2+x:5=14+x:15.$

$$2+x:1=14+x:3.$$
$$6+3x=14+x.$$
$$2x=8.$$
$$\therefore x=4.$$

Therefore, the number required is 4.

32. In a continued proportion the sum of the three proportionals is 39, and the sum of their squares is 741. What is the proportion?

Let $x=$ the first proportional,

and $y=$ the third proportional.

Then $\sqrt{xy}=$ the second proportional,

and $x:\sqrt{xy}=\sqrt{xy}:y=$ the proportion.

$$\therefore x+\sqrt{xy}+y=39, \qquad (1)$$

and $x^2+xy+y^2=741. \qquad (2)$

Divide (2) by (1), $x-\sqrt{xy}+y=19. \qquad (3)$

Add (1) and (3), $2x+2y=58.$

$$\therefore x+y=29. \qquad (4)$$

Substitute the value of $x+y$ in (1),

$$\sqrt{xy}+29=39.$$
$$\sqrt{xy}=10.$$

$$xy = 100.$$

$$3xy = 300. \qquad (5)$$

Subtract (5) from (2), $x^2 - 2xy + y^2 = 441.$

$$x - y = \pm 21. \qquad (6)$$

Add (4) and (6), $2x = 50, \text{ or } 8.$

$$\therefore x = 25, \text{ or } 4.$$

Subtract (6) from (4), $2y = 8, \text{ or } 50.$

$$\therefore y = 4, \text{ or } 25.$$

Then $\sqrt{xy} = 10.$

Therefore, the proportion required is $4 : 10 = 10 : 25.$

33. Find two numbers such that their sum is to their difference as $5 : 1$, and their sum to their product as $5 : 4$.

Let $x =$ the larger number,

and $y =$ the smaller number.

$$\therefore x + y : x - y = 5 : 1, \qquad (1)$$

and $x + y : xy = 5 : 4. \qquad (2)$

From (1), $x + y = 5x - 5y.$

$$4x = 6y.$$

$$\therefore x = \frac{3y}{2}. \qquad (3)$$

From (2), $4x + 4y = 5xy. \qquad (4)$

Substitute in (4) the value of x from (3),

$$6y + 4y = \frac{15y^2}{2}.$$

$$12y + 8y = 15y^2.$$

$$15y^2 - 20y = 0.$$

$$3y^2 - 4y = 0.$$

$$y(3y - 4) = 0.$$

$$\therefore y = 0, \text{ or } \tfrac{4}{3}.$$

Substitute the value of y in (3), $x = 0, \text{ or } 2.$

Therefore, the required numbers are 2 and $1\frac{1}{3}$.

34. If 3 is subtracted from the greater of two numbers and 2 is added to the smaller, the results are in the ratio $2 : 3$; but if 1 is added to the greater number and 2 is subtracted from the smaller, the results are in the ratio $3 : 2$. Find the numbers.

Let $x =$ the greater number,
and $y =$ the smaller number.

$$\therefore\ x - 3 : y + 2 = 2 : 3, \qquad (1)$$

and
$$x + 1 : y - 2 = 3 : 2. \qquad (2)$$

From (1), $3x - 9 = 2y + 4.$
$$3x - 2y = 13. \qquad (3)$$

From (2), $2x + 2 = 3y - 6.$
$$2x - 3y = -8. \qquad (4)$$

Multiply (3) by 3, $9x - 6y = 39. \qquad (5)$

Multiply (4) by 2, $4x - 6y = -16. \qquad (6)$

Subtract (6) from (5), $5x = 55.$
$$\therefore\ x = 11.$$

Substitute the value of x in (3), $y = 10.$

Therefore, the required numbers are 11 and 10.

35. On a division of the house when all members present must vote, if 50 members more had voted for the motion, it would have been carried in the ratio of $5:3$; but if 60 members more had voted against it, it would have been lost in the ratio of $4:3$. How many members attended at the division of the house?

Let $x =$ the number of members voting for the motion,
and $y =$ the number of members voting against the motion.
Then $x + y =$ the number of members present.

$$\therefore\ x + 50 : y - 50 = 5 : 3, \qquad (1)$$

and
$$x - 60 : y + 60 = 3 : 4. \qquad (2)$$

From (1), $3x + 150 = 5y - 250.$
$$3x - 5y = -400. \qquad (3)$$

From (2), $4x - 240 = 3y + 180.$
$$4x - 3y = 420. \qquad (4)$$

Multiply (3) by 4, $12x - 20y = -1600. \qquad (5)$

Multiply (4) by 3, $12x - 9y = 1260. \qquad (6)$

Subtract (5) from (6), $11y = 2860.$
$$\therefore\ y = 260.$$

Substitute the value of y in (4), $4x - 780 = 420.$
$$4x = 1200.$$
$$\therefore\ x = 300.$$

Then $x + y = 560.$

Therefore, 560 members attended.

36. A pendulum that beats seconds at the sea level is taken to a place where the force of gravity is 0.9 that at the surface of the earth. In what time will the pendulum make one vibration? If the pendulum was attached to a clock, how much time would the clock lose in 24 hours?

Let $t =$ the number of seconds required for one vibration.

By Formula 4, § 350, p. 290,

$$1 = \pi\sqrt{\frac{l}{g}} \text{ and } t = \pi\sqrt{\frac{l}{0.9g}}.$$

$$\therefore t : 1 = \pi\sqrt{\frac{l}{0.9g}} : \pi\sqrt{\frac{l}{g}} = \pi\frac{\sqrt{l}}{\sqrt{0.9g}} : \pi\frac{\sqrt{l}}{\sqrt{g}}$$

$$= \frac{1}{\sqrt{0.9g}} : \frac{1}{\sqrt{g}} = \frac{1}{\sqrt{0.9}} : 1.$$

$$\therefore t = \frac{1}{\sqrt{0.9}} = \frac{1}{0.9}\sqrt{0.9} = \frac{10}{9} \times 0.94868 = 1.0541.$$

While the pendulum at the sea level beats $24 \times 60 \times 60$ times; the required pendulum beats $\frac{24 \times 60 \times 60}{1.0541} = \frac{86400}{1.0541} = 81965.66$ times.

Hence, in 24 hours the required clock would lose $(86400 - 81965.66)$ seconds $= 4434.34$ seconds $= 1$ hour 13 minutes 54.34 seconds.

Therefore, the time of vibration is 1.0541 seconds, and the clock in 24 hours would lose 1 hour 13 minutes 54.34 seconds.

Exercise 128. Page 331

1. If $y \propto x$, and $y = 6$ when $x = 9$, find y when $x = 15$.

Here $y = mx.$

When $x = 9$, then $y = 6.$

$\therefore 6 = 9m.$

$m = \frac{2}{3}.$

$\therefore y = \frac{2}{3}x.$

When $x = 15$, then $y = \frac{2}{3}$ of $15 = 10.$

2. If $y \propto \frac{1}{x}$, and $y = 32$ when $x = 6$, find y when $x = 21$.

Here $y = \frac{m}{x}$.

When $x = 6$, then $y = 32$.

$$\therefore 32 = \frac{m}{6}.$$

$$m = 192.$$

$$\therefore y = \frac{192}{x}.$$

When $x = 21$, then $y = \frac{192}{21} = \frac{64}{7} = 9\frac{1}{7}$.

3. If z varies jointly as x and y, and 4, 5, 6, are simultaneous values of x, y, z, find z when $x = 8$ and $y = 9$.

Here $z = mxy$.

When $x = 4$ and $y = 5$, then $z = 6$.

$$\therefore 6 = m \times 4 \times 5.$$

$$m = \frac{3}{10}.$$

$$\therefore z = \frac{3xy}{10}.$$

When $x = 8$ and $y = 9$,

then $z = \frac{3 \times 8 \times 9}{10} = \frac{108}{5} = 21\frac{3}{5}$.

4. If the cube of x varies inversely as the square of y, and $x = 2$ when $y = 3$, find the equation between x and y.

Here $x^3 = \frac{m}{y^2}$.

When $y = 3$, then $x = 2$.

$$\therefore 8 = \frac{m}{9}.$$

$$m = 72.$$

$$\therefore x^3 = \frac{72}{y^2}.$$

$$\therefore x^3y^2 = 72.$$

5. If z varies directly as x and inversely as y, and if $x = 6$ and $y = 15$ when $z = 7$, find z when $x = 12$ and $y = 8$.

Here $z = \dfrac{mx}{y}$.

When $z = 7$, then $x = 6$ and $y = 15$.

$$\therefore 7 = \frac{6m}{15}.$$

$$m = \tfrac{35}{2}.$$

$$\therefore z = \frac{35x}{2y}.$$

When $x = 12$ and $y = 8$,

then $z = \dfrac{35 \times 12}{2 \times 8} = \dfrac{105}{4} = 26\tfrac{1}{4}$.

6. The volume of a rectangular solid varies jointly as the length, breadth, and height. A cube of clay 12 inches on an edge is molded into a right prism whose base is a rectangle 16 inches by 8 inches. Find the height of the prism.

Let x = the number of inches in the length of a rectangular solid,
y = the number of inches in the breadth of a rectangular solid,
z = the number of inches in the height of a rectangular solid,
and v = the number of cubic inches in the volume of a rectangular solid.

Then $v \propto xyz$,

and $z \propto \dfrac{v}{xy}$.

Here $z = \dfrac{mv}{xy}$.

When $v = 12^3 = 1728$, $x = 12$, and $y = 12$,

then $z = 12$.

$$\therefore 12 = \frac{m \times 1728}{12 \times 12}.$$

$$m = 1.$$

$$\therefore z = \frac{v}{xy}.$$

When $v = 1728$, $x = 16$, and $y = 8$,

then $z = \dfrac{1728}{16 \times 8} = 13\tfrac{1}{2}$.

Therefore, the height of the prism is $13\frac{1}{2}$ inches.

7. The volume of a sphere varies as the cube of the diameter. The diameter of a sphere whose volume is 75 cubic feet 1377 cubic inches is 5 feet 3 inches. Find the diameter of a sphere whose volume is 179 cubic feet 1152 cubic inches.

75 cubic feet 1377 cubic inches = 130,977 cubic inches.
179 cubic feet 1152 cubic inches = 310,464 cubic inches.
5 feet 3 inches = 63 inches.

Let $D =$ the number of inches in the diameter of a sphere,
and $v =$ the number of cubic inches in the volume of a sphere.

Then $v \propto D^3.$

Here $v = mD^3.$

When $v = 130977$, then $D = 63.$

$$\therefore 130977 = m \times 63^3.$$

$$m = \frac{130977}{63^3} = \frac{11}{21}.$$

$$\therefore v = \tfrac{11}{21} D^3.$$

When $v = 310464$, then $310464 = \tfrac{11}{21} D^3.$

$$\begin{aligned} \therefore D &= \sqrt[3]{\tfrac{21}{11} \times 310464} \\ &= \sqrt[3]{21 \times 28224} \\ &= \sqrt[3]{2^6 \times 3^3 \times 7^3} \\ &= 2^2 \times 3 \times 7 \\ &= 84. \end{aligned}$$

84 inches = 7 feet.

Therefore, the required diameter is 7 feet.

8. The intensity of light varies inversely as the square of the distance from the source. How far from a lamp is a point that receives half as much light as another point 16 feet away?

Let $d =$ the number of feet in the distance from the lamp,
and $i =$ the intensity of the light.

Then $i \propto \frac{1}{d^2}.$

Here $i = \frac{m}{d^2}.$

When $i = 1$, then $d = 16.$

$$\therefore 1 = \frac{m}{16^2} = \frac{m}{256}.$$

$$m = 256.$$

$$\therefore i = \frac{256}{d^2}.$$

$$d^2 = \frac{256}{i}.$$

$$d = \sqrt{\frac{256}{i}}.$$

When $i = \frac{1}{2}$, then $d = \sqrt{\frac{256}{\frac{1}{2}}} = \sqrt{512} = 22.627.$

Therefore, the required distance is 22.63 feet.

9. The volume of a right cylinder varies jointly as its height and the square of its radius. One cylindrical vessel is 5 inches high and $1\frac{1}{2}$ inches in radius, and another is 6 inches high and 2 inches in radius. Find the radius of a third cylindrical vessel 7 inches high that will hold as much as the other two vessels together.

Let v = the number of cubic inches in the volume of a right cylinder,

h = the number of inches in the height,

and r = the number of inches in the radius.

Then $v \propto hr^2$.

Here $v = mhr^2$.

Let v', v'', v''' = respectively the number of cubic inches in the volumes of the first, second, and third cylindrical vessels.

Then
$$v' = m \times 5 \times (1\tfrac{1}{2})^2 = \frac{45m}{4},$$
$$v'' = m \times 6 \times 2^2 = 24m,$$
$$v''' = m \times 7 \times r^2 = 7mr^2.$$

Since
$$v''' = v' + v'',$$
$$7mr^2 = \frac{45m}{4} + 24m.$$
$$7r^2 = \tfrac{45}{4} + 24.$$
$$28r^2 = 45 + 96.$$
$$28r^2 = 141.$$
$$r^2 = \tfrac{141}{28}.$$

$$r = \sqrt{\frac{141}{28}} = \sqrt{\frac{141 \times 7}{28 \times 7}} = \sqrt{\frac{987}{196}} = \frac{1}{14}\sqrt{987} = \frac{1}{14} \text{ of } 31.416 = 2.244.$$

Therefore, the radius of the third cylinder is 2.244 inches.

10. If a body falling freely falls 144 feet in 3 seconds and acquires a velocity of 96 feet per second, find the velocity acquired and the distance fallen in 5 seconds; in 7 seconds; in 9 seconds (see p. 289, § 350, Law 1).

Let $v =$ the velocity in feet per second,
$t =$ the number of seconds,
and $s =$ the number of feet in the distance fallen.

Then $v \propto t.$
Here $v = mt.$
When $t = 3$, then $v = 96.$

$$\therefore 96 = m \times 3.$$
$$m = 32.$$
$$\therefore v = 32t.$$

When $t = 5,\ v = 32 \times 5 = 160;$
when $t = 7,\ v = 32 \times 7 = 224;$
when $t = 9,\ v = 32 \times 9 = 288.$

Again, $s \propto t^2.$
Here $s = nt^2.$
When $t = 3$, then $s = 144.$

$$\therefore 144 = n \times 3^2.$$
$$n = 16.$$
$$\therefore s = 16t^2.$$

When $t = 5,\ s = 16 \times 5^2 = 16 \times 25 = 400;$
when $t = 7,\ s = 16 \times 7^2 = 16 \times 49 = 784;$
when $t = 9,\ s = 16 \times 9^2 = 16 \times 81 = 1296.$

Therefore, the velocity acquired in 5 seconds is 160 feet per second; in 7 seconds is 224 feet per second; in 9 seconds is 288 feet per second. The distance fallen in 5 seconds is 400 feet; in 7 seconds is 784 feet; in 9 seconds is 1296 feet.

11. The deflection of a horizontal beam bent by a heavy weight varies directly as the weight, directly as the cube of the length of the beam, inversely as the breadth, and inversely as the cube of the thickness. If a pine beam 5 feet long, 2 inches wide, 3 inches thick is bent 1 inch by a weight of 600 pounds, how much will a pine beam 15 feet long, 4 inches wide, 6 inches thick be bent by a weight of 2 tons?

Let $x =$ the number of inches in the deflection,
$w =$ the number of pounds in the weight,
$l =$ the number of feet in the length of the beam,
$b =$ the number of inches in the breadth,
and $t =$ the number of inches in the thickness.

Then $$x \propto \frac{wl^3}{bt^3}.$$

Here $$x = \frac{mwl^3}{bt^3}.$$

When $l = 5$, $b = 2$, $t = 3$, and $w = 600$, then $x = 1$.

$$\therefore\ 1 = \frac{m \times 600 \times 5^3}{2 \times 3^3} = \frac{12500\,m}{9}.$$

$$m = \frac{9}{12500}.$$

$$\therefore\ x = \frac{9\,wl^3}{12500\,bt^3}.$$

When $l = 15$, $b = 4$, $t = 6$, and $w = 4000$,

then $$x = \frac{9 \times 4000 \times 15^3}{12500 \times 4 \times 6^3} = 11\tfrac{1}{4}.$$

Therefore, the deflection of the required beam will be $11\frac{1}{4}$ inches.

Table on Page 336

1. The series is a, $a + d$, $a + 2\,d$, $\cdots$

Hence, the nth term is $a + (n - 1)\,d$.

Let l denote the last, or nth, term.

Then $$l = a + (n - 1)\,d.$$

2. $$l = a + (n - 1)\,d, \tag{1}$$

and $$s = \frac{n}{2}(a + l). \tag{2}$$

From (1), $$l = a + nd - d.$$

$$nd = l - a + d.$$

$$\therefore\ n = \frac{l - a + d}{d}. \tag{3}$$

From (2), $$2\,s = n(a + l).$$

$$\therefore\ n = \frac{2\,s}{a + l}. \tag{4}$$

Equate the values of n, $$\frac{2\,s}{a + l} = \frac{l - a + d}{d}.$$

$$2\,ds = l^2 - a^2 + ad + dl.$$

$$l^2 + dl = 2\,ds + a^2 - ad.$$

Complete the square, $$4\,l^2 + (\ ) + d^2 = 8\,ds + 4\,a^2 - 4\,ad + d^2.$$

$$2\,l + d = \pm\sqrt{8\,ds + 4\,a^2 - 4\,ad + d^2}.$$

$$2\,l = -\,d \pm \sqrt{8\,ds + (2\,a - d)^2}.$$

$$\therefore\ l = \tfrac{1}{2}\left[-\,d + \sqrt{8\,ds + (2\,a - d)^2}\right].$$

3.
$$s = \frac{n}{2}(a + l).$$
$$2s = na + nl.$$
$$2s - na = nl.$$
$$\therefore\ l = \frac{2s}{n} - a.$$

4.
$$l = a + (n-1)d, \tag{1}$$
and
$$s = \frac{n}{2}(a + l). \tag{2}$$
From (2),
$$2s = na + nl.$$
$$na = 2s - nl.$$
$$\therefore\ a = \frac{2s}{n} - l.$$
Substitute the value of a in (1),
$$l = \frac{s}{n} + \frac{(n-1)d}{2}.$$

5.
$$l = a + (n-1)d, \tag{1}$$
and
$$s = \frac{n}{2}(a + l). \tag{2}$$
Substitute in (2) the value of l from (1),
$$s = \tfrac{1}{2}n[2a + (n-1)d].$$

6.
$$l = a + (n-1)d, \tag{1}$$
and
$$s = \frac{n}{2}(a + l). \tag{2}$$
From (1),
$$l = a + nd - d.$$
$$nd = l - a + d.$$
$$\therefore\ n = \frac{l - a + d}{d}. \tag{3}$$
From (2),
$$2s = n(a + l).$$
$$\therefore\ n = \frac{2s}{a + l}. \tag{4}$$
Equate the values of n,
$$\frac{l - a + d}{d} = \frac{2s}{a + l}.$$
$$l^2 - a^2 + ad + dl = 2ds.$$
$$\therefore\ s = \frac{dl + ad + l^2 - a^2}{2d}.$$
$$\therefore\ s = \frac{l + a}{2} + \frac{l^2 - a^2}{2d}.$$

7. $s = a + (a+d) + (a+2d) + \cdots + (l-d) + l,$ (1)

or $s = l + (l-d) + (l-2d) + \cdots + (a+d) + a.$ (2)

Add (1) and (2),

$$2s = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l).$$

$$\therefore 2s = n(a+l).$$

$$\therefore s = \frac{n}{2}(a+l).$$

8. $l = a + (n-1)d,$ (1)

and $s = \frac{n}{2}(a+l).$ (2)

From (1), $a = l - (n-1)d.$ (3)

From (2), $2s = an + ln.$

$an = 2s - ln.$

$\therefore a = \frac{2s - ln}{n}.$ (4)

Equate the values of a, $\frac{2s - ln}{n} = l - (n-1)d.$

$$2s - ln = ln - (n-1)dn.$$

$$2s = 2ln - (n-1)dn.$$

$$\therefore s = \tfrac{1}{2}n[2l - (n-1)d].$$

9. $l = a + (n-1)d.$

Transpose, $a = l - (n-1)d.$

10. $l = a + (n-1)d,$ (1)

and $s = \frac{n}{2}(a+l).$ (2)

From (2), $2s = na + nl.$

$nl = 2s - na.$

$\therefore l = \frac{2s - na}{n}.$ (3)

Equate the values of l, $a + (n-1)d = \frac{2s - na}{n}.$

$$na + (n-1)dn = 2s - na.$$

$$2na = 2s - (n-1)dn.$$

$$\therefore a = \frac{s}{n} - \frac{(n-1)d}{2}.$$

11.
$$l = a + (n-1)d, \tag{1}$$

and
$$s = \frac{n}{2}(a + l). \tag{2}$$

From (1),
$$l = a + dn - d.$$
$$dn = l - a + d.$$
$$\therefore\ n = \frac{l - a + d}{d}. \tag{3}$$

From (2),
$$2s = n(a + l).$$
$$\therefore\ n = \frac{2s}{a + l}. \tag{4}$$

Equate the values of n, $\dfrac{l - a + d}{d} = \dfrac{2s}{a + l}.$
$$l^2 - a^2 + ad + ld = 2ds.$$
$$a^2 - ad = l^2 + ld - 2ds.$$

Complete the square, $4a^2 - (\ ) + d^2 = 4l^2 + 4ld + d^2 - 8ds.$
$$2a - d = \pm\sqrt{4l^2 + 4ld + d^2 - 8ds}.$$
$$2a = d \pm \sqrt{(2l + d)^2 - 8ds}.$$
$$\therefore\ a = \tfrac{1}{2}\left[d \pm \sqrt{(2l + d)^2 - 8ds}\right].$$

12.
$$s = \frac{n}{2}(a + l).$$
$$2s = na + ln.$$
$$na = 2s - ln.$$
$$\therefore\ a = \frac{2s}{n} - l.$$

13.
$$l = a + (n-1)d.$$
$$(n-1)d = l - a.$$
$$\therefore\ d = \frac{l - a}{n - 1}.$$

14.
$$l = a + (n-1)d, \tag{1}$$

and
$$s = \frac{n}{2}(a + l). \tag{2}$$

From (2),
$$2s = an + nl.$$
$$nl = 2s - an.$$
$$\therefore\ l = \frac{2s - an}{n}. \tag{3}$$

Equate the values of l, $a + (n-1)d = \dfrac{2s - an}{n}.$

$$an + n^2d - nd = 2s - an.$$
$$n^2d - nd = 2s - 2an.$$
$$n(n-1)d = 2(s - an).$$
$$\therefore d = \frac{2(s-an)}{n(n-1)}.$$

15.
$$l = a + (n-1)d, \tag{1}$$
and
$$s = \frac{n}{2}(a + l). \tag{2}$$
From (1),
$$l = a + nd - d.$$
$$nd = l - a + d.$$
$$\therefore n = \frac{l - a + d}{d}. \tag{3}$$
From (2),
$$2s = n(a + l).$$
$$\therefore n = \frac{2s}{a + l}. \tag{4}$$
Equate the values of n,
$$\frac{l - a + d}{d} = \frac{2s}{a + l}.$$
$$l^2 - a^2 + ad + dl = 2ds.$$
$$2ds - dl - ad = l^2 - a^2.$$
$$(2s - l - a)d = l^2 - a^2.$$
$$\therefore d = \frac{l^2 - a^2}{2s - l - a}.$$

16.
$$l = a + (n-1)d, \tag{1}$$
and
$$s = \frac{n}{2}(a + l). \tag{2}$$
From (1),
$$a = l - (n-1)d. \tag{3}$$
From (2),
$$2s = na + nl.$$
$$na = 2s - nl.$$
$$\therefore a = \frac{2s - nl}{n}. \tag{4}$$
Equate the values of a,
$$l - (n-1)d = \frac{2s - nl}{n}.$$
$$nl - dn^2 + dn = 2s - nl.$$
$$dn^2 - dn = 2nl - 2s.$$
$$n(n-1)d = 2(nl - s).$$
$$\therefore d = \frac{2(nl - s)}{n(n-1)}.$$

17.

$$l = a + (n - 1)d.$$
$$l = a + dn - d.$$
$$dn = l - a + d.$$
$$\therefore\ n = \frac{l - a}{d} + 1.$$

18.

$$l = a + (n - 1)d, \quad (1)$$

and

$$s = \frac{n}{2}(a + l). \quad (2)$$

From (2),

$$2s = an + nl.$$
$$nl = 2s - an.$$
$$\therefore\ l = \frac{2s - an}{n}. \quad (3)$$

Equate the values of l, $a + (n - 1)d = \dfrac{2s - an}{n}$.

$$an + dn^2 - dn = 2s - an.$$
$$dn^2 + (2a - d)n = 2s.$$

Complete the square,

$$4d^2n^2 + (\ ) + (2a - d)^2 = (2a - d)^2 + 8ds.$$
$$2dn + (2a - d) = \pm\sqrt{(2a - d)^2 + 8ds}.$$
$$2dn = d - 2a \pm \sqrt{(2a - d)^2 + 8ds}.$$
$$\therefore\ n = \frac{d - 2a \pm \sqrt{(2a - d)^2 + 8ds}}{2d}.$$

19.

$$s = \frac{n}{2}(a + l).$$
$$2s = n(a + l).$$
$$\therefore\ n = \frac{2s}{l + a}.$$

20.

$$l = a + (n - 1)d, \quad (1)$$

and

$$s = \frac{n}{2}(a + l). \quad (2)$$

From (1),

$$a = l - (n - 1)d. \quad (3)$$

From (2),

$$2s = na + ln.$$
$$na = 2s - ln.$$
$$\therefore\ a = \frac{2s - ln}{n}. \quad (4)$$

Equate the values of a, $l - (n - 1)d = \dfrac{2s - ln}{n}$.

$$ln - dn^2 + dn = 2s - ln.$$

$$dn^2 - (2l + d)n = -2s.$$

Complete the square,

$$4d^2n^2 - (\ ) + (2l + d)^2 = (2l + d)^2 - 8ds.$$

$$2dn - (2l + d) = \pm\sqrt{(2l + d)^2 - 8ds}.$$

$$2dn = 2l + d \pm\sqrt{(2l + d)^2 - 8ds}.$$

$$\therefore\ n = \frac{2l + d \pm\sqrt{(2l + d)^2 - 8ds}}{2d}.$$

Exercise 129. Page 337

1. Find the eighth term of 7, 10, 13, ⋯

Here $a = 7,\ d = 3,\ n = 8.$

$$l = a + (n - 1)d = 7 + (8 - 1)3 = 7 + 7 \times 3 = 7 + 21 = 28.$$

2. Find the tenth term of 2, 11, 20, ⋯

Here $a = 2,\ d = 9,\ n = 10.$

$$l = a + (n - 1)d = 2 + (10 - 1)9 = 2 + 9 \times 9 = 2 + 81 = 83.$$

3. Find the fifteenth term of 4, 16, 28, ⋯

Here $a = 4,\ d = 12,\ n = 15.$

$$l = a + (n - 1)d = 4 + (15 - 1)12 = 4 + 14 \times 12$$
$$= 4 + 168 = 172.$$

4. Find the twelfth term of 144, 138, 132, ⋯

Here $a = 144,\ d = -6,\ n = 12.$

$$l = a + (n - 1)d = 144 + (12 - 1)(-6) = 144 + 11(-6)$$
$$= 144 - 66 = 78.$$

5. Find the sixth term of 5, −1, −7, ⋯

Here $a = 5,\ d = -6,\ n = 6.$

$$l = a + (n - 1)d = 5 + (6 - 1)(-6) = 5 + 5(-6)$$
$$= 5 - 30 = -25.$$

6. Find the fourteenth term of 12, 4, −4, ⋯

Here $a = 12,\ d = -8,\ n = 14.$

$$l = a + (n - 1)d = 12 + (14 - 1)(-8) = 12 + 13(-8)$$
$$= 12 - 104 = -92.$$

7. Find the sum of seven terms of 18, 21, 24, $\cdots$

Here $a = 18,\ d = 3,\ n = 7.$

$$l = a + (n-1)d = 18 + (7-1)3 = 18 + 6 \times 3$$
$$= 18 + 18 = 36.$$
$$s = \frac{n}{2}(a + l) = \frac{7}{2}(18 + 36) = \frac{7}{2} \times 54 = 189.$$

8. Find the sum of ten terms of $-6,\ -2,\ 2,\ \cdots$

Here $a = -6,\ d = 4,\ n = 10.$

$$l = a + (n-1)d = -6 + (10-1)4 = -6 + 9 \times 4$$
$$= -6 + 36 = 30.$$
$$s = \frac{n}{2}(a + l) = \frac{10}{2}(-6 + 30) = 5 \times 24 = 120.$$

9. Find the sum of eighteen terms of $3\frac{1}{8},\ 3\frac{7}{8},\ 4\frac{5}{8},\ \cdots$

Here $a = 3\frac{1}{8},\ d = \frac{3}{4},\ n = 18.$

$$l = a + (n-1)d = 3\tfrac{1}{8} + (18-1)(\tfrac{3}{4}) = 3\tfrac{1}{8} + 17 \times \tfrac{3}{4}$$
$$= 3\tfrac{1}{8} + 12\tfrac{3}{4} = 15\tfrac{7}{8}.$$
$$s = \frac{n}{2}(a + l) = \frac{18}{2}(3\tfrac{1}{8} + 15\tfrac{7}{8}) = 9 \times 19 = 171.$$

10. Find the sum of twenty-four terms of $1\frac{5}{9},\ 3,\ 4\frac{4}{9},\ \cdots$

Here $a = 1\frac{5}{9},\ d = 1\frac{4}{9},\ n = 24.$

$$l = a + (n-1)d = 1\tfrac{5}{9} + (24-1)(1\tfrac{4}{9}) = 1\tfrac{5}{9} + 23 \times 1\tfrac{4}{9}$$
$$= 1\tfrac{5}{9} + 33\tfrac{2}{9} = 34\tfrac{7}{9}.$$
$$s = \frac{n}{2}(a + l) = \frac{24}{2}(1\tfrac{5}{9} + 34\tfrac{7}{9}) = 12 \times 36\tfrac{1}{3} = 436.$$

11. Find the sum of n terms of $3,\ 5\frac{2}{3},\ 8\frac{1}{3},\ \cdots$

Here $a = 3,\ d = 2\frac{2}{3},\ n = n.$

$$l = a + (n-1)d = 3 + (n-1)(2\tfrac{2}{3}) = 3 + \frac{8n-8}{3}.$$
$$s = \frac{n}{2}(a + l) = \frac{n}{2}\left(3 + 3 + \frac{8n-8}{3}\right) = \frac{n}{2}\left(6 + \frac{8n-8}{3}\right)$$
$$= 3n + \frac{4n^2 - 4n}{3} = \frac{4n^2 + 5n}{3}.$$

12. Find the sum of n terms of a, $a+4b$, $a+8b$, $\cdots$

Here $a=a,\ d=4b,\ n=n.$

$$l=a+(n-1)d=a+(n-1)4b=a+4bn-4b.$$

$$s=\frac{n}{2}(a+l)=\frac{n}{2}(a+a+4bn-4b)=\frac{n}{2}(2a+4bn-4b)$$

$$=an+2bn^2-2bn.$$

13. Given $a=4,\ l=34,\ n=11$; find d and s.

By Formula 13, page 336,

$$d=\frac{l-a}{n-1}=\frac{34-4}{11-1}=\frac{30}{10}=3.$$

By Formula 7, page 336,

$$s=\frac{n}{2}(a+l)=\tfrac{11}{2}(4+34)=\tfrac{11}{2}\times 38=209.$$

14. Given $a=120,\ n=16,\ s=960$; find d and l.

By Formula 14, page 336,

$$d=\frac{2(s-an)}{n(n-1)}=\frac{2(960-120\times 16)}{16(16-1)}=\frac{2(960-1920)}{16\times 15}$$

$$=\frac{-1920}{16\times 15}=-8.$$

By Formula 3, page 336,

$$l=\frac{2s}{n}-a=\frac{2\times 960}{16}-120=120-120=0.$$

15. Given $a=-12,\ d=4,\ l=40$; find n and s.

By Formula 17, page 336,

$$n=\frac{l-a}{d}+1=\frac{40+12}{4}+1=13+1=14.$$

By Formula 6, page 336,

$$s=\frac{l+a}{2}+\frac{l^2-a^2}{2d}=\frac{40-12}{2}+\frac{1600-144}{2\times 4}=\frac{28}{2}+\frac{1456}{8}$$

$$=14+182=196.$$

16. Given $a = 7$, $l = 49$, $s = 812$; find d and n.

By Formula 15, page 336,

$$d = \frac{l^2 - a^2}{2s - l - a} = \frac{49^2 - 7^2}{2 \times 812 - 49 - 7} = \frac{2352}{1568} = \frac{3}{2} = 1\tfrac{1}{2}.$$

By Formula 19, page 336,

$$n = \frac{2s}{l + a} = \frac{2 \times 812}{49 + 7} = \frac{1624}{56} = 29.$$

17. Given $d = \frac{2}{3}$, $n = 24$, $s = 56$; find a and l.

By Formula 10, page 336,

$$a = \frac{s}{n} - \frac{(n-1)d}{2} = \frac{56}{24} - \frac{23 \times \frac{2}{3}}{2} = \frac{7}{3} - \frac{23}{3} = -\frac{16}{3} = -5\tfrac{1}{3}.$$

By Formula 4, page 336,

$$l = \frac{s}{n} + \frac{(n-1)d}{2} = \frac{7}{3} + \frac{23}{3} = \frac{30}{3} = 10.$$

18. Given $a = 4$, $d = 3$, $s = 246$; find l and n.

By Formula 2, page 336,

$$\begin{aligned} l &= \tfrac{1}{2}[-d \pm \sqrt{8ds + (2a - d)^2}] \\ &= \tfrac{1}{2}[-3 \pm \sqrt{8 \times 3 \times 246 + (2 \times 4 - 3)^2}] \\ &= \tfrac{1}{2}[-3 \pm \sqrt{5904 + 25}] = \tfrac{1}{2}[-3 \pm \sqrt{5929}] \\ &= \tfrac{1}{2}[-3 \pm 77] = 37, \text{ or } -40. \end{aligned}$$

By Formula 18, page 336,

$$\begin{aligned} n &= \frac{d - 2a \pm \sqrt{(2a - d)^2 + 8ds}}{2d} = \frac{3 - 8 \pm 77}{2 \times 3} \\ &= \frac{-5 \pm 77}{6} = 12, \text{ or } -13\tfrac{2}{3}. \end{aligned}$$

Only the positive solutions apply.

Therefore, $l = 37$, and $n = 12$.

19. Insert four arithmetical means between 8 and 23.

Here $n = 6$, $a = 8$, and $l = 23$.

$$l = a + (n - 1)d = 8 + (6 - 1)d.$$

$$l = 8 + 5d.$$
$$\therefore\ 23 = 8 + 5d.$$
$$5d = 15.$$
$$\therefore\ d = 3.$$

Hence, the complete series is 8, 11, 14, 17, 20, 23.

Therefore, the four means required are 11, 14, 17, 20.

20. Insert six arithmetical means between 96 and 47.

Here $n = 8$, $a = 96$, and $l = 47$.

$$l = a + (n-1)d = 96 + (8-1)d.$$
$$l = 96 + 7d.$$
$$\therefore\ 47 = 96 + 7d.$$
$$7d = -49.$$
$$\therefore\ d = -7.$$

Hence, the complete series is 96, 89, 82, 75, 68, 61, 54, 47.

Therefore, the six means required are 89, 82, 75, 68, 61, 54.

21. Insert eight arithmetical means between 4 and 58.

Here $n = 10$, $a = 4$, and $l = 58$.

$$l = a + (n-1)d = 4 + (10-1)d.$$
$$l = 4 + 9d.$$
$$\therefore\ 58 = 4 + 9d.$$
$$9d = 54.$$
$$\therefore\ d = 6.$$

Hence, the complete series is 4, 10, 16, 22, 28, 34, 40, 46, 52, 58.

Therefore, the eight means required are 10, 16, 22, 28, 34, 40, 46, 52.

22. Insert twelve arithmetical means between 1 and 10.

Here $n = 14$, $a = 1$, and $l = 10$.

$$l = a + (n-1)d = 1 + (14-1)d.$$
$$l = 1 + 13d.$$
$$\therefore\ 10 = 1 + 13d.$$
$$13d = 9.$$
$$\therefore\ d = \tfrac{9}{13}.$$

Hence, the complete series is 1, $1\frac{9}{13}$, $2\frac{5}{13}$, $3\frac{1}{13}$, $3\frac{10}{13}$, $4\frac{6}{13}$, $5\frac{2}{13}$, $5\frac{11}{13}$, $6\frac{7}{13}$, $7\frac{3}{13}$, $7\frac{12}{13}$, $8\frac{8}{13}$, $9\frac{4}{13}$, 10.

Therefore, the twelve means required are $1\frac{9}{13}$, $2\frac{5}{13}$, $3\frac{1}{13}$, $3\frac{10}{13}$, $4\frac{6}{13}$, $5\frac{2}{13}$, $5\frac{11}{13}$, $6\frac{7}{13}$, $7\frac{3}{13}$, $7\frac{12}{13}$, $8\frac{8}{13}$, $9\frac{4}{13}$.

23. The first term of an arithmetical progression is 7, and the third term is 23. Find the sum of ten terms.

$$
\begin{aligned}
l &= a + (n-1)d.\\
\therefore\ 23 &= 7 + (3-1)d.\\
23 &= 7 + 2d.\\
2d &= 16.\\
\therefore\ d &= 8.
\end{aligned}
$$

$$l = a + (n-1)d = 7 + (10-1)8 = 7 + 9 \times 8 = 7 + 72 = 79.$$

$$s = \frac{n}{2}(a + l) = \tfrac{10}{2}(7 + 79) = 5 \times 86 = 430.$$

24. The first term of an arithmetical progression is 4, and the sum of seven terms is 175. What term is 60?

$$
\begin{aligned}
s &= \frac{n}{2}(a + l).\\
\therefore\ 175 &= \tfrac{7}{2}(4 + l).\\
350 &= 28 + 7l.\\
7l &= 322.\\
\therefore\ l &= 46, \text{ seventh term.}\\
l &= a + (n-1)d.\\
46 &= 4 + (7-1)d.\\
46 &= 4 + 6d.\\
6d &= 42.\\
\therefore\ d &= 7.
\end{aligned}
$$

If 60 is the nth term, $60 = 4 + (n-1)7.$

$$
\begin{aligned}
7(n-1) &= 56.\\
n - 1 &= 8.\\
\therefore\ n &= 9.
\end{aligned}
$$

Therefore, 60 is the ninth term.

25. How many terms of the series $-9, -3, 3, \cdots$ must be taken in order that their sum may be 135?

Here $a = -9$, $d = 6$, and $s = 135$.

$$
\begin{aligned}
l &= a + (n-1)d.\\
l &= -9 + (n-1)6. &\quad (1)\\
s &= \frac{n}{2}(a + l).
\end{aligned}
$$

$$135 = \frac{n}{2}(-9 + l). \qquad (2)$$

From (1), $l = 6n - 15.$

Substitute the value of n in (2), $135 = \frac{n}{2}(6n - 24).$

$$135 = 3n^2 - 12n.$$
$$45 = n^2 - 4n.$$
$$n^2 - 4n - 45 = 0.$$
$$(n - 9)(n + 5) = 0.$$
$$\therefore n = 9, \text{ or } -5.$$

Therefore, nine terms must be taken.

26. The sum of three numbers in arithmetical progression is 27, and the sum of their squares is 293. Find the numbers.

Let $x - y$, x, and $x + y =$ the numbers.

Then $x - y + x + x + y = 27, \qquad (1)$

and $(x - y)^2 + x^2 + (x + y)^2 = 293. \qquad (2)$

From (1), $3x = 27.$

$$\therefore x = 9. \qquad (3)$$

Expand (2),

$$x^2 - 2xy + y^2 + x^2 + x^2 + 2xy + y^2 = 293.$$

Combine, $3x^2 + 2y^2 = 293. \qquad (4)$

Substitute in (4) the value of x from (3),

$$243 + 2y^2 = 293.$$
$$2y^2 = 50.$$
$$y^2 = 25.$$
$$y = \pm 5.$$

Then $x - y = 4$, or 14,

and $x + y = 14$, or 4.

Therefore, the three numbers required are 4, 9, 14, or 14, 9, 4.

27. The sum of four numbers in arithmetical progression is 42, and the sum of their squares is 686. Find the numbers.

Let $x - 3y, x - y, x + y, x + 3y =$ the numbers.

Then $x - 3y + x - y + x + y + x + 3y = 42, \qquad (1)$

and $(x - 3y)^2 + (x - y)^2 + (x + y)^2 + (x + 3y)^2 = 686. \qquad (2)$

Combine (1), $4x = 42.$

$\therefore x = \frac{21}{2}.$ (3)

Expand (2),

$x^2 - 6xy + 9y^2 + x^2 - 2xy + y^2 + x^2 + 2xy + y^2 + x^2 + 6xy + 9y^2 = 686.$

Combine, $4x^2 + 20y^2 = 686.$ (4)

Substitute in (4) the value of x from (3),

$$441 + 20y^2 = 686.$$
$$20y^2 = 245.$$
$$y^2 = \frac{49}{4}.$$
$$\therefore y = \pm \frac{7}{2}.$$

Then $x - 3y = 0$, or 21,

$x - y = 7$, or 14,

$x + y = 14$, or 7,

and $x + 3y = 21$, or 0.

Therefore, the numbers required are 0, 7, 14, 21, or 21, 14, 7, 0.

28. How many terms of the arithmetical series 3, 7, 11, $\cdots$ must be taken that the sum of the first half may be to the sum of the second half in the ratio 11 : 31?

Let $2n =$ the number of terms,

$s =$ the sum of the whole series,

and $s' =$ the sum of the first half.

For the first half,

$$l = a + (n - 1)d = 3 + (n - 1)4 = 4n - 1.$$

$$s' = \frac{n}{2}(a + l) = \frac{n}{2}(3 + 4n - 1) = \frac{n}{2}(4n + 2) = 2n^2 + n.$$

For the whole series,

$$l = a + (n - 1)d = 3 + (2n - 1)4 = 8n - 1$$
$$s = n(3 + 8n - 1) = n(8n + 2) = 8n^2 + 2n.$$

Now $s' : s - s' = 11 : 31.$

$\therefore 2n^2 + n : 8n^2 + 2n - (2n^2 + n) = 11 : 31.$

$$2n^2 + n : 6n^2 + n = 11 : 31.$$
$$2n + 1 : 6n + 1 = 11 : 31.$$
$$62n + 31 = 66n + 11.$$
$$4n = 20.$$
$$2n = 10.$$

Therefore, ten terms must be taken.

29. The sum of the squares of the extremes of four numbers in arithmetical progression is 450, and the sum of the squares of the means is 306. What are the numbers?

Let $x-3y,\ x-y,\ x+y,\ x+3y =$ the numbers.

Then $$(x-3y)^2+(x+3y)^2=450, \tag{1}$$

and $$(x-y)^2+(x+y)^2=306. \tag{2}$$

Expand (1),

$$x^2-6xy+9y^2+x^2+6xy+9y^2=450. \tag{3}$$

Expand (2),

$$x^2-2xy+y^2+x^2+2xy+y^2=306. \tag{4}$$

Combine (3), $$2x^2+18y^2=450. \tag{5}$$

Combine (4), $$2x^2+2y^2=306. \tag{6}$$

Subtract (6) from (5), $$16y^2=144.$$

$$y^2=9.$$

$$\therefore\ y=\pm 3.$$

Substitute the value of y in (6),

$$2x^2+18=306.$$

$$2x^2=288.$$

$$x^2=144.$$

$$\therefore\ x=\pm 12.$$

Then $$x-3y=\pm 3, \text{ or } \pm 21,$$

$$x-y=\pm 9, \text{ or } \pm 15,$$

$$x+y=\pm 15, \text{ or } \pm 9,$$

and $$x+3y=\pm 21, \text{ or } \pm 3.$$

Therefore, the numbers required are 3, 9, 15, 21; -3, -9, -15, -21; 21, 15, 9, 3; or -21, -15, -9, -3.

30. If a number composed of three digits in arithmetical progression is divided by the sum of the digits, the quotient is 63 and the remainder 6; if 792 is subtracted from the number, the order of the digits is reversed. Find the number.

Let $x+y =$ the digit in the hundreds' place,

$x =$ the digit in the tens' place,

and $x-y =$ the digit in the units' place.

Then $100(x+y)+10x+x-y =$ the number,

and $100(x-y)+10x+x+y =$ the number with digits reversed.

$$\therefore \frac{100(x+y)+10x+x-y-6}{x+y+x+x-y}=63, \qquad (1)$$

and $$100(x+y)+10x+x-y-792=100(x-y)+10x+x+y. \qquad (2)$$

From (1),

$$\frac{100x+100y+10x+x-y-6}{x+y+x+x-y}=63.$$

$$\frac{111x+99y-6}{3x}=63.$$

$$\frac{37x+33y-2}{x}=63.$$

$$37x+33y-2=63x.$$

$$26x=33y-2. \qquad (3)$$

From (2),

$$100x+100y+10x+x-y-792=100x-100y+10x+x+y.$$

Transpose and combine, $198y=792$.

$$\therefore y=4.$$

Substitute the value of y in (3), $26x=130$.

$$\therefore x=5.$$

Then $$x+y=9,$$

$$x-y=1,$$

and $$100(x+y)+10x+x-y=951.$$

Therefore, the required number is 951.

31. In a potato race 100 potatoes are placed 3 feet apart in a straight line. A runner picks up one potato at a time and carries it to a basket in the line of the potatoes, and 3 feet from the first potato. How far must the contestant run?

In picking up the potatoes the contestant goes successively 6 feet, 12 feet, 18 feet, $\cdots$

Hence, these distances form an arithmetical series in which $a=6$, $d=6$, $n=100$.

$$l=a+(n-1)d=6+(100-1)6=6+99\times 6$$
$$=100\times 6=600.$$

$$s=\frac{n}{2}(a+l)=\frac{100}{2}(6+600)=50\times 606=30300.$$

Therefore, the contestant must run 30,300 feet.

32. In a potato race, if there are 30 potatoes 4 feet apart, and the basket is 4 feet from the first potato, how far must the contestant run?

In picking up the potatoes the contestant goes successively 8 feet, 16 feet, 24 feet, $\cdots$

Hence, these distances form an arithmetical series in which $a = 8$, $d = 8$, $n = 30$.

$$l = a + (n-1)d = 8 + (30-1)8 = 8 + 29 \times 8$$
$$= 30 \times 8 = 240.$$
$$s = \frac{n}{2}(a + l) = \frac{30}{2}(8 + 240) = 15 \times 248 = 3720.$$

Therefore, the contestant must run 3720 feet.

33. A body falling freely falls 16.08 feet in the first second, and in each succeeding second 32.16 feet more than in the second immediately preceding. If a stone dropped from a stationary balloon reaches the ground in 12 seconds, how far does it fall in the last second, and how high is the balloon?

Here $a = 16.08$, $d = 32.16$, $n = 12$.

$$l = a + (n-1)d = 16.08 + 11 \times 32.16$$
$$= 16.08 + 353.76 = 369.84.$$
$$s = \frac{n}{2}(a + l) = \frac{12}{2}(16.08 + 369.84) = 6 \times 385.92 = 2315.52.$$

Therefore, the stone falls 369.84 feet in the last second, and the balloon is 2315.52 feet high.

34. A stone is dropped from the top of a tower 402 feet high. In how many seconds does it reach the ground?

Here $a = 16.08$, $d = 32.16$, $s = 402$.

By Formula 18, page 336,

$$n = \frac{d - 2a \pm \sqrt{(2a - d)^2 + 8ds}}{2d}$$
$$= \frac{32.16 - 32.16 \pm \sqrt{(32.16 - 32.16)^2 + 8 \times 32.16 \times 402}}{2 \times 32.16}$$
$$= \frac{\pm\sqrt{8 \times 32.16 \times 402}}{64.32}$$
$$= \pm\sqrt{\frac{8 \times 32.16 \times 402}{64.32 \times 64.32}}$$
$$= \pm\sqrt{25}$$
$$= \pm 5.$$

Therefore, the stone reaches the ground in 5 seconds.

35. A stone is dropped from a stationary balloon 3618 feet high. In how many seconds does it reach the ground?

Here $a = 16.08$, $d = 32.16$, $s = 3618$.

By Formula 18, page 336,

$$\begin{aligned}
n &= \frac{d - 2a \pm \sqrt{(2a - d)^2 + 8ds}}{2d} \\
&= \frac{32.16 - 32.16 \pm \sqrt{(32.16 - 32.16)^2 + 8 \times 32.16 \times 3618}}{2 \times 32.16} \\
&= \frac{\pm\sqrt{8 \times 32.16 \times 3618}}{64.32} \\
&= \pm\sqrt{\frac{8 \times 32.16 \times 3618}{64.32 \times 64.32}} \\
&= \pm\sqrt{225} \\
&= \pm 15.
\end{aligned}$$

Therefore, the stone reaches the ground in 15 seconds.

36. In astronomical time the hours of a day are numbered from 1 to 24. If a clock should strike the hours of an astronomical day, how many strokes would it strike in one day?

Here $a = 1$, $l = 24$, $n = 24$.

$$s = \frac{n}{2}(a + l) = \frac{24}{2}(1 + 24) = 12 \times 25 = 300.$$

Therefore, the clock would strike 300 times in one day.

37. A body falling freely falls 4.9 meters the first second, and each succeeding second 9.8 meters more than in the second immediately preceding. How far will a body fall in 1 minute?

Here $a = 4.9$, $d = 9.8$, $n = 60$.

$$\begin{aligned}
l &= a + (n - 1)d = 4.9 + (60 - 1)9.8 = 4.9 + 59 \times 9.8 \\
&= 4.9 + 578.2 = 583.1.
\end{aligned}$$

$$s = \frac{n}{2}(a + l) = \frac{60}{2}(4.9 + 583.1) = 30 \times 588 = 17640.$$

Therefore, a body will fall 17,640 meters in 1 minute.

38. If a bullet, when fired vertically upward, traverses 485.1 meters the first second, how high will it rise, and how long will it be before it reaches the earth again?

By Example 37, the bullet traverses in succeeding seconds 485.1, $485.1 - 9.8$, $485.1 - 2 \times 9.8$, $485.1 - 3 \times 9.8$, $\cdots$ meters.

Here $a = 485.1$, $d = -9.8$, $l = 4.9$.

By Formula 6, page 336,

$$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d} = \frac{4.9 + 485.1}{2} + \frac{4.9^2 - 485.1^2}{2(-9.8)}$$
$$= 245 + 12005 = 12250.$$

By Formula 17, page 336,

$$n = \frac{l-a}{d} + 1 = \frac{4.9 - 485.1}{-9.8} + 1 = 49 + 1 = 50.$$

At the very instant that the bullet reaches the highest point of its flight it begins to fall.

Since it takes the bullet as long to fall as it takes to reach the highest point, the bullet is in the air 2 × 50 seconds = 100 seconds.

Therefore, the bullet rises 12,250 meters and is 100 seconds in the air.

39. In order to sell a football a boy marks tickets from 1 to 25 and obtains twenty-five persons to draw each a ticket. Each person pays as many cents as the number on the ticket he draws, and the owner of the football is determined by lot. How much does the boy receive for the football?

Here $a = 1,\ l = 25,\ n = 25.$

$$s = \frac{n}{2}(a + l) = \frac{25}{2}(1 + 25) = \frac{25}{2} \times 26 = 325.$$

325 cents = \$3.25.

Therefore, the boy receives \$3.25 for the football.

40. In selling a gold watch a man used the same system as in Example 39, but sold 80 tickets marked from 1 to 80. How much did he receive for the watch?

Here $a = 1,\ l = 80,\ n = 80.$

$$s = \frac{n}{2}(a + l) = \frac{80}{2}(1 + 80) = 40 \times 81 = 3240.$$

3240 cents = \$32.40.

Therefore, the man received \$32.40 for the watch.

41. In selling a Persian rug a dealer used the same system as in Example 39, but sold 250 tickets marked from 1 to 250. How much did he receive for the rug?

Here $a = 1,\ l = 250,\ n = 250.$

$$s = \frac{n}{2}(a + l) = \frac{250}{2}(1 + 250) = 125 \times 251 = 31375.$$

31375 cents = \$313.75.

Therefore, the dealer received \$313.75 for the rug.

Table on Page 344

1. The 1st term $= a$,

2d term $= ar$,

3d term $= ar^2$,

4th term $= ar^3$,

. . . .

nth term $= ar^{n-1}$.

$$\therefore l = ar^{n-1}.$$

2.
$$s = \frac{rl - a}{r - 1}.$$
$$(r-1)s = rl - a.$$
$$rl = a + (r-1)s.$$
$$\therefore l = \frac{a + (r-1)s}{r}.$$

3.
$$l = ar^{n-1}, \quad (1)$$
and
$$s = \frac{a(r^n - 1)}{r - 1}. \quad (2)$$
From (1), $rl = ar^n$, (3)

and
$$r^{n-1} = \frac{l}{a}.$$
$$\therefore r = \sqrt[n-1]{\frac{l}{a}}. \quad (4)$$
From (2),
$$sr - s = ar^n - a.$$
$$sr - ar^n = s - a.$$
Substitute the value of ar^n from (3),
$$sr - rl = s - a.$$
$$r(s - l) = s - a.$$
$$\therefore r = \frac{s - a}{s - l}. \quad (5)$$
Equate the values of r in (4) and (5),
$$\sqrt[n-1]{\frac{l}{a}} = \frac{s - a}{s - l}.$$
$$\frac{l}{a} = \frac{(s-a)^{n-1}}{(s-l)^{n-1}}.$$
$$\therefore l(s - l)^{n-1} - a(s - a)^{n-1} = 0.$$

4.
$$l = ar^{n-1}, \quad (1)$$
and
$$s = \frac{a(r^n - 1)}{r - 1}. \quad (2)$$
From (1),
$$a = \frac{l}{r^{n-1}}. \quad (3)$$
From (2),
$$s(r-1) = a(r^n - 1).$$
$$\therefore a = \frac{s(r-1)}{r^n - 1}. \quad (4)$$
Equate the values of a in (3) and (4),
$$\frac{l}{r^{n-1}} = \frac{s(r-1)}{r^n - 1}.$$
$$l(r^n - 1) = sr^{n-1}(r - 1).$$
$$\therefore l = \frac{(r-1)sr^{n-1}}{r^n - 1}.$$

5. $s = a + ar + ar^2 + ar^3$
$+ \cdots + ar^{n-1}$. (1)

Multiply (1) by r,
$$rs = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \quad (2)$$
Subtract (1) from (2),
$$rs - s = ar^n - a.$$
$$(r-1)s = a(r^n - 1).$$
$$\therefore s = \frac{a(r^n - 1)}{r - 1}.$$

6.
$$s = \frac{a(r^n - 1)}{r - 1}, \quad (1)$$
and
$$l = ar^{n-1}. \quad (2)$$
From (1),
$$s = \frac{ar^n - a}{r - 1}. \quad (3)$$
From (2), $rl = ar^n$.

Substitute the value of ar^n in (3),
$$s = \frac{rl - a}{r - 1}.$$

7. $$l = ar^{n-1}, \qquad (1)$$

and $$s = \frac{a(r^n - 1)}{r - 1}. \qquad (2)$$

From (1), $r^{n-1} = \frac{l}{a}$.

$$\therefore r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}. \qquad (3)$$

From (2),

$$s(r - 1) = ar^n - a. \qquad (4)$$

Substitute in (4) the value of r from (3),

$$s\left[\left(\frac{l}{a}\right)^{\frac{1}{n-1}} - 1\right] = a\left(\frac{l}{a}\right)^{\frac{n}{n-1}} - a.$$

$$\therefore s = \frac{a\left(\frac{l}{a}\right)^{\frac{n}{n-1}} - a}{\left(\frac{l}{a}\right)^{\frac{1}{n-1}} - 1}.$$

Multiply both numerator and denominator of the fraction by $a^{\frac{1}{n-1}}$,

$$s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$$

8. $$l = ar^{n-1}, \qquad (1)$$

and $$s = \frac{a(r^n - 1)}{r - 1}. \qquad (2)$$

From (1), $$a = \frac{l}{r^{n-1}}. \qquad (3)$$

From (2),

$$s(r - 1) = a(r^n - 1).$$

$$\therefore a = \frac{s(r - 1)}{r^n - 1}. \qquad (4)$$

Equate the values of a in (3) and (4),

$$\frac{s(r - 1)}{r^n - 1} = \frac{l}{r^{n-1}}.$$

$$s(r^n - r^{n-1}) = lr^n - l.$$

$$\therefore s = \frac{lr^n - l}{r^n - r^{n-1}}.$$

9. $$l = ar^{n-1}.$$

$$\therefore a = \frac{l}{r^{n-1}}.$$

10. $$s = \frac{a(r^n - 1)}{r - 1}.$$

$$(r - 1)s = (r^n - 1)a.$$

$$\therefore a = \frac{(r - 1)s}{r^n - 1}.$$

11. $$s = \frac{rl - a}{r - 1}.$$

$$(r - 1)s = rl - a.$$

$$\therefore a = rl - (r - 1)s.$$

12. $$l = ar^{n-1}, \qquad (1)$$

and $$s = \frac{rl - a}{r - 1}. \qquad (2)$$

From (1), $r^{n-1} = \frac{l}{a}$.

$$\therefore r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}. \qquad (3)$$

From (2),

$$rs - s = rl - a.$$

$$rs - rl = s - a.$$

$$\therefore r = \frac{s - a}{s - l}. \qquad (4)$$

Equate the values of r in (3) and (4),

$$\left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \frac{s - a}{s - l}.$$

$$\therefore \frac{l}{a} = \frac{(s - a)^{n-1}}{(s - l)^{n-1}}.$$

$$\therefore a(s - a)^{n-1} - l(s - l)^{n-1} = 0.$$

13. $$l = ar^{n-1}.$$

$$r^{n-1} = \frac{l}{a}.$$

$$\therefore r = \sqrt[n-1]{\frac{l}{a}}.$$

14.

$$s = \frac{a(r^n - 1)}{r - 1}.$$

$$rs - s = ar^n - a.$$

$$ar^n - rs + s - a = 0.$$

$$\therefore r^n - \frac{s}{a}r + \frac{s - a}{a} = 0.$$

15.

$$s = \frac{rl - a}{r - 1}.$$

$$rs - s = rl - a.$$

$$rs - rl = s - a.$$

$$\therefore r = \frac{s - a}{s - l}.$$

16.

$$l = ar^{n-1}, \qquad (1)$$

and

$$s = \frac{a(r^n - 1)}{r - 1}. \qquad (2)$$

From (1), $a = \frac{l}{r^{n-1}}$. (3)

From (2),

$$s(r - 1) = a(r^n - 1).$$

$$\therefore a = \frac{s(r - 1)}{r^n - 1}. \qquad (4)$$

Equate the values of a in (3) and (4),

$$\frac{l}{r^{n-1}} = \frac{s(r - 1)}{r^n - 1}.$$

$$lr^n - l = sr^n - sr^{n-1}.$$

$$sr^n - lr^n - sr^{n-1} + l = 0.$$

$$(s - l)r^n - sr^{n-1} + l = 0.$$

$$\therefore r^n - \frac{s}{s - l}r^{n-1} + \frac{l}{s - l} = 0.$$

Exercise 130. Page 343

1. Find the sixth term of 4, 8, 16, ...

Here $a = 4, r = 2, n = 6.$

$$l = ar^{n-1} = 4 \times 2^5 = 4 \times 32 = 128.$$

2. Find the ninth term of 2, 14, 98, ...

Here $a = 2, r = 7, n = 9.$

$$l = ar^{n-1} = 2 \times 7^8 = 2 \times 5764801 = 11{,}529{,}602.$$

3. Find the tenth term of 128, − 64, 32, ...

Here $a = 128, r = -\frac{1}{2}, n = 10.$

$$l = ar^{n-1} = 128 \times (-\tfrac{1}{2})^9 = 128 \times (-\tfrac{1}{512}) = -\tfrac{1}{4}.$$

4. Find the twelfth term of 4, − 3, 2¼, ...

Here $a = 4, r = -\frac{3}{4}, n = 12.$

$$l = ar^{n-1} = 4 \times \left(-\frac{3}{4}\right)^{11} = 4 \times -\frac{3^{11}}{4^{11}} = -\frac{3^{11}}{4^{10}}.$$

5. Find the twenty-fifth term of 2, 6, 18, ...

Here $a = 2, r = 3, n = 25.$

$$l = ar^{n-1} = 2 \times 3^{24}.$$

6. Find the nth term of $2,\ -1\frac{1}{9},\ \frac{50}{81},\ \cdots$

Here $a = 2,\ r = -\frac{5}{9},\ n = n.$

$$l = ar^{n-1} = 2\left(-\tfrac{5}{9}\right)^{n-1}.$$

7. Find the sum of nine terms of $8,\ 24,\ 72,\ \cdots$

Here $a = 8,\ r = 3,\ n = 9.$

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{8(3^9 - 1)}{3 - 1} = \frac{8 \times 19682}{2} = 78{,}728.$$

8. Find the sum of twelve terms of $9,\ -3,\ 1,\ \cdots$

Here $a = 9,\ r = -\frac{1}{3},\ n = 12.$

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{9[(-\frac{1}{3})^{12} - 1]}{-\frac{1}{3} - 1} = 9 \times \left(-\frac{531440}{531441}\right) \times \left(-\frac{3}{4}\right)$$

$$= \tfrac{132860}{19683} = 6\tfrac{14762}{19683}.$$

9. Find the sum of sixteen terms of $81,\ 27,\ 9,\ \cdots$

Here $a = 81,\ r = \frac{1}{3},\ n = 16.$

$$l = ar^{n-1} = 81 \times \left(\frac{1}{3}\right)^{15} = 3^4 \times \frac{1}{3^{15}} = \frac{1}{3^{11}}.$$

$$s = \frac{rl - a}{r - 1} = \frac{\frac{1}{3} \times \frac{1}{3^{11}} - 3^4}{\frac{1}{3} - 1} = \frac{\frac{1}{3^{12}} - 3^4}{-\frac{2}{3}} = \frac{3}{2}\left(3^4 - \frac{1}{3^{12}}\right)$$

$$= \frac{1}{2}\left(3^5 - \frac{1}{3^{11}}\right).$$

10. Find the sum of nineteen terms of $5,\ -10,\ 20,\ \cdots$

Here $a = 5,\ r = -2,\ n = 19.$

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{5[(-2)^{19} - 1]}{-2 - 1} = -\tfrac{5}{3}[(-2)^{19} - 1]$$

$$= \tfrac{5}{3}(2^{19} + 1).$$

11. Find the sum of n terms of $2\frac{1}{4},\ 1\frac{1}{2},\ 1,\ \cdots$

Here $a = 2\frac{1}{4},\ r = \frac{2}{3},\ n = n.$

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{\frac{9}{4}[(\frac{2}{3})^n - 1]}{\frac{2}{3} - 1} = \frac{\frac{3^2}{2^2}\left(1 - \frac{2^n}{3^n}\right)}{\frac{1}{3}} = \frac{3^3}{2^2}\left(1 - \frac{2^n}{3^n}\right)$$

$$= \frac{27}{4} - \frac{2^{n-2}}{3^{n-3}}.$$

12. Find the sum of the infinite series $6 - 3 + 1\frac{1}{2} - \cdots$

Here $a = 6,\ r = -\frac{1}{2}$.

$$s = \frac{a}{1-r} = \frac{6}{1+\frac{1}{2}} = \frac{6}{\frac{3}{2}} = \frac{2}{3} \times 6 = 4.$$

13. Find the sum of the infinite series $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots$

Here $a = \frac{1}{3},\ r = \frac{1}{2}$.

$$s = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{2}} = 2 \times \frac{1}{3} = \frac{2}{3}.$$

14. Find the sum of the infinite series $1 - \frac{2}{3} + \frac{4}{9} - \cdots$

Here $a = 1,\ r = -\frac{2}{3}$.

$$s = \frac{a}{1-r} = \frac{1}{1+\frac{2}{3}} = \frac{3}{3+2} = \frac{3}{5}.$$

15. Find the sum of the infinite series $2 + \frac{4}{5} + \frac{8}{25} + \cdots$

Here $a = 2,\ r = \frac{2}{5}$.

$$s = \frac{a}{1-r} = \frac{2}{1-\frac{2}{5}} = \frac{10}{5-2} = \frac{10}{3} = 3\frac{1}{3}.$$

16. Find the sum of the infinite series $2 - \frac{2}{5} + \frac{2}{25} - \cdots$

Here $a = 2,\ r = -\frac{1}{5}$.

$$s = \frac{a}{1-r} = \frac{2}{1+\frac{1}{5}} = \frac{10}{5+1} = \frac{10}{6} = \frac{5}{3} = 1\frac{2}{3}.$$

17. Find the sum of the infinite series $4 + 2\frac{2}{3} + 1\frac{7}{9} + \cdots$

Here $a = 4,\ r = \frac{2}{3}$.

$$s = \frac{a}{1-r} = \frac{4}{1-\frac{2}{3}} = \frac{4}{\frac{1}{3}} = 12.$$

18. Find the value of the recurring decimal $0.454545\cdots$

Here $a = 0.45,\ r = 0.01$.

$$s = \frac{a}{1-r} = \frac{0.45}{1-0.01} = \frac{0.45}{0.99} = \frac{45}{99} = \frac{5}{11}.$$

19. Find the value of the recurring decimal $0.020303\cdots$

Consider first the part that recurs, $0.000303\cdots$

Here $a = 0.0003$, $r = 0.01$.

$$s = \frac{a}{1-r} = \frac{0.0003}{1-0.01} = \frac{0.0003}{0.99} = \frac{3}{9900} = \frac{1}{3300}.$$

$$\therefore 0.020303\cdots = 0.02 + \tfrac{1}{3300} = \tfrac{1}{50} + \tfrac{1}{3300} = \tfrac{67}{3300}.$$

20. Find the value of the recurring decimal $0.7283283\cdots$

Consider first the part that recurs, $0.0283283\cdots$

Here $a = 0.0283$, $r = 0.001$.

$$s = \frac{a}{1-r} = \frac{0.0283}{1-0.001} = \frac{0.0283}{0.999} = \frac{283}{9990}.$$

$$\therefore 0.7283283\cdots = 0.7 + \tfrac{283}{9990} = \tfrac{7}{10} + \tfrac{283}{9990} = \tfrac{7276}{9990} = \tfrac{3638}{4995}.$$

21. Find the value of the recurring decimal $0.11342342\cdots$

Consider first the part that recurs, $0.00342342\cdots$

Here $a = 0.00342$, $r = 0.001$.

$$s = \frac{a}{1-r} = \frac{0.00342}{1-0.001} = \frac{0.00342}{0.999} = \frac{342}{99900} = \frac{19}{5550}.$$

$$\therefore 0.11342342\cdots = 0.11 + \tfrac{19}{5550} = \tfrac{11}{100} + \tfrac{19}{5550} = \tfrac{1259}{11100}.$$

22. Find the value of the recurring decimal $0.64389389\cdots$

Consider first the part that recurs, $0.00389389\cdots$

Here $a = 0.00389$, $r = 0.001$.

$$s = \frac{a}{1-r} = \frac{0.00389}{1-0.001} = \frac{0.00389}{0.999} = \frac{389}{99900}.$$

$$\therefore 0.64389389\cdots = 0.64 + \tfrac{389}{99900} = \tfrac{64}{100} + \tfrac{389}{99900} = \tfrac{64325}{99900} = \tfrac{2573}{3996}.$$

23. Find the value of the recurring decimal $0.55862862\cdots$

Consider first the part that recurs, $0.00862862\cdots$

Here $a = 0.00862$, $r = 0.001$.

$$s = \frac{a}{1-r} = \frac{0.00862}{1-0.001} = \frac{0.00862}{0.999} = \frac{862}{99900} = \frac{431}{49950}.$$

$$\therefore 0.55862862 = 0.55 + \tfrac{431}{49950} = \tfrac{55}{100} + \tfrac{431}{49950} = \tfrac{55807}{99900}.$$

24. Find the value of the recurring decimal 2.4336336 · · ·

Consider first the part that recurs, 0.0336336 · · ·

Here $a = 0.0336$, $r = 0.001$.

$$s = \frac{a}{1-r} = \frac{0.0336}{1-0.001} = \frac{0.0336}{0.999} = \frac{336}{9990} = \frac{56}{1665}.$$

$$\therefore\ 2.4336336\cdots = 2.4 + \tfrac{56}{1665} = 2\tfrac{2}{5} + \tfrac{56}{1665} = 2\tfrac{722}{1665}.$$

25. Find the value of the recurring decimal 3.731843184 · · ·

Consider first the part that recurs, 0.031843184 · · ·

Here $a = 0.03184$, $r = 0.0001$.

$$s = \frac{a}{1-r} = \frac{0.03184}{1-0.0001} = \frac{0.03184}{0.9999} = \frac{3184}{99990}.$$

$$\therefore\ 3.731843184\cdots = 3.7 + \tfrac{3184}{99990} = 3\tfrac{7}{10} + \tfrac{3184}{99990} = 3\tfrac{73177}{99990}.$$

26. Insert three geometrical means between 3 and 768.

Here the whole number of terms is 5, the first term is 3, and the fifth term is 768.

$$l = ar^{n-1}.$$
$$768 = 3\,r^4.$$
$$r^4 = 256.$$
$$\therefore\ r = \pm 4.$$

Hence, the series is 3, 12, 48, 192, 768, or 3, -12, 48, -192, 768.

Therefore, the means required are 12, 48, 192, or -12, 48, -192.

27. Insert four geometrical means between 2 and 6250.

Here the whole number of terms is 6, the first term is 2, and the sixth term is 6250.

$$l = ar^{n-1}.$$
$$6250 = 2\,r^5.$$
$$\therefore\ r^5 = 3125.$$
$$\therefore\ r = 5.$$

Hence, the series is 2, 10, 50, 250, 1250, 6250.

Therefore, the means required are 10, 50, 250, 1250.

28. Insert five geometrical means between 243 and $\frac{1}{3}$.

Here the whole number of terms is 7, the first term is 243, and the seventh term is $\frac{1}{3}$.

$$l = ar^{n-1}.$$
$$\tfrac{1}{3} = 243\,r^6.$$
$$\therefore\ r^6 = \tfrac{1}{729}.$$
$$\therefore\ r = \pm\tfrac{1}{3}.$$

Hence, the series is 243, 81, 27, 9, 3, 1, $\frac{1}{3}$, or 243, -81, 27, -9, 3, -1, $\frac{1}{3}$.

Therefore, the means required are 81, 27, 9, 3, 1, or -81, 27, -9, 3, -1.

29. Given $a = 8$, $r = 2$, $s = 248$; find l and n.

By Formula 2, page 344,

$$l = \frac{a + (r-1)s}{r} = \frac{8 + (2-1)248}{2} = \frac{8 + 248}{2} = 4 + 124 = 128.$$
$$l = ar^{n-1}.$$
$$128 = 8 \times 2^{n-1}.$$
$$2^{n-1} = 16.$$
$$2^{n-1} = 2^4.$$
$$\therefore\ n - 1 = 4.$$
$$\therefore\ n = 5.$$

Therefore, $l = 128$, and $n = 5$.

30. Given $a = 243$, $n = 9$, $l = \frac{1}{27}$; find r and s.

By Formula 13, page 344,

$$r = \sqrt[n-1]{\frac{l}{a}} = \sqrt[8]{\frac{1}{27} \div 243} = \sqrt[8]{\frac{1}{27} \times \frac{1}{243}} = \sqrt[8]{\frac{1}{3^3} \times \frac{1}{3^5}} = \sqrt[8]{\frac{1}{3^8}} = \frac{1}{3}.$$

By Formula 6, page 344,

$$s = \frac{rl - a}{r - 1} = \frac{a - rl}{1 - r} = \frac{243 - \frac{1}{3} \times \frac{1}{27}}{\frac{2}{3}} = \tfrac{3}{2} \times 243 - \tfrac{3}{2} \times \tfrac{1}{3} \times \tfrac{1}{27}$$
$$= 364\tfrac{1}{2} - \tfrac{1}{54} = 364\tfrac{13}{27}.$$

Therefore, $r = \frac{1}{3}$, and $s = 364\frac{13}{27}$.

31. Given $r = 1\frac{1}{2}$, $n = 8$, $s = 1050\frac{5}{6}$; find a and l.

By Formula 10, page 344,

$$a = \frac{(r-1)s}{r^n - 1} = \frac{(1\frac{1}{2} - 1)\,1050\frac{5}{6}}{(\frac{3}{2})^8 - 1} = \frac{\frac{1}{2} \times 1050\frac{5}{6}}{\frac{6561}{256} - 1} = \frac{\frac{1}{2} \times 1050\frac{5}{6}}{\frac{6305}{256}}$$
$$= \tfrac{1}{2} \times \tfrac{6305}{6} \times \tfrac{256}{6305} = \tfrac{64}{3} = 21\tfrac{1}{3}.$$

By Formula 2, page 344,

$$l=\frac{a+(r-1)s}{r}=\frac{21\frac{1}{3}+(1\frac{1}{2}-1)1050\frac{5}{6}}{1\frac{1}{2}}=\frac{21\frac{1}{3}+\frac{1}{2}\times 1050\frac{5}{6}}{\frac{3}{2}}$$
$$=\tfrac{2}{3}\times\tfrac{64}{3}+\tfrac{2}{3}\times\tfrac{1}{2}\times\tfrac{6305}{6}=\tfrac{128}{9}+\tfrac{6305}{18}=14\tfrac{2}{9}+350\tfrac{5}{18}=364\tfrac{1}{2}.$$

Therefore, $a=21\frac{1}{3}$, and $l=364\frac{1}{2}$.

32. Given $a=2$, $l=486$, $s=728$; find r and n.

By Formula 15, page 344,

$$r=\frac{s-a}{s-l}=\frac{728-2}{728-486}=\frac{726}{242}=3.$$
$$l=ar^{n-1}.$$
$$486=2\times 3^{n-1}.$$
$$3^{n-1}=243.$$
$$3^{n-1}=3^5.$$
$$\therefore\ n-1=5.$$
$$\therefore\ n=6.$$

Therefore, $r=3$, and $n=6$.

33. Given $r=6$, $n=5$, $l=1296$; find a and s.

By Formula 9, page 344,

$$a=\frac{l}{r^{n-1}}=\frac{1296}{6^4}=\frac{1296}{1296}=1.$$
$$s=\frac{a(r^n-1)}{r-1}=\frac{1\times(6^5-1)}{6-1}=\frac{7776-1}{5}=\frac{7775}{5}=1555.$$

Therefore, $a=1$, and $s=1555$.

34. If the first term of a geometrical series is 5 and the ratio 3, what term is 1215?

Here $a=5$, $r=3$, $l=1215$; to find n.

$$l=ar^{n-1}.$$
$$1215=5\times 3^{n-1}.$$
$$3^{n-1}=243.$$
$$3^{n-1}=3^5.$$
$$\therefore\ n-1=5.$$
$$\therefore\ n=6.$$

Therefore, 1215 is the sixth term.

35. The fourth term of a geometrical series is 160 and the ratio is 4. Find the sixth term and the eighth term.

The 4th term $= ar^3 = 160$.
The 6th term $= ar^5 = r^2(ar^3) = 4^2 \times 160 = 2560$.
The 8th term $= ar^7 = r^2(ar^5) = 4^2 \times 2560 = 40960$.

Therefore, the sixth term is 2560, and the eighth term 40,960.

36. Four numbers are in geometrical progression. The sum of the first and fourth is 130, and the sum of the second and third is 40. Find the numbers.

Let $x =$ the first number,
and $y =$ the ratio.
Then $x, xy, xy^2, xy^3 =$ the numbers.

$$\therefore x + xy^3 = 130, \tag{1}$$

and

$$xy + xy^2 = 40. \tag{2}$$

From (1),

$$x(1 + y^3) = 130.$$

$$x(1 + y)(1 - y + y^2) = 130. \tag{3}$$

From (2),

$$xy(1 + y) = 40.$$

$$\therefore x = \frac{40}{y(1 + y)}. \tag{4}$$

Substitute the value of x in (3),

$$\frac{40}{y(1 + y)}(1 + y)(1 - y + y^2) = 130.$$

$$\frac{40}{y}(1 - y + y^2) = 130.$$

$$\frac{4}{y}(1 - y + y^2) = 13.$$

$$4 - 4y + 4y^2 = 13y.$$

$$4y^2 - 17y + 4 = 0.$$

$$(y - 4)(4y - 1) = 0.$$

$$\therefore y = 4, \text{ or } \tfrac{1}{4}.$$

Substitute the value of y in (4), $x = \dfrac{40}{4 \times 5}$, or $\dfrac{40}{\frac{1}{4} \times \frac{5}{4}}$.

$$\therefore x = 2, \text{ or } 128.$$

Then

$$xy = 8, \text{ or } 32;$$

$$xy^2 = 32, \text{ or } 8;$$

and

$$xy^3 = 128, \text{ or } 2.$$

Therefore, the numbers required are 2, 8, 32, 128, or 128, 32, 8, 2.

37. **The sum of three numbers in arithmetical progression is 45. If 2 is added to the first number, 3 to the second, and 7 to the third, the new numbers are in geometrical progression. Find the numbers.**

Let $x - y =$ the first number,

$x =$ the second number,

and $x + y =$ the third number.

$$\therefore x - y + x + x + y = 45, \qquad (1)$$

and
$$\frac{x+3}{x-y+2} = \frac{x+y+7}{x+3}. \qquad (2)$$

From (2),
$$x^2 + 6x + 9 = x^2 - y^2 + 9x - 5y + 14.$$
$$y^2 + 5y = 3x + 5. \qquad (3)$$

From (1),
$$3x = 45.$$
$$\therefore x = 15. \qquad (4)$$

Substitute the value of x in (3),
$$y^2 + 5y = 45 + 5.$$
$$y^2 + 5y - 50 = 0.$$
$$(y - 5)(y + 10) = 0.$$
$$\therefore y = 5, \text{ or } -10.$$

Then
$$x - y = 10, \text{ or } 25;$$

and
$$x + y = 20, \text{ or } 5.$$

Therefore, the numbers required are 10, 15, 20, or 25, 15, 5.

38. A man deposited $1000 in a savings bank that pays 3 per cent interest, compounded annually. If the depositor withdrew no money from the bank for 6 years, what was the average amount he had on deposit?

Here $a = 1000$, $r = 1.03$, $n = 6$.

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{1000(1.03^6 - 1)}{1.03 - 1} = \frac{1000(1.194052296529 - 1)}{0.03}$$
$$= \frac{1000 \times 0.194052296529}{0.03} = \frac{194.052296529}{0.03} = 6468.4098843.$$

The average amount on deposit was $\frac{1}{6}$ of \$6468.41 = \$1078.07.

Therefore, the average amount on deposit was $1078.07.

39. A man deposited $1000 in a savings bank that pays 4 per cent interest, compounded semiannually. If the depositor withdrew no money from the bank for 6 years, what was the average amount he had on deposit?

4 per cent annually = 2 per cent semiannually.

Here $a = 1000,\ r = 1.02,\ n = 12.$

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{1000(1.02^{12} - 1)}{1.02 - 1}$$

$$= \frac{1000(1.268241794562545318301696 - 1)}{0.02}$$

$$= \frac{268.241794562545318301696}{0.02}$$

$$= 13412.0897281272659150848.$$

The average amount on deposit was $\frac{1}{12}$ of \$13412.09 = \$1117.67.

Therefore, the average amount on deposit was \$1117.67.

40. In using an air pump to exhaust air from a receiver, if the capacity of the piston barrel is a and that of the receiver is b, and the height of the barometer is c inches, the pressure within the receiver is reduced by one stroke of the piston to $\frac{b}{a+b}$ of c inches, by two strokes to $\left[\frac{b}{a+b}\right]^2$ of c inches, and so on. If the capacity of the receiver is 5 cubic feet, that of the piston barrel 1 cubic foot, and the original pressure of the air is 30 inches, what is the pressure in the receiver after five strokes of the piston?

The first term is c, the ratio $\frac{b}{a+b}$, and the number of terms $n + 1$; to find the last term.

Here $a = 30$ inches, $r = \frac{5}{1+5} = \frac{5}{6},\ n = 6.$

$$l = ar^{n-1} = 30 \times (\tfrac{5}{6})^5 = \tfrac{15625}{1296} = 12.056.$$

Therefore, the pressure required is 12.056 inches.

41. If the capacity of the receiver of an air pump is 8 liters, and that of the piston barrel is 1 liter, and the original pressure of the air is 76 centimeters, what is the pressure within the receiver after six strokes of the piston?

Here $a = 76$ centimeters, $r = \frac{8}{1+8} = \frac{8}{9},\ n = 7.$

$$l = ar^{n-1} = 76 \times (\tfrac{8}{9})^6 = 76 \times \tfrac{262144}{531441} = \tfrac{19922944}{531441} = 37.489.$$

Therefore, the pressure required is 37.489 centimeters.

42. If a rubber ball, falling freely upon a marble floor, rebounds to one third the height from which it has fallen, how great a distance will it traverse if tossed to a height of 50 feet?

The ball rebounds $\frac{1}{3}$ of 50 feet $= 16\frac{2}{3}$ feet, then $\frac{1}{3}$ of $16\frac{2}{3}$ feet $= (\frac{1}{3})^2$ of 50 feet, then $(\frac{1}{3})^3$ of 50 feet, and so on, — an infinite geometrical series.

After the ball rebounds a certain distance it must fall that same distance. Hence, the ball traverses a distance equal to 100 feet $+ \frac{1}{3}$ of 100 feet $+ (\frac{1}{3})^2$ of 100 feet, and so on, in an infinite geometrical series in which $a = 100$ and $r = \frac{1}{3}$.

$$s = \frac{a}{1-r} = \frac{100}{1-\frac{1}{3}} = \frac{100}{\frac{2}{3}} = \frac{3}{2} \text{ of } 100 = 150.$$

Therefore, the ball traverses 150 feet.

43. If a rubber ball rebounds to three fifths of the height from which it has fallen, how great a distance will it traverse if tossed to a height of 30 feet?

In the infinite geometrical series $a = 60$ and $r = \frac{3}{5}$ (see Example 42).

$$s = \frac{a}{1-r} = \frac{60}{1-\frac{3}{5}} = \frac{60}{\frac{2}{5}} = \frac{5}{2} \times 60 = 150.$$

Therefore, the ball traverses 150 feet.

Exercise 131. Page 348

1. Insert four harmonical means between 2 and 5.

First find four arithmetical means between $\frac{1}{2}$ and $\frac{1}{5}$.

Here $a = \frac{1}{2},\ l = \frac{1}{5},\ n = 6.$

$$l = a + (n-1)d.$$

$$\tfrac{1}{5} = \tfrac{1}{2} + 5d.$$

$$5d = -\tfrac{3}{10}.$$

$$\therefore d = -\tfrac{3}{50}.$$

Hence, the arithmetical means are $\frac{11}{25}$, $\frac{19}{50}$, $\frac{8}{25}$, $\frac{13}{50}$.

Therefore, the harmonical means required are $2\frac{3}{11}$, $2\frac{12}{19}$, $3\frac{1}{8}$, $3\frac{11}{13}$.

2. Insert seven harmonical means between $\frac{2}{3}$ and $\frac{2}{7}$.

First find seven arithmetical means between $\frac{3}{2}$ and $\frac{7}{2}$.

Here
$$a = \tfrac{3}{2},\ l = \tfrac{7}{2},\ n = 9.$$
$$l = a + (n-1)d.$$
$$\tfrac{7}{2} = \tfrac{3}{2} + 8d.$$
$$8d = 2.$$
$$\therefore\ d = \tfrac{1}{4}.$$

Hence, the arithmetical means are $\frac{7}{4}$, 2, $\frac{9}{4}$, $\frac{5}{2}$, $\frac{11}{4}$, 3, $\frac{13}{4}$.

Therefore, the harmonical means required are $\frac{4}{7}$, $\frac{1}{2}$, $\frac{4}{9}$, $\frac{2}{5}$, $\frac{4}{11}$, $\frac{1}{3}$, $\frac{4}{13}$.

3. Find the eighth term of the harmonical series 4, $4\frac{4}{5}$, 6, $\cdots$

First find the eighth term of the arithmetical series $\frac{1}{4}$, $\frac{5}{24}$, $\frac{1}{6}$, $\cdots$

Here $a = \frac{1}{4},\ d = -\frac{1}{24},\ n = 8.$
$$l = a + (n-1)d = \tfrac{1}{4} + 7\left(-\tfrac{1}{24}\right) = \tfrac{1}{4} - \tfrac{7}{24} = -\tfrac{1}{24}.$$

Hence, the eighth term of the arithmetical series is $-\frac{1}{24}$.

Therefore, the eighth term required is -24.

4. Find the sixth term of the harmonical series $9\frac{3}{5}$, $6\frac{6}{7}$, $5\frac{1}{3}$, $\cdots$

First find the sixth term of the arithmetical series $\frac{5}{48}$, $\frac{7}{48}$, $\frac{3}{16}$, $\cdots$

Here $a = \frac{5}{48},\ d = \frac{1}{24},\ n = 6.$
$$l = a + (n-1)d = \tfrac{5}{48} + 5\left(\tfrac{1}{24}\right) = \tfrac{5}{48} + \tfrac{5}{24} = \tfrac{15}{48} = \tfrac{5}{16}.$$

Hence, the sixth term of the arithmetical series is $\frac{5}{16}$.

Therefore, the sixth term required is $3\frac{1}{5}$.

5. The fourth and seventh terms of a harmonical progression are $\frac{1}{10}$ and $\frac{1}{16}$. Find the first seven terms.

First find the first seven terms of an arithmetical progression whose fourth and seventh terms are 10 and 16.

The fourth term is $a + 3d$, and the seventh term is $a + 6d$.
$$\therefore\ a + 3d = 10, \qquad (1)$$
and
$$a + 6d = 16. \qquad (2)$$
Subtract (1) from (2),
$$3d = 6.$$
$$\therefore\ d = 2.$$
Substitute the value of d in (1),
$$a + 6 = 10.$$
$$\therefore\ a = 4.$$

Hence, the first seven terms of the arithmetical series are 4, 6, 8, 10, 12, 14, 16.

Therefore, the first seven terms required are $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$, $\frac{1}{14}$, $\frac{1}{16}$.

6. The difference between two numbers is 12, and the harmonical mean between them is $6\frac{2}{5}$. Find the numbers.

Let $x =$ the larger number.

Then $x - 12 =$ the smaller number.

By § 423, $H = \dfrac{2ab}{a+b}$.

$$\therefore \frac{2x(x-12)}{x+x-12} = 6\tfrac{2}{5}.$$

$$\frac{2x(x-12)}{2x-12} = 3\tfrac{2}{5}.$$

$$\frac{x^2-12x}{x-6} = \frac{32}{5}.$$

$$5x^2 - 60x = 32x - 192.$$

$$5x^2 - 92x = -192.$$

$$25x^2 - (\ ) + 2116 = 1156.$$

$$5x - 46 = \pm 34.$$

$$5x = 80, \text{ or } 12.$$

$$\therefore x = 16, \text{ or } 2\tfrac{2}{5}.$$

$$x - 12 = 4, \text{ or } -9\tfrac{3}{5}.$$

Therefore, the numbers required are 16 and 4, or $2\frac{2}{5}$ and $-9\frac{3}{5}$.

7. The difference between the arithmetical and harmonical means between two numbers is $2\frac{1}{2}$, and one of the numbers is three times the other. Find the numbers.

Let $x =$ the one number.

Then $3x =$ the other number,

$\dfrac{x+3x}{2} = 2x =$ the arithmetical mean,

and $\dfrac{2(x)(3x)}{x+3x} = \dfrac{3x}{2} =$ the harmonical mean.

$$\therefore 2x - \frac{3x}{2} = 2\tfrac{1}{2}, \text{ or } \frac{3x}{2} - 2x = 2\tfrac{1}{2}.$$

$$4x - 3x = 5, \text{ or } 3x - 4x = 5.$$

$$\therefore x = 5, \text{ or } -5.$$

Then $3x = 15, \text{ or } -15.$

Therefore, the numbers required are 5 and 15, or -5 and -15.

8. The arithmetical mean between two numbers exceeds the geometrical mean by 68, and the geometrical mean exceeds the harmonical mean by 60. Find the numbers.

Let $x =$ one number,

and $y =$ the other number.

$$\therefore \frac{x+y}{2} - \sqrt{xy} = 68, \tag{1}$$

and

$$\sqrt{xy} - \frac{2xy}{x+y} = 60. \tag{2}$$

Add (1) and (2),

$$\frac{x+y}{2} + \frac{2xy}{x+y} = 128. \tag{3}$$

From (1),

$$\frac{x+y}{2} = 68 + \sqrt{xy}.$$

$$x + y = 136 + 2\sqrt{xy}. \tag{4}$$

Substitute the value of $x + y$ in (3),

$$68 + \sqrt{xy} - \frac{xy}{68 + \sqrt{xy}} = 128.$$

$$4624 + 136\sqrt{xy} + xy - xy = 8704 + 128\sqrt{xy}.$$

$$8\sqrt{xy} = 4080.$$

$$\sqrt{xy} = 510.$$

$$xy = 260100.$$

$$\therefore x = \frac{260100}{y}. \tag{5}$$

Substitute the value of $\sqrt{xy}$ in (4),

$$x + y = 1156.$$

$$\therefore x = 1156 - y. \tag{6}$$

Equate the values of x in (5) and (6),

$$\frac{260100}{y} = 1156 - y.$$

$$260100 = 1156y - y^2.$$

$$y^2 - 1156y = -260100.$$

$$y^2 - 1156y + 334084 = 73984.$$

$$y - 578 = \pm 272.$$

$$\therefore y = 850, \text{ or } 306.$$

Substitute the value of y in (6), $x = 306$, or 850.

Therefore, the numbers required are 306 and 850.

9. The sum of three numbers in harmonical progression is 39, and the square of the first is greater by 99 than the sum of the squares of the second and third. Find the numbers.

Let $x =$ the first number,

and $y =$ the third number.

Then $\frac{2xy}{x+y} =$ the second number.

$$\therefore\ x+y+\frac{2xy}{x+y}=39, \qquad (1)$$

$$x^2-99=\left(\frac{2xy}{x+y}\right)^2+y^2. \qquad (2)$$

Square (1), $x^2+2xy+y^2+4xy+\left(\dfrac{2xy}{x+y}\right)^2=1521.$

$$x^2+6xy+y^2+\left(\frac{2xy}{x+y}\right)^2=1521. \qquad (3)$$

From (2), $x^2-y^2-\left(\dfrac{2xy}{x+y}\right)^2=99. \qquad (4)$

Add (3) and (4), $2x^2+6xy=1620.$

$$x^2+3xy=810. \qquad (5)$$

$$3xy=810-x^2.$$

$$\therefore\ y=\frac{810-x^2}{3x}. \qquad (6)$$

Substitute the value of y in (1),

$$x+\frac{810-x^2}{3x}+\frac{2x\left(\dfrac{810-x^2}{3x}\right)}{x+\dfrac{810-x^2}{3x}}=39.$$

$$\frac{3x^2+810-x^2}{3x}+\frac{\dfrac{1620-2x^2}{3}}{\dfrac{2x^2+810}{3x}}=39.$$

$$\frac{2x^2+810}{3x}+\frac{810x-x^3}{x^2+405}=39.$$

Clear of fractions,

$$2x^4+1620x^2+328050+2430x^2-3x^4=117x^3+47385x.$$

$$x^4+117x^3-4050x^2+47385x-328050=0.$$

$$(x-18)(x^3+135x^2-1620x+18225)=0.$$

$$\therefore\ x-18=0, \text{ or } x^3+135x^2-1620x+18225=0.$$

$$\therefore\ x=18.$$

The expression $x^3+135x^2-1620x+18225$ cannot be resolved into rational factors. Hence, the three roots of the equation

$$x^3+135x^2-1620x+18225=0$$

cannot be found by methods that the pupil has already taken.

Substitute the value of x in (6), $y = \frac{810 - 324}{54} = 9.$

Then $$\frac{2xy}{x+y} = \frac{2 \times 18 \times 9}{27} = 12.$$

Therefore, the numbers required are 18, 12, and 9.

10. Show that a, b, and c are in arithmetical progression, in geometrical progression, or in harmonical progression, according as $a - b : b - c$ is equal to $a : a$, to $a : b$, or to $a : c$.

If a, b, c are in arithmetical progression,

then $$a - b = b - c.$$

$$\therefore \frac{a-b}{b-c} = \frac{a}{a}.$$

If a, b, c are in geometrical progression,

then $$\frac{a}{b} = \frac{b}{c}.$$

$$\therefore \frac{a-b}{a} = \frac{b-c}{b}. \qquad (\S\ 383)$$

$$\therefore \frac{a-b}{b-c} = \frac{a}{b}. \qquad (\S\ 385)$$

If a, b, c are in harmonical progression,

then $$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

$$\frac{a-b}{ab} = \frac{b-c}{bc}.$$

$$\frac{a-b}{a} = \frac{b-c}{c}.$$

$$\therefore \frac{a-b}{b-c} = \frac{a}{c}. \qquad (\S\ 385)$$

11. Show that if a, b, c are in harmonical progression, then $a - \frac{1}{2}b$, $\frac{1}{2}b$, $c - \frac{1}{2}b$ are in geometrical progression.

If a, b, c are in harmonical progression,

then $$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

Multiply by abc, $$ac - bc = ab - ac.$$

$$2ac = bc + ab.$$

$$4ac = 2bc + 2ab.$$

$$4ac - 2bc - 2ab = 0.$$

$$b^2 = 4ac - 2bc - 2ab + b^2.$$

$$b^2 = (2a - b)(2c - b).$$

$$\therefore\ 2a - b : b = b : 2c - b. \qquad \S\ 380$$

$$\therefore\ a - \tfrac{1}{2}b : \tfrac{1}{2}b = \tfrac{1}{2}b : c - \tfrac{1}{2}b.$$

That is, $a - \frac{1}{2}b$, $\frac{1}{2}b$, $c - \frac{1}{2}b$ are in geometrical progression.

12. Show that if a, b, c are in harmonical series, a, $a - c$, $a - b$ are in harmonical series, as are c, $c - a$, $c - b$.

If a, b, c are in harmonical series,

then $$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

$$ac - bc = ab - ac.$$

$$c(a - b) = a(b - c).$$

$$\frac{c}{a} = \frac{b - c}{a - b}.$$

$$\frac{c}{a(a - c)} = \frac{b - c}{(a - b)(a - c)}.$$

$$\frac{a - (a - c)}{a(a - c)} = \frac{a - c - a + b}{(a - b)(a - c)}.$$

$$\therefore\ \frac{1}{a - c} - \frac{1}{a} = \frac{1}{a - b} - \frac{1}{a - c}.$$

That is, a, $a - c$, $a - b$ are in harmonical series.

Again, $$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

$$ac - bc = ab - ac.$$

$$ac - ab = bc - ac.$$

$$a(c - b) = c(b - a).$$

$$\frac{a}{c} = \frac{b - a}{c - b}.$$

$$\frac{a}{c(c - a)} = \frac{b - a}{(c - b)(c - a)}.$$

$$\frac{c - c + a}{c(c - a)} = \frac{c - a - c + b}{(c - b)(c - a)}.$$

$$\therefore\ \frac{1}{c - a} - \frac{1}{c} = \frac{1}{c - b} - \frac{1}{c - a}.$$

That is, c, $c - a$, $c - b$ are in harmonical series.

Exercise 132. Page 357

1. Find the limiting value of $\dfrac{x^2 - 5x + 4}{x^2 - 6x + 5}$ as $x \doteq 1$.

When x has the value 1, the fraction has the form $\frac{0}{0}$.

Put $x = 1 + z$.

$$\frac{x^2 - 5x + 4}{x^2 - 6x + 5} = \frac{(1 + z)^2 - 5(1 + z) + 4}{(1 + z)^2 - 6(1 + z) + 5}$$

$$= \frac{1 + 2z + z^2 - 5 - 5z + 4}{1 + 2z + z^2 - 6 - 6z + 5} = \frac{z^2 - 3z}{z^2 - 4z} = \frac{z - 3}{z - 4}.$$

As z is indefinitely decreased, $\dfrac{z - 3}{z - 4}$ approaches $\dfrac{-3}{-4}$, or $\dfrac{3}{4}$, as a limit.

Therefore, $\frac{3}{4}$ is the limiting value of the fraction as $x \doteq 1$.

2. Find the limiting value of $\frac{27-x^3}{3-x}$ as $x \doteq 3$.

When x has the value 3, the fraction has the form $\frac{0}{0}$.

Put $x = 3 + z$.

$$\frac{27-x^3}{3-x} = \frac{27-(3+z)^3}{3-(3+z)} = \frac{27-27-27z-9z^2-z^3}{3-3-z}$$

$$= \frac{-27z-9z^2-z^3}{-z} = 27 + 9z + z^2.$$

As z is indefinitely decreased, $27 + 9z + z^2$ approaches 27 as a limit.

Therefore, 27 is the limiting value of the fraction as $x \doteq 3$.

3. Find the limiting value of $\frac{\sqrt{x+1}-\sqrt{2}}{x-1}$ as $x \doteq 1$.

When x has the value 1 the fraction has the form $\frac{0}{0}$.

If $1 + z$ is put for x,

then $$\frac{\sqrt{x+1}-\sqrt{2}}{x-1} = \frac{\sqrt{1+z+1}-\sqrt{2}}{1+z-1} = \frac{\sqrt{z+2}-\sqrt{2}}{z}.$$

As z is indefinitely decreased, $\frac{\sqrt{z+2}-\sqrt{2}}{z}$ approaches $\frac{0}{0}$ as a limit, and the limit is indeterminate.

Hence, to obtain a determinate value of the given fraction, it is necessary to change its form by rationalizing the numerator.

Multiply both terms of the given fraction by $\sqrt{x+1}+\sqrt{2}$,

$$\frac{\sqrt{x+1}-\sqrt{2}}{x-1} = \frac{x-1}{(x-1)(\sqrt{x+1}+\sqrt{2})} = \frac{1}{\sqrt{x+1}+\sqrt{2}}.$$

As $x \doteq 1$, $$\frac{1}{\sqrt{x+1}+\sqrt{2}} \doteq \frac{1}{\sqrt{1+1}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{1}{4}\sqrt{2}.$$

Therefore, $\frac{1}{4}\sqrt{2}$ is the limiting value of the fraction as $x \doteq 1$.

4. Find the limiting value of $\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a-x}}$ as $x \doteq a$.

When x has the value a the fraction has the form $\frac{0}{0}$.

Multiply both terms of the given fraction by $\sqrt{a}+\sqrt{x}$,

$$\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a-x}} = \frac{a-x}{\sqrt{a-x}(\sqrt{a}+\sqrt{x})} = \frac{\sqrt{a-x}}{\sqrt{a}+\sqrt{x}}.$$

As $x \doteq a$, $$\frac{\sqrt{a-x}}{\sqrt{a}+\sqrt{x}} \doteq \frac{\sqrt{a-a}}{\sqrt{a}+\sqrt{a}} = \frac{0}{2\sqrt{a}} = 0.$$

Therefore, 0 is the limiting value of the fraction as $x \doteq a$.

5. Find the limiting value of $\frac{x^2-3ax+2a^2}{x^2+4ax-5a^2}$ as $x \doteq a$.

When x has the value a the fraction has the form $\frac{0}{0}$.

Put $x = a + z$.

$$\begin{aligned}\frac{x^2-3ax+2a^2}{x^2+4ax-5a^2} &= \frac{(a+z)^2-3a(a+z)+2a^2}{(a+z)^2+4a(a+z)-5a^2}\\ &= \frac{a^2+2az+z^2-3a^2-3az+2a^2}{a^2+2az+z^2+4a^2+4az-5a^2}\\ &= \frac{z^2-az}{z^2+6az} = \frac{z-a}{z+6a}.\end{aligned}$$

As z is indefinitely decreased, $\frac{z-a}{z+6a}$ approaches $\frac{-a}{6a}$, or $-\frac{1}{6}$, as a limit.

Therefore, $-\frac{1}{6}$ is the limiting value of the fraction as $x \doteq a$.

6. Find the limiting value of $\frac{3x^3-4x^2-1}{2x^3+2x-5}$ as $x \doteq \infty$.

Divide each term of the numerator and denominator by x^3,

$$\frac{3x^3-4x^2-1}{2x^3+2x-5} = \frac{3-\frac{4}{x}-\frac{1}{x^3}}{2+\frac{2}{x^2}-\frac{5}{x^3}}.$$

As x increases indefinitely $\frac{4}{x}$, $\frac{1}{x^3}$, $\frac{2}{x^2}$, $\frac{5}{x^3}$ each approaches 0 as a limit (§ 434, IV), and the fraction approaches $\frac{3}{2}$ as a limit.

Therefore, $\frac{3}{2}$ is the limiting value of the fraction as $x \doteq \infty$.

7. Find the limiting value of $\frac{\sqrt[4]{5x-1}-\sqrt{2}}{x-1}$ as $x \doteq 1$.

When x has the value 1 the fraction has the form $\frac{0}{0}$.

Multiply both terms of the given fraction by $\sqrt[4]{5x-1}+\sqrt{2}$,

$$\frac{\sqrt[4]{5x-1}-\sqrt{2}}{x-1} = \frac{\sqrt{5x-1}-2}{(x-1)(\sqrt[4]{5x-1}+\sqrt{2})}.$$

Multiply both terms of this last fraction by $\sqrt{5x-1}+2$,

$$\frac{\sqrt[4]{5x-1}-\sqrt{2}}{x-1} = \frac{5x-1-4}{(x-1)(\sqrt[4]{5x-1}+\sqrt{2})(\sqrt{5x-1}+2)}$$

$$= \frac{5(x-1)}{(x-1)(\sqrt[4]{5x-1}+\sqrt{2})(\sqrt{5x-1}+2)}$$

$$= \frac{5}{(\sqrt[4]{5x-1}+\sqrt{2})(\sqrt{5x-1}+2)}.$$

As $x \doteq 1$, this last fraction approaches as a limit

$$\frac{5}{(\sqrt[4]{4}+\sqrt{2})(\sqrt{4}+2)} = \frac{5}{(2\sqrt{2})(4)} = \frac{5}{8\sqrt{2}} = \frac{5}{16}\sqrt{2}.$$

Therefore, $\frac{5}{16}\sqrt{2}$ is the limiting value of the fraction as $x \doteq 1$.

8. Find the limiting value of $\dfrac{y^3-2y^2-y+2}{y^2-5y+6}$ as $y \doteq 2$.

When y has the value 2, the fraction has the form $\frac{0}{0}$.

Put $y = 2 + z$.

$$\frac{y^3-2y^2-y+2}{y^2-5y+6} = \frac{(2+z)^3-2(2+z)^2-(2+z)+2}{(2+z)^2-5(2+z)+6}$$

$$= \frac{8+12z+6z^2+z^3-8-8z-2z^2-2-z+2}{4+4z+z^2-10-5z+6}$$

$$= \frac{z^3+4z^2+3z}{z^2-z} = \frac{z^2+4z+3}{z-1}.$$

As z is indefinitely decreased, $\dfrac{z^2+4z+3}{z-1}$ approaches $\dfrac{3}{-1}$, or -3, as a limit.

Therefore, -3 is the limiting value of the fraction as $y \doteq 2$.

9. Find the limiting value of $\dfrac{x^2+y^2+2xy-4}{x+y-2}$ as $x \doteq 1$ and $y \doteq 1$.

When x has the value 1 and y has the value 1, the fraction has the form $\frac{0}{0}$.

Put $x = 1 + z$, and $y = 1 + u$.

$$\frac{x^2+y^2+2xy-4}{x+y-2} = \frac{(1+z)^2+(1+u)^2+2(1+z)(1+u)-4}{(1+z)+(1+u)-2}$$

$$= \frac{1+2z+z^2+1+2u+u^2+2+2u+2z+2uz-4}{1+z+1+u-2}$$

$$= \frac{z^2 + 2uz + u^2 + 4z + 4u}{z + u}$$

$$= \frac{(z+u)^2 + 4(z+u)}{z+u} = z + u + 4.$$

As z and u are indefinitely decreased $z + u + 4$ approaches 4 as a limit.

Therefore, 4 is the limiting value of the fraction as $x \doteq 1$ and $y \doteq 1$.

10. Find the limiting value of $\frac{2xz - x^2 + y^2 - z^2}{2xy - x^2 - y^2 + z^2}$ as $z \doteq x - y$.

When z has the value $x - y$, the fraction has the form $\frac{0}{0}$.

Put $z = x - y + u$.

$$\frac{2xz - x^2 + y^2 - z^2}{2xy - x^2 - y^2 + z^2}$$

$$= \frac{2x(x - y + u) - x^2 + y^2 - (x - y + u)^2}{2xy - x^2 - y^2 + (x - y + u)^2}$$

$$= \frac{2x^2 - 2xy + 2xu - x^2 + y^2 - x^2 - y^2 - u^2 + 2xy - 2xu + 2yu}{2xy - x^2 - y^2 + x^2 + y^2 + u^2 - 2xy + 2xu - 2yu}$$

$$= \frac{2yu - u^2}{u^2 + 2xu - 2yu} = \frac{2y - u}{u + 2x - 2y}.$$

As u is indefinitely decreased $\frac{2y - u}{u + 2x - 2y}$ approaches $\frac{2y}{2x - 2y}$, or $\frac{y}{x - y}$, as a limit.

Therefore, $\frac{y}{x - y}$ is the limiting value of the fraction as $z \doteq x - y$.

11. Find the limiting value of $\frac{4x^3 + 3x^2 + 2x + 1}{5x^3 - 3x^2 + 7x + 3}$ as $x \doteq \infty$.

Divide each term of the numerator and denominator by x^3,

$$\frac{4x^3 + 3x^2 + 2x + 1}{5x^3 - 3x^2 + 7x + 3} = \frac{4 + \frac{3}{x} + \frac{2}{x^2} + \frac{1}{x^3}}{5 - \frac{3}{x} + \frac{7}{x^2} + \frac{3}{x^3}}.$$

As x increases indefinitely $\frac{3}{x}, \frac{2}{x^2}, \frac{1}{x^3}, \frac{3}{x}, \frac{7}{x^2}, \frac{3}{x^3}$ each approaches 0 as a limit (§ 434, IV), and the fraction approaches $\frac{4}{5}$ as a limit.

Therefore, $\frac{4}{5}$ is the limiting value of the fraction as $x \doteq \infty$.

Exercise 133. Page 359

1. Determine what kind of series is the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

This series is evidently an infinite geometrical progression in which the first term is 1 and the ratio is $-\frac{1}{2}$.

Let $s =$ the sum of n terms.

By § 415, $$s = \frac{a(r^n - 1)}{r - 1} = \frac{1[(-\frac{1}{2})^n - 1]}{-\frac{1}{2} - 1} = \frac{-\frac{1}{2^n} - 1}{-\frac{3}{2}} = \frac{1 + \frac{1}{2^n}}{\frac{3}{2}}$$
$$= \frac{2}{3} + \frac{2}{3 \times 2^n} = \frac{2}{3} + \frac{1}{3 \times 2^{n-1}}.$$

As n is indefinitely increased 2^{n-1} increases indefinitely, $3 \times 2^{n-1}$ increases indefinitely, and $\frac{1}{3 \times 2^{n-1}}$ decreases indefinitely.

Hence, as n is indefinitely increased s approaches $\frac{2}{3}$ as a limit.

Therefore, the series is convergent.

2. Determine what kind of series is the series $1 - 2 + 4 - 8 + 16 - \cdots$

This series is evidently an infinite geometrical series in which the first term is 1 and the ratio is -2.

Let $s =$ the sum of n terms.

By § 415, $$s = \frac{a(r^n - 1)}{r - 1} = \frac{1[(-2)^n - 1]}{-2 - 1} = \frac{-2^n - 1}{-3} = \frac{1 + 2^n}{3}.$$

As n is indefinitely increased 2^n increases indefinitely, and $\frac{1 + 2^n}{3}$ increases indefinitely.

Hence, as n is indefinitely increased s increases without limit.

Therefore, the series is divergent.

3. Determine what kind of series is the series

$$1 - 2 - 1 + 2 + 1 - 2 - 1 + 2 + \cdots$$

Let $s =$ the sum of n terms.

When $n = 1$, $s = 1$; $n = 5$, $s = 1$;
$n = 2$, $s = -1$; $n = 6$, $s = -1$;
$n = 3$, $s = -2$; $n = 7$, $s = -2$;
$n = 4$, $s = 0$; $n = 8$, $s = 0$; and so on.

Evidently, as n is increased indefinitely by one unit at a time s is repeatedly equal to $1, -1, -2, 0$.

Therefore, the series is oscillating.

4. Show that the harmonic series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$ is divergent.

The series may be written $\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+\cdots$
In the new form of writing the series

$$\frac{1}{2}=\frac{1}{2},$$
$$\frac{1}{3}+\frac{1}{4}=\frac{7}{12},$$
$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}=\frac{533}{840},$$
$$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}=\frac{96163}{180180},\text{ and so on.}$$

That is, each term is equal to $\frac{1}{2}$, or is greater than $\frac{1}{2}$.
Hence, as n is indefinitely increased s increases without limit.

Therefore, the series is divergent.

5. Show that the series $\frac{1}{1}+\frac{1}{1\times 2}+\frac{1}{1\times 2\times 3}+\frac{1}{1\times 2\times 3\times 4}+\cdots$ is convergent.

Compare the given series with the series $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$
The given series may be written $\frac{1}{1}+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\cdots$
Hence, the two first terms of the given series are identical with the two first terms of the other series, and every term of the given series after the second is evidently less than the corresponding term of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$
Now, by § 428, it was shown that the sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$ as n is indefinitely increased is 2.
Therefore, the sum of any number of terms of the given series is less than 2.

Therefore, the series is convergent.

Exercise 134. Page 363

1. Expand $\frac{1}{1+2x}$ to four terms.

Assume $$\frac{1}{1+2x}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$$

Multiply by $1+2x$,

$$1=A+(B+2A)x+(C+2B)x^2+(D+2C)x^3$$
$$+(E+2D)x^4+\cdots$$

$$\therefore\ A=1,\ B+2A=0,\ C+2B=0,\ D+2C=0,\ E+2D=0,\cdots;$$

whence $A=1,\ B=-2,\ C=4,\ D=-8,\ E=16,\cdots$

$$\therefore\ \frac{1}{1+2x}=1-2x+4x^2-8x^3+\cdots$$

2. Expand $\dfrac{1}{3-5x}$ to four terms.

Assume $$\frac{1}{3-5x}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$$

Multiply by $3-5x$,

$$1=3A+(3B-5A)x+(3C-5B)x^2+(3D-5C)x^3$$
$$+(3E-5D)x^4+\cdots$$
$$\therefore\ 3A=1,\ 3B-5A=0,\ 3C-5B=0,\ 3D-5C=0,$$
$$3E-5D=0,\cdots;$$

whence $A=\frac{1}{3},\ B=\frac{5}{9},\ C=\frac{25}{27},\ D=\frac{125}{81},\ E=\frac{625}{243},\cdots$

$$\therefore\ \frac{1}{3-5x}=\frac{1}{3}+\frac{5}{9}x+\frac{25}{27}x^2+\frac{125}{81}x^3+\cdots$$

3. Expand $\dfrac{1-x}{1+x+x^2}$ to four terms.

Assume $$\frac{1-x}{1+x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$$

Multiply by $1+x+x^2$,

$$1-x=A+(B+A)x+(C+B+A)x^2+(D+C+B)x^3$$
$$+(E+D+C)x^4+\cdots$$
$$\therefore\ A=1,\ B+A=-1,\ C+B+A=0,\ D+C+B=0,$$
$$E+D+C=0,\cdots;$$

whence $A=1,\ B=-2,\ C=1,\ D=1,\ E=-2,\cdots$

$$\therefore\ \frac{1-x}{1+x+x^2}=1-2x+x^2+x^3-\cdots$$

4. Expand $\dfrac{5-x}{1+x-x^2}$ to four terms.

Assume $$\frac{5-x}{1+x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$$

Multiply by $1+x-x^2$,

$$5-x=A+(B+A)x+(C+B-A)x^2+(D+C-B)x^3$$
$$+(E+D-C)x^4+\cdots$$
$$\therefore\ A=5,\ B+A=-1,\ C+B-A=0,\ D+C-B=0,$$
$$E+D-C=0,\cdots;$$

whence $A=5,\ B=-6,\ C=11,\ D=-17,\ E=28,\cdots$

$$\therefore\ \frac{5-x}{1+x-x^2}=5-6x+11x^2-17x^3+\cdots$$

5. Expand $\frac{3-x}{1+x+x^2}$ to four terms.

Assume $$\frac{3-x}{1+x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$$

Multiply by $1+x+x^2$,

$$3-x=A+(B+A)x+(C+B+A)x^2+(D+C+B)x^3+(E+D+C)x^4+\cdots$$

$$\therefore A=3,\ B+A=-1,\ C+B+A=0,\ D+C+B=0,\ E+D+C=0,\cdots;$$

whence $A=3,\ B=-4,\ C=1,\ D=3,\ E=-4,\cdots$

$$\therefore \frac{3-x}{1+x+x^2}=3-4x+x^2+3x^3-\cdots$$

6. Expand $\frac{4-5x}{1-2x+3x^2}$ to four terms.

Assume $$\frac{4-5x}{1-2x+3x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$$

Multiply by $1-2x+3x^2$,

$$4-5x=A+(B-2A)x+(C-2B+3A)x^2+(D-2C+3B)x^3+(E-2D+3C)x^4+\cdots$$

$$\therefore A=4,\ B-2A=-5,\ C-2B+3A=0,\ D-2C+3B=0,\ E-2D+3C=0,\cdots;$$

whence $A=4,\ B=3,\ C=-6,\ D=-21,\ E=-24,\cdots$

$$\therefore \frac{4-5x}{1-2x+3x^2}=4+3x-6x^2-21x^3-\cdots$$

7. Expand $\frac{5}{3+x}$ to five terms.

Assume $$\frac{5}{3+x}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\cdots$$

Multiply by $3+x$,

$$5=3A+(3B+A)x+(3C+B)x^2+(3D+C)x^3+(3E+D)x^4+(3F+E)x^5+\cdots$$

$$\therefore 3A=5,\ 3B+A=0,\ 3C+B=0,\ 3D+C=0,\ 3E+D=0,\ 3F+E=0,\cdots;$$

whence $A=\frac{5}{3},\ B=-\frac{5}{9},\ C=\frac{5}{27},\ D=-\frac{5}{81},\ E=\frac{5}{243},\ F=-\frac{5}{729},\cdots$

$$\therefore \frac{5}{3+x}=\frac{5}{3}-\frac{5}{9}x+\frac{5}{27}x^2-\frac{5}{81}x^3+\frac{5}{243}x^4-\cdots$$

8. Expand $\frac{3-x}{4+x}$ to five terms.

Assume $$\frac{3-x}{4+x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \cdots$$

Multiply by $4 + x$,

$$3 - x = 4A + (4B + A)x + (4C + B)x^2 + (4D + C)x^3 + (4E + D)x^4 + (4F + E)x^5 + \cdots$$

$$\therefore 4A = 3, 4B + A = -1, 4C + B = 0, 4D + C = 0, 4E + D = 0, 4F + E = 0, \cdots;$$

whence $$A = \tfrac{3}{4}, B = -\tfrac{7}{16}, C = \tfrac{7}{64}, D = -\tfrac{7}{256}, E = \tfrac{7}{1024}, F = -\tfrac{7}{4096}, \cdots$$

$$\therefore \frac{3-x}{4+x} = \frac{3}{4} - \frac{7}{16}x + \frac{7}{64}x^2 - \frac{7}{256}x^3 + \frac{7}{1024}x^4 - \cdots$$

9. Expand $\frac{7-3x}{1+5x-x^2}$ to five terms.

Assume $$\frac{7-3x}{1+5x-x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \cdots$$

Multiply by $1 + 5x - x^2$,

$$7 - 3x = A + (B + 5A)x + (C + 5B - A)x^2 + (D + 5C - B)x^3 + (E + 5D - C)x^4 + (F + 5E - D)x^5 + \cdots$$

$$\therefore A = 7, B + 5A = -3, C + 5B - A = 0, D + 5C - B = 0, E + 5D - C = 0, F + 5E - D = 0, \cdots;$$

whence $$A = 7, B = -38, C = 197, D = -1023, E = 5312, F = -27583, \cdots$$

$$\therefore \frac{7-3x}{1+5x-x^2} = 7 - 38x + 197x^2 - 1023x^3 + 5312x^4 - \cdots$$

10. Expand $\frac{x^2+x-1}{x(x+2)}$ to five terms.

$$\frac{x^2+x-1}{x(x+2)} = \frac{x^2+x-1}{x^2+2x}.$$

Assume $$\frac{x^2+x-1}{x^2+2x} = A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{E}{x^4} + \frac{F}{x^5} + \cdots$$

Multiply by $x^2 + 2x$,

$$x^2 + x - 1 = Ax^2 + (B + 2A)x + (C + 2B) + \frac{D + 2C}{x} + \frac{E + 2D}{x^2} + \frac{F + 2E}{x^3} + \cdots$$

$$\therefore\ A=1, B+2A=1, C+2B=-1, D+2C=0, E+2D=0,\quad F+2E=0, \cdots;$$

whence $A=1,\ B=-1,\ C=1,\ D=-2,\ E=4,\ F=-8, \cdots$

$$\therefore\ \frac{x^2+x-1}{x(x+2)}=1-\frac{1}{x}+\frac{1}{x^2}-\frac{2}{x^3}+\frac{4}{x^4}-\cdots$$

11. Expand $\dfrac{1-8x}{1-x-6x^2}$ to five terms.

Assume $$\frac{1-8x}{1-x-6x^2}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\cdots$$

Multiply by $1-x-6x^2$,

$$1-8x=A+(B-A)x+(C-B-6A)x^2+(D-C-6B)x^3 + (E-D-6C)x^4+(F-E-6D)x^5+\cdots$$

$$\therefore\ A=1,\ B-A=-8,\ C-B-6A=0,\ D-C-6B=0,\quad E-D-6C=0,\ F-E-6D=0, \cdots;$$

whence $A=1, B=-7, C=-1, D=-43, E=-49, F=-307, \cdots$

$$\therefore\ \frac{1-8x}{1-x-6x^2}=1-7x-x^2-43x^3-49x^4-\cdots$$

12. Expand $\dfrac{x^2+x+1}{(x+1)(x^2+1)}$ to five terms.

$$\frac{x^2+x+1}{(x+1)(x^2+1)}=\frac{x^2+x+1}{x^3+x^2+x+1}.$$

Assume $$\frac{x^2+x+1}{x^3+x^2+x+1}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}+\frac{E}{x^5}+\frac{F}{x^6}+\cdots$$

Multiply by x^3+x^2+x+1,

$$x^2+x+1=Ax^2+(B+A)x+(C+B+A)+(D+C+B+A)\frac{1}{x} + (E+D+C+B)\frac{1}{x^2}+(F+E+D+C)\frac{1}{x^3}+\cdots$$

$$\begin{aligned}\therefore\ A=1,\ B+A=1,\ C+B+A=1,\ D+C+B+A=0,\\ E+D+C+B=0,\ F+E+D+C=0,\\ G+F+E+D=0,\ H+G+F+E=0,\\ I+H+G+F=0,\ J+I+H+G=0,\\ K+J+I+H=0,\ L+K+J+I=0, \cdots;\end{aligned}$$

whence $A=1,\ B=0,\ C=0,\ D=-1,\ E=1,\ F=0,\ G=0,$
$H=-1,\ I=1,\ J=0,\ K=0,\ L=-1, \cdots$

$$\therefore\ \frac{x^2+x+1}{(x+1)(x^2+1)}=\frac{1}{x}-\frac{1}{x^4}+\frac{1}{x^5}-\frac{1}{x^8}+\frac{1}{x^9}-\cdots$$

13. Expand $\sqrt{1+2a}$ to four terms.

Assume $\qquad \sqrt{1+2a} = A + Ba + Ca^2 + Da^3 + Ea^4 + \cdots$

Square,

$$1 + 2a = A^2 + 2ABa + (2AC + B^2)a^2 + (2AD + 2BC)a^3 + (2AE + 2BD + C^2)a^4 + \cdots$$

$$\therefore A^2 = 1,\ 2AB = 2,\ 2AC + B^2 = 0,\ 2AD + 2BC = 0,\ 2AE + 2BD + C^2 = 0, \cdots;$$

whence $\qquad A = 1,\ B = 1,\ C = -\frac{1}{2},\ D = \frac{1}{2},\ E = -\frac{5}{8}, \cdots$

$\therefore \sqrt{1+2a} = 1 + a - \frac{1}{2}a^2 + \frac{1}{2}a^3 - \cdots$

14. Expand $\sqrt{1-2a}$ to four terms.

Assume $\qquad \sqrt{1-2a} = A + Ba + Ca^2 + Da^3 + Ea^4 + \cdots$

Square,

$$1 - 2a = A^2 + 2ABa + (2AC + B^2)a^2 + (2AD + 2BC)a^3 + (2AE + 2BD + C^2)a^4 + \cdots$$

$$\therefore A^2 = 1,\ 2AB = -2,\ 2AC + B^2 = 0,\ 2AD + 2BC = 0,\ 2AE + 2BD + C^2 = 0, \cdots;$$

whence $\qquad A = 1,\ B = -1,\ C = -\frac{1}{2},\ D = -\frac{1}{2},\ E = -\frac{5}{8}, \cdots$

$\therefore \sqrt{1-2a} = 1 - a - \frac{1}{2}a^2 - \frac{1}{2}a^3 - \cdots$

15. Expand $\sqrt{4-3a}$ to four terms.

Assume $\qquad \sqrt{4-3a} = A + Ba + Ca^2 + Da^3 + Ea^4 + \cdots$

Square,

$$4 - 3a = A^2 + 2ABa + (2AC + B^2)a^2 + (2AD + 2BC)a^3 + (2AE + 2BD + C^2)a^4 + \cdots$$

$$\therefore A^2 = 4,\ 2AB = -3,\ 2AC + B^2 = 0,\ 2AD + 2BC = 0,\ 2AE + 2BD + C^2 = 0, \cdots;$$

whence $\qquad A = 2,\ B = -\frac{3}{4},\ C = -\frac{9}{64},\ D = -\frac{27}{512},\ E = -\frac{405}{16384}, \cdots$

$\therefore \sqrt{4-3a} = 2 - \frac{3}{4}a - \frac{9}{64}a^2 - \frac{27}{512}a^3 - \cdots$

16. Expand $\sqrt{1+x}$ to four terms.

Assume $\qquad \sqrt{1+x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$

Square,

$$1 + x = A^2 + 2ABx + (2AC + B^2)x^2 + (2AD + 2BC)x^3 + (2AE + 2BD + C^2)x^4 + \cdots$$

$$\therefore A^2 = 1,\ 2AB = 1,\ 2AC + B^2 = 0,\ 2AD + 2BC = 0,\ 2AE + 2BD + C^2 = 0, \cdots;$$

whence $\qquad A = 1,\ B = \frac{1}{2},\ C = -\frac{1}{8},\ D = \frac{1}{16},\ E = -\frac{5}{128}, \cdots$

$\therefore \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$

17. Expand $\sqrt{1-x+x^2}$ to four terms.

Assume $\sqrt{1-x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$

Square,

$$1-x+x^2=A^2+2ABx+(2AC+B^2)x^2+(2AD+2BC)x^3$$
$$+(2AE+2BD+C^2)x^4+\cdots$$
$$\therefore\ A^2=1,\ 2AB=-1,\ 2AC+B^2=1,\ 2AD+2BC=0$$
$$2AE+2BD+C^2=0,\ \cdots;$$

whence $A=1,\ B=-\frac{1}{2},\ C=\frac{3}{8},\ D=\frac{3}{16},\ E=\frac{3}{128},\cdots$

$\therefore\ \sqrt{1-x+x^2}=1-\frac{1}{2}x+\frac{3}{8}x^2+\frac{3}{16}x^3+\cdots$

18. Expand $\sqrt[3]{1-x+x^2}$ to four terms.

Assume $\sqrt[3]{1-x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$

Raise to the third power,

$$1-x+x^2=A^3+3A^2Bx+(3A^2C+3AB^2)x^2$$
$$+(3A^2D+6ABC+B^3)x^3$$
$$+(3A^2E+3AC^2+3B^2C+6ABD)x^4+\cdots$$
$$\therefore\ A^3=1,\ 3A^2B=-1,\ 3A^2C+3AB^2=1,$$
$$3A^2D+6ABC+B^3=0,$$
$$3A^2E+3AC^2+3B^2C+6ABD=0,\ \cdots;$$

whence $A=1,\ B=-\frac{1}{3},\ C=\frac{2}{9},\ D=\frac{13}{81},\ E=\frac{8}{243},\cdots$

$\therefore\ \sqrt[3]{1-x+x^2}=1-\frac{1}{3}x+\frac{2}{9}x^2+\frac{13}{81}x^3+\cdots$

Exercise 135. Page 366

1. Resolve $\dfrac{2x+1}{x^2+x-30}$ into partial fractions.

$$\frac{2x+1}{x^2+x-30}=\frac{2x+1}{(x+6)(x-5)}.$$

Assume $$\frac{2x+1}{x^2+x-30}=\frac{A}{x+6}+\frac{B}{x-5}.$$

Clear of fractions, $2x+1=Ax-5A+Bx+6B.$

$$2x+1=(A+B)x+(-5A+6B).$$
$$\therefore\ A+B=2,\text{ and }-5A+6B=1;$$

whence $A=1$, and $B=1.$

$$\therefore\ \frac{2x+1}{x^2+x-30}=\frac{1}{x+6}+\frac{1}{x-5}.$$

2. Resolve $\frac{2x}{x^2-1}$ into partial fractions.

$$\frac{2x}{x^2-1}=\frac{2x}{(x+1)(x-1)}.$$

Assume $$\frac{2x}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1}.$$

Clear of fractions, $$2x=Ax-A+Bx+B.$$

$$2x=(A+B)x+(-A+B).$$

$$\therefore A+B=2,\text{ and }-A+B=0;$$

whence $$A=1,\text{ and }B=1.$$

$$\therefore \frac{2x}{x^2-1}=\frac{1}{x+1}+\frac{1}{x-1}.$$

3. Resolve $\frac{9ac}{a^3-27c^3}$ into partial fractions.

$$\frac{9ac}{a^3-27c^3}=\frac{9ac}{(a-3c)(a^2+3ac+9c^2)}.$$

Assume $$\frac{9ac}{a^3-27c^3}=\frac{A}{a-3c}+\frac{Ba+Cc}{a^2+3ac+9c^2}.$$

Clear of fractions, $$9ac=Aa^2+3Aac+9Ac^2+Ba^2-3Bac+Cac-3Cc^2.$$

$$9ac=(A+B)a^2+(3A-3B+C)ac+(9A-3C)c^2.$$

$$\therefore A+B=0,\ 3A-3B+C=9,\ 9A-3C=0;$$

whence $$A=1,\ B=-1,\ C=3.$$

$$\therefore \frac{9ac}{a^3-27c^3}=\frac{1}{a-3c}+\frac{-a+3c}{a^2+3ac+9c^2}$$

$$=\frac{1}{a-3c}-\frac{a-3c}{a^2+3ac+9c^2}.$$

4. Resolve $\frac{2a^3}{1+a^2+a^4}$ into partial fractions.

$$\frac{2a^3}{1+a^2+a^4}=\frac{2a^3}{(1+a+a^2)(1-a+a^2)}.$$

Assume $$\frac{2a^3}{1+a^2+a^4}=\frac{Aa+B}{1+a+a^2}+\frac{Ca+D}{1-a+a^2}.$$

Clear of fractions, $2a^3 = Aa - Aa^2 + Aa^3 + B - Ba + Ba^2 + Ca + Ca^2 + Ca^3 + D + Da + Da^2.$

$$2a^3 = (A + C)a^3 + (-A + B + C + D)a^2 + (A - B + C + D)a + (B + D).$$

$$\therefore A + C = 2, \ -A + B + C + D = 0,$$

$$A - B + C + D = 0, \ B + D = 0;$$

whence $A = 1, B = 1, C = 1, D = -1.$

$$\therefore \frac{2a^3}{1 + a^2 + a^4} = \frac{a + 1}{1 + a + a^2} + \frac{a - 1}{1 - a + a^2}.$$

5. Resolve $\dfrac{2}{x^3 + 3x^2 + 2x}$ into partial fractions.

$$\frac{2}{x^3 + 3x^2 + 2x} = \frac{2}{x(x^2 + 3x + 2)} = \frac{2}{x(x + 1)(x + 2)}.$$

Assume $\dfrac{2}{x^3 + 3x^2 + 2x} = \dfrac{A}{x} + \dfrac{B}{x + 1} + \dfrac{C}{x + 2}.$

Clear of fractions, $2 = Ax^2 + 3Ax + 2A + Bx^2 + 2Bx + Cx^2 + Cx.$

$$2 = (A + B + C)x^2 + (3A + 2B + C)x + 2A.$$

$$\therefore A + B + C = 0, \ 3A + 2B + C = 0, \ 2A = 2;$$

whence $A = 1, B = -2, C = 1.$

$$\therefore \frac{2}{x^3 + 3x^2 + 2x} = \frac{1}{x} - \frac{2}{x + 1} + \frac{1}{x + 2}.$$

6. Resolve $\dfrac{2 - d}{d(d^2 - 1)}$ into partial fractions.

$$\frac{2 - d}{d(d^2 - 1)} = \frac{2 - d}{d(d + 1)(d - 1)}.$$

Assume $\dfrac{2 - d}{d(d^2 - 1)} = \dfrac{A}{d} + \dfrac{B}{d + 1} + \dfrac{C}{d - 1}.$

Clear of fractions, $2 - d = Ad^2 - A + Bd^2 - Bd + Cd^2 + Cd.$

$$2 - d = (A + B + C)d^2 + (-B + C)d - A.$$

$$\therefore A + B + C = 0, \ -B + C = -1, \ -A = 2;$$

whence $A = -2, B = \frac{3}{2}, C = \frac{1}{2}.$

$$\therefore \frac{2 - d}{d(d^2 - 1)} = \frac{-2}{d} + \frac{\frac{3}{2}}{d + 1} + \frac{\frac{1}{2}}{d - 1}$$

$$= \frac{3}{2(d + 1)} + \frac{1}{2(d - 1)} - \frac{2}{d}.$$

7. Resolve $\frac{x^2+2x+6}{x^3-1}$ into partial fractions.

$$\frac{x^2+2x+6}{x^3-1}=\frac{x^2+2x+6}{(x-1)(x^2+x+1)}.$$

Assume $$\frac{x^2+2x+6}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}.$$

Clear of fractions,

$$x^2+2x+6=Ax^2+Ax+A+Bx^2-Bx+Cx-C.$$
$$x^2+2x+6=(A+B)x^2+(A-B+C)x+(A-C).$$
$$\therefore A+B=1,\ A-B+C=2,\ A-C=6;$$

whence $$A=3,\ B=-2,\ C=-3.$$

$$\therefore \frac{x^2+2x+6}{x^3-1}=\frac{3}{x-1}+\frac{-2x-3}{x^2+x+1}=\frac{3}{x-1}-\frac{2x+3}{x^2+x+1}.$$

8. Resolve $\frac{4x-3x^2-8}{x^3+1}$ into partial fractions.

$$\frac{4x-3x^2-8}{x^3+1}=\frac{4x-3x^2-8}{(x+1)(x^2-x+1)}.$$

Assume $$\frac{4x-3x^2-8}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}.$$

Clear of fractions,

$$4x-3x^2-8=Ax^2-Ax+A+Bx^2+Bx+Cx+C.$$
$$-3x^2+4x-8=(A+B)x^2+(-A+B+C)x+(A+C).$$
$$\therefore A+B=-3,\ -A+B+C=4,\ A+C=-8;$$

whence $$A=-5,\ B=2,\ C=-3.$$

$$\therefore \frac{4x-3x^2-8}{x^3+1}=\frac{-5}{x+1}+\frac{2x-3}{x^2-x+1}=\frac{2x-3}{x^2-x+1}-\frac{5}{x+1}.$$

9. Resolve $\frac{7x-19}{x^2-6x+9}$ into partial fractions.

$$\frac{7x-19}{x^2-6x+9}=\frac{7x-19}{(x-3)^2}.$$

Assume $$\frac{7x-19}{x^2-6x+9}=\frac{A}{x-3}+\frac{B}{(x-3)^2}.$$

Clear of fractions, $7x-19=Ax-3A+B.$

$$\therefore A=7,\text{ and } -3A+B=-19;$$

whence $$A=7,\text{ and } B=2.$$

$$\therefore \frac{7x-19}{x^2-6x+9}=\frac{7}{x-3}+\frac{2}{(x-3)^2}.$$

10. Resolve $\dfrac{7x^2-44x+32}{x^3-8x^2+16x}$ into partial fractions.

$$\frac{7x^2-44x+32}{x^3-8x^2+16x}=\frac{7x^2-44x+32}{x(x^2-8x+16)}=\frac{x^2-44x+32}{x(x-4)^2}.$$

Assume $\dfrac{7x^2-44x+32}{x^3-8x^2+16x}=\dfrac{A}{x}+\dfrac{B}{x-4}+\dfrac{C}{(x-4)^2}.$

Clear of fractions,

$$7x^2-44x+32=Ax^2-8Ax+16A+Bx^2-4Bx+Cx.$$

$$7x^2-44x+32=(A+B)x^2+(-8A-4B+C)x+16A.$$

$$\therefore\ A+B=7,\ -8A-4B+C=-44,\ 16A=32;$$

whence $A=2,\ B=5,\ C=-8.$

$$\therefore\ \frac{7x^2-44x+32}{x^3-8x^2+16x}=\frac{2}{x}+\frac{5}{x-4}-\frac{8}{(x-4)^2}.$$

11. Resolve $\dfrac{5x^3-11x^2+8x-6}{x^4-2x^3+4x-4}$ into partial fractions.

Now
$$\begin{aligned}x^4-2x^3+4x-4&=(x^4-4)-(2x^3-4x)\\&=(x^2+2)(x^2-2)-2x(x^2-2)\\&=(x^2-2x+2)(x^2-2).\end{aligned}$$

$$\therefore\ \frac{5x^3-11x^2+8x-6}{{}^4x-2x^3+4x-4}=\frac{5x^3-11x^2+8x-6}{(x^2-2x+2)(x^2-2)}.$$

Assume

$$\frac{5x^3-11x^2+8x-6}{x^4-2x^3+4x-4}=\frac{Ax+B}{x^2-2x+2}+\frac{Cx+D}{x^2-2}.$$

Clear of fractions,

$$\begin{aligned}5x^3-11x^2+8x-6&=Ax^3-2Ax+Bx^2-2B+Cx^3-2Cx^2\\&\quad+2Cx+Dx^2-2Dx+2D.\end{aligned}$$

$$\begin{aligned}5x^3-11x^2+8x-6&=(A+C)x^3+(B-2C+D)x^2\\&\quad+(-2A+2C-2D)x+(-2B+2D).\end{aligned}$$

$$\therefore\ A+C=5,\ B-2C+D=-11,$$
$$-2A+2C-2D=8,\ -2B+2D=-6;$$

whence $A=3,\ B=-2,\ C=2,\ D=-5.$

$$\therefore\ \frac{5x^3-11x^2+8x-6}{x^4-2x^3+4x-4}=\frac{3x-2}{x^2-2x+2}+\frac{2x-5}{x^2-2}.$$

12. Resolve $\frac{4x^2}{1-x^4}$ into partial fractions.

$$\frac{4x^2}{1-x^4}=\frac{4x^2}{(1-x^2)(1+x^2)}=\frac{4x^2}{(1-x)(1+x)(1+x^2)}.$$

Assume
$$\frac{4x^2}{1-x^4}=\frac{A}{1-x}+\frac{B}{1+x}+\frac{Cx+D}{1+x^2}.$$

Clear of fractions,
$$\begin{aligned}4x^2 &= A+Ax+Ax^2+Ax^3+B-Bx+Bx^2\\&\quad -Bx^3+Cx-Cx^3+D-Dx^2.\\4x^2 &= (A-B-C)x^3+(A+B-D)x^2\\&\quad +(A-B+C)x+(A+B+D).\end{aligned}$$

$$\therefore\ A-B-C=0,\ A+B-D=4,\ A-B+C=0,$$
$$A+B+D=0;$$

whence
$$A=1,\ B=1,\ C=0,\ D=-2.$$

$$\therefore\ \frac{4x^2}{1-x^4}=\frac{1}{1-x}+\frac{1}{1+x}-\frac{2}{1+x^2}.$$

13. Resolve $\frac{5x^3-13x^2-4x+35}{x^2(x^2-5x+7)}$ into partial fractions.

Assume
$$\frac{5x^3-13x^2-4x+35}{x^2(x^2-5x+7)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2-5x+7}.$$

Clear of fractions,
$$\begin{aligned}5x^3-13x^2-4x+35 &= Ax^3-5Ax^2+7Ax+Bx^2-5Bx+7B\\&\quad +Cx^3+Dx^2.\\5x^3-13x^2-4x+35 &= (A+C)x^3+(-5A+B+D)x^2\\&\quad +(7A-5B)x+7B.\end{aligned}$$

$$\therefore\ A+C=5,\ -5A+B+D=-13,$$
$$7A-5B=-4,\ 7B=35;$$

whence
$$A=3,\ B=5,\ C=2,\ D=-3.$$

$$\therefore\ \frac{5x^3-13x^2-4x+35}{x^2(x^2-5x+7)}=\frac{3}{x}+\frac{5}{x^2}+\frac{2x-3}{x^2-5x+7}.$$

14. Resolve $\frac{x^3+1}{x(x-1)^3}$ into partial fractions.

Assume
$$\frac{x^3+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}.$$

Clear of fractions,
$$\begin{aligned}x^3+1 &= Ax^3-3Ax^2+3Ax-A+Bx^3-2Bx^2\\&\quad +Bx+Cx^2-Cx+Dx.\\x^3+1 &= (A+B)x^3+(-3A-2B+C)x^2\\&\quad +(3A+B-C+D)x+(-A).\end{aligned}$$

$$\therefore A + B = 1,\ -3A - 2B + C = 0,$$
$$3A + B - C + D = 0,\ -A = 1;$$

whence $A = -1,\ B = 2,\ C = 1,\ D = 2.$

$$\therefore \frac{x^3+1}{x(x-1)^3} = \frac{-1}{x} + \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3}$$
$$= \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3} - \frac{1}{x}.$$

15. Resolve $\dfrac{3x^3 - 7x^2 - 35x + 6}{x^4 - 2x^3 - 3x^2 + 4x + 4}$ into partial fractions.

Now
$$\begin{aligned} x^4 - 2x^3 - 3x^2 + 4x + 4 &= x^2(x^2 - 2x - 3) + 4(x+1) \\ &= x^2(x+1)(x-3) + 4(x+1) \\ &= (x+1)(x^3 - 3x^2 + 4) \\ &= (x+1)(x^3 + x^2 - 4x^2 + 4) \\ &= (x+1)[x^2(x+1) - 4(x^2-1)] \\ &= (x+1)[x^2(x+1) - 4(x+1)(x-1)] \\ &= (x+1)(x+1)(x^2 - 4x + 4) \\ &= (x+1)^2(x-2)^2. \end{aligned}$$

Assume
$$\frac{3x^3 - 7x^2 - 35x + 6}{x^4 - 2x^3 - 3x^2 + 4x + 4} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}.$$

Clear of fractions,
$$\begin{aligned} 3x^3 - 7x^2 - 35x + 6 &= A(x+1)(x-2)^2 + B(x-2)^2 \\ &\quad + C(x+1)^2(x-2) + D(x+1)^2. \\ 3x^3 - 7x^2 - 35x + 6 &= Ax^3 - 3Ax^2 + 4A + Bx^2 - 4Bx + 4B \\ &\quad + Cx^3 - 3Cx - 2C + Dx^2 + 2Dx + D. \\ 3x^3 - 7x^2 - 35x + 6 &= (A + C)x^3 + (-3A + B + D)x^2 \\ &\quad + (-4B - 3C + 2D)x \\ &\quad + (4A + 4B - 2C + D). \end{aligned}$$

$$\therefore A + C = 3,\ -3A + B + D = -7,$$
$$-4B - 3C + 2D = -35,\ 4A + 4B - 2C + D = 6;$$

whence $A = \frac{26}{27},\ B = \frac{31}{9},\ C = \frac{55}{27},\ D = -\frac{68}{9}.$

$$\therefore \frac{3x^3 - 7x^2 - 35x + 6}{x^4 - 2x^3 - 3x^2 + 4x + 4}$$
$$= \frac{\frac{26}{27}}{x+1} + \frac{\frac{31}{9}}{(x+1)^2} + \frac{\frac{55}{27}}{x-2} + \frac{-\frac{68}{9}}{(x-2)^2}$$
$$= \frac{26}{27(x+1)} + \frac{31}{9(x+1)^2} + \frac{55}{27(x-2)} - \frac{68}{9(x-2)^2}.$$

16. Resolve $\frac{3x^3-21x^2-6x+51}{x^4-6x^3+3x^2+8x-6}$ into partial fractions.

Now

$$\begin{aligned} x^4-6x^3+3x^2+8x-6 &= (x-1)(x^3-5x^2-2x+6) && \S\ 147 \\ &= (x-1)(x-1)(x^2-4x-6) && \S\ 147 \\ &= (x-1)^2(x^2-4x-6). \end{aligned}$$

Assume

$$\frac{3x^3-21x^2-6x+51}{x^4-6x^3+3x^2+8x-6}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2-4x-6}.$$

Clear of fractions,

$$\begin{aligned} 3x^3-21x^2-6x+51 &= A(x-1)(x^2-4x-6)+B(x^2-4x-6) \\ &\quad +(Cx+D)(x-1)^2. \\ 3x^3-21x^2-6x+51 &= Ax^3-5Ax^2-2Ax+6A+Bx^2 \\ &\quad -4Bx-6B+Cx^3-2Cx^2+Cx \\ &\quad +Dx^2-2Dx+D. \\ 3x^3-21x^2-6x+51 &= (A+C)x^3+(-5A+B-2C+D)x^2 \\ &\quad +(-2A-4B+C-2D)x \\ &\quad +(6A-6B+D). \end{aligned}$$

$$\therefore A+C=3,\ -5A+B-2C+D=-21,$$
$$-2A-4B+C-2D=-6,\ 6A-6B+D=51;$$

whence $A=5,\ B=-3,\ C=-2,\ D=3.$

$$\begin{aligned} \therefore \frac{3x^3-21x^2-6x+51}{x^4-6x^3+3x^2+8x-6} &= \frac{5}{x-1}+\frac{-3}{(x-1)^2}+\frac{-2x+3}{x^2-4x-6} \\ &= \frac{5}{x-1}-\frac{3}{(x-1)^2}-\frac{2x-3}{x^2-4x-6}. \end{aligned}$$

Exercise 136. Page 368

1. What is the logarithm of 24 if $24=10^{1.3802}$?

If $24=10^{1.3802}$, the logarithm of 24 is 1.3802.

2. What is the logarithm of $\frac{1}{10}$?

Now $\frac{1}{10}=10^{-1}$.

Therefore, $\log\frac{1}{10}=-1$.

3. What is the logarithm of $10^8\times10^4$?

Now $10^8\times10^4=10^{12}$.

Therefore, $\log(10^8\times10^4)=12$.

4. What is the logarithm of 8 to the base 2?

Now $$8 = 2^3.$$

Therefore, $\log_2 8 = 3$.

5. May 1 be taken as the base of a system of logarithms? Why?

No. Because no number except 1 may be represented as a power of 1.

6. What is the logarithm of -27 to the base -3?

Now $$-27 = (-3)^3.$$

Therefore, $\log_{-3}(-27) = 3$.

7. May a negative number be used as the base of a complete system of logarithms? Why?

No. Because even integral powers of a negative number are positive numbers, and odd integral powers of a negative number are negative numbers, and such powers do not represent all numbers.

8. Write the common logarithm of each of the following: 1000, 0.001, 10^8, $\sqrt{10}$, $\sqrt[3]{10^2}$, $10^{\frac{2}{3}} \div 10^{\frac{3}{4}}$.

$$\begin{aligned}
1000 &= 10^3; & \therefore\ \log 1000 &= 3.\\
0.001 &= 10^{-3}; & \therefore\ \log 0.001 &= -3.\\
10^8 &= 10^8; & \therefore\ \log 10^8 &= 8.\\
\sqrt{10} &= 10^{\frac{1}{2}}; & \therefore\ \log \sqrt{10} &= \tfrac{1}{2}.\\
\sqrt[3]{10^2} &= 10^{\frac{2}{3}}; & \therefore\ \log \sqrt[3]{10^2} &= \tfrac{2}{3}.\\
10^{\frac{2}{3}} \div 10^{\frac{3}{4}} &= 10^{-\frac{1}{12}}; & \therefore\ \log (10^{\frac{2}{3}} \div 10^{\frac{3}{4}}) &= -\tfrac{1}{12}.
\end{aligned}$$

9. If the logarithm of 4.94 is 0.6937, what is the logarithm of 494? of 0.494?

$$\begin{aligned}
4.94 &= 10^{0.6937}.\\
494 &= 100 \times 4.94 = 10^2 \times 10^{0.6937} = 10^{2.6937}.\\
\therefore\ \log 494 &= 2.6937.\\
0.494 &= \tfrac{1}{10} \text{ of } 4.94 = 10^{-1} \times 10^{0.6937} = 10^{\bar{1}.6937}.\\
\therefore\ \log 0.494 &= \bar{1}.6937.
\end{aligned}$$

10. If the logarithm of 2 is 0.3010, find the logarithm of 2^6; of $\sqrt{2}$; of $\sqrt[3]{2}$.

If $\log 2 = 0.3010$, then $2 = 10^{0.3010}$.

$$\begin{aligned}
2^6 &= (10^{0.3010})^6 = 10^{1.8060}.\\
\sqrt{2} &= (10^{0.3010})^{\frac{1}{2}} = 10^{0.1505}.\\
\sqrt[3]{2} &= (10^{0.3010})^{\frac{1}{3}} = 10^{0.1003}.
\end{aligned}$$

Therefore, $\log 2^6 = 1.8060$; $\log \sqrt{2} = 0.1505$; $\log \sqrt[3]{2} = 0.1003$.

Exercise 137. Page 379

Given $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$.

Find the logarithms of the following numbers by resolving the numbers into factors and taking the sum of the logarithms of the factors:

1. $6 = 2 \times 3$.

$$\begin{aligned} \log 2 &= 0.3010\\ \log 3 &= \underline{0.4771}\\ \therefore \log 6 &= 0.7781 \end{aligned}$$

2. $15 = 3 \times 5$.

$$\begin{aligned} \log 3 &= 0.4771\\ \log 5 &= \underline{0.6990}\\ \therefore \log 15 &= 1.1761 \end{aligned}$$

3. $21 = 3 \times 7$.

$$\begin{aligned} \log 3 &= 0.4771\\ \log 7 &= \underline{0.8451}\\ \therefore \log 21 &= 1.3222 \end{aligned}$$

4. $49 = 7 \times 7$.

$$\begin{aligned} \log 7 &= 0.8451\\ \log 7 &= \underline{0.8451}\\ \therefore \log 49 &= 1.6902 \end{aligned}$$

5. $60 = 2 \times 2 \times 3 \times 5$.

$$\begin{aligned} \log 2 &= 0.3010\\ \log 2 &= 0.3010\\ \log 3 &= 0.4771\\ \log 5 &= \underline{0.6990}\\ \therefore \log 60 &= 1.7781 \end{aligned}$$

6. $7.5 = 3 \times 5 \times 5 \times 0.1$.

$$\begin{aligned} \log 3 &= 0.4771\\ \log 5 &= 0.6990\\ \log 5 &= 0.6990\\ \log 0.1 &= \underline{9.0000 - 10}\\ \therefore \log 7.5 &= 0.8751 \end{aligned}$$

7. $0.021 = 3 \times 7 \times 0.001$.

$$\begin{aligned} \log 3 &= 0.4771\\ \log 7 &= 0.8451\\ \log 0.001 &= \underline{7.0000 - 10}\\ \therefore \log 0.021 &= 8.3222 - 10 \end{aligned}$$

8. $0.35 = 5 \times 7 \times 0.01$.

$$\begin{aligned} \log 5 &= 0.6990\\ \log 7 &= 0.8451\\ \log 0.01 &= \underline{8.0000 - 10}\\ \therefore \log 0.35 &= 9.5441 - 10 \end{aligned}$$

9. $0.0035 = 5 \times 7 \times 0.0001$.

$$\begin{aligned} \log 5 &= 0.6990\\ \log 7 &= 0.8451\\ \log 0.0001 &= \underline{6.0000 - 10}\\ \therefore \log 0.0035 &= 7.5441 - 10 \end{aligned}$$

10. $12.5 = 5^3 \times 0.1$.

$$\begin{aligned} \log 5^3 &= 2.0970\\ \log 0.1 &= \underline{9.0000 - 10}\\ \therefore \log 12.5 &= 1.0970 \end{aligned}$$

11. $420 = 2^2 \times 3 \times 5 \times 7$.

$$\begin{aligned} \log 2^2 &= 0.6020\\ \log 3 &= 0.4771\\ \log 5 &= 0.6990\\ \log 7 &= \underline{0.8451}\\ \therefore \log 420 &= 2.6232 \end{aligned}$$

12. $0.004 = 2^2 \times 0.001.$

$$\begin{aligned} \log 2^2 &= 0.6020 \\ \log 0.001 &= \underline{7.0000 - 10} \\ \therefore \log 0.004 &= 7.6020 - 10 \end{aligned}$$

13. $0.63 = 3^2 \times 7 \times 0.01.$

$$\begin{aligned} \log 3^2 &= 0.9542 \\ \log 7 &= 0.8451 \\ \log 0.01 &= \underline{8.0000 - 10} \\ \therefore \log 0.63 &= 9.7993 - 10 \end{aligned}$$

14. $105 = 3 \times 5 \times 7.$

$$\begin{aligned} \log 3 &= 0.4771 \\ \log 5 &= 0.6990 \\ \log 7 &= \underline{0.8451} \\ \therefore \log 105 &= 2.0212 \end{aligned}$$

15. $0.0105 = 3 \times 5 \times 7 \times 0.0001.$

$$\begin{aligned} \log 3 &= 0.4771 \\ \log 5 &= 0.6990 \\ \log 7 &= 0.8451 \\ \log 0.0001 &= \underline{6.0000 - 10} \\ \therefore \log 0.0105 &= 8.0212 - 10 \end{aligned}$$

16. $0.05 = 5 \times 0.01.$

$$\begin{aligned} \log 5 &= 0.6990 \\ \log 0.01 &= \underline{8.0000 - 10} \\ \therefore \log 0.05 &= 8.6990 - 10 \end{aligned}$$

17. $1.05 = 3 \times 5 \times 7 \times 0.01.$

$$\begin{aligned} \log 3 &= 0.4771 \\ \log 5 &= 0.6990 \\ \log 7 &= 0.8451 \\ \log 0.01 &= \underline{8.0000 - 10} \\ \therefore \log 1.05 &= 0.0212 \end{aligned}$$

18. $0.056 = 2^3 \times 7 \times 0.001.$

$$\begin{aligned} \log 2^3 &= 0.9030 \\ \log 7 &= 0.8451 \\ \log 0.001 &= \underline{7.0000 - 10} \\ \therefore \log 0.056 &= 8.7481 - 10 \end{aligned}$$

19. $8 \times 9 = 2^3 \times 3^2.$

$$\begin{aligned} \log 2^3 &= 0.9030 \\ \log 3^2 &= \underline{0.9542} \\ \therefore \log (8 \times 9) &= 1.8572 \end{aligned}$$

20. $25 \times 32 = 5^2 \times 2^5.$

$$\begin{aligned} \log 5^2 &= 1.3980 \\ \log 2^5 &= \underline{1.5050} \\ \therefore \log (25 \times 32) &= 2.9030 \end{aligned}$$

21. Find log 5, knowing that log 2 = 0.3010 and log 10 = 1.

$$5 = \tfrac{10}{2}.$$

$$\therefore \log 5 = \log 10 - \log 2.$$

$$\begin{aligned} \log 10 &= 1.0000 \\ \log 2 &= \underline{0.3010} \\ \therefore \log 5 &= 0.6990 \end{aligned}$$

22. Find the logarithm of $\frac{2}{5}$.

$$\begin{aligned} \log 2 &= 0.3010 \\ \log 5 &= \underline{0.6990} \\ \therefore \log \tfrac{2}{5} &= \bar{1}.6020 \end{aligned}$$

23. Find the logarithm of $\frac{3}{7}$.

$$\begin{aligned} \log 3 &= 0.4771 \\ \log 7 &= \underline{0.8451} \\ \therefore \log \tfrac{3}{7} &= \bar{1}.6320 \end{aligned}$$

24. Find the logarithm of $\frac{7}{5}$.

$$\begin{aligned} \log 7 &= 0.8451 \\ \log 5 &= \underline{0.6990} \\ \therefore \log \tfrac{7}{5} &= 0.1461 \end{aligned}$$

25. Find the logarithm of $\frac{3}{5}$.

$$\begin{aligned} \log 3 &= 0.4771 \\ \log 5 &= \underline{0.6990} \\ \therefore \log \tfrac{3}{5} &= \bar{1}.7781 \end{aligned}$$

26. Find the logarithm of $\frac{7}{0.5}$.

$$\log 7 = 0.8451$$
$$\log 0.5 = \underline{\bar{1}.6990}$$
$$\therefore \log \frac{7}{0.5} = 1.1461$$

27. Find the logarithm of $\frac{0.05}{3}$.

$$\log 0.05 = \bar{2}.6990$$
$$\log 3 = \underline{0.4771}$$
$$\therefore \log \frac{0.05}{3} = \bar{2}.2219$$

28. Find the logarithm of $\frac{0.0007}{0.2}$.

$$\log 0.0007 = \bar{4}.8451$$
$$\log 0.2 = \underline{\bar{1}.3010}$$
$$\therefore \log \frac{0.0007}{0.2} = \bar{3}.5441$$

29. Find the logarithm of $\frac{0.003}{7}$.

$$\log 0.003 = \bar{3}.4771$$
$$\log 7 = \underline{0.8451}$$
$$\therefore \log \frac{0.003}{7} = \bar{4}.6320$$

30. Find the logarithm of $\frac{420}{125}$.

$$420 = 2^2 \times 3 \times 5 \times 7;\ 125 = 5^3.$$

$$\log 2^2 = 0.6020$$
$$\log 3 = 0.4771$$
$$\log 5 = 0.6990$$
$$\log 7 = \underline{0.8451}$$
$$\therefore \log 420 = 2.6232$$
$$\log 5^3 = \underline{2.0970}$$
$$\therefore \log \frac{420}{125} = 0.5262$$

31. Find the logarithm of $\frac{0.02}{0.005}$.

$$\log 0.02 = 8.3010 - 10$$
$$\log 0.005 = \underline{7.6990 - 10}$$
$$\therefore \log \frac{0.02}{0.005} = 0.6020$$

32. Find the logarithm of $\frac{27}{10}$.

$$27 = 3^3.$$

$$\log 3^3 = 1.4313$$
$$\log 10 = \underline{1.0000}$$
$$\therefore \log \frac{27}{10} = 0.4313$$

33. Find the logarithm of $\frac{100}{0.5}$.

$$\log 100 = 2.0000$$
$$\log 0.5 = \underline{\bar{1}.6990}$$
$$\therefore \log \frac{100}{0.5} = 2.3010$$

34. Find the logarithm of $\frac{84}{25}$.

$$84 = 2^2 \times 3 \times 7;\ 25 = 5^2.$$

$$\log 2^2 = 0.6020$$
$$\log 3 = 0.4771$$
$$\log 7 = \underline{0.8451}$$
$$\therefore \log 84 = 1.9242$$
$$\log 5^2 = \underline{1.3980}$$
$$\therefore \log \frac{84}{25} = 0.5262$$

35. Find the logarithm of $\frac{0.05}{0.7}$.

$$\log 0.05 = \bar{2}.6990$$
$$\log 0.7 = \underline{\bar{1}.8451}$$
$$\therefore \log \frac{0.05}{0.7} = \bar{2}.8539$$

36. Find the logarithm of $\frac{6\times 7}{5\times 5}$.

$6 \times 7 = 2 \times 3 \times 7$; $5 \times 5 = 5^2$.

$$\begin{aligned} \log 2 &= 0.3010 \\ \log 3 &= 0.4771 \\ \log 7 &= \underline{0.8451} \\ \therefore \log (6 \times 7) &= 1.6232 \\ \log 5^2 &= \underline{1.3980} \\ \therefore \log \frac{6 \times 7}{5 \times 5} &= 0.2252 \end{aligned}$$

37. Find the logarithm of $\frac{7\times 9}{8\times 5}$.

$7 \times 9 = 7 \times 3^2$; $8 \times 5 = 2^3 \times 5$.

$$\begin{aligned} \log 7 &= 0.8451 \\ \log 3^2 &= \underline{0.9542} \\ \therefore \log (7 \times 9) &= 1.7993 \end{aligned}$$

$$\begin{aligned} \log 2^3 &= 0.9030 \\ \log 5 &= \underline{0.6990} \\ \therefore \log (8 \times 5) &= 1.6020 \end{aligned}$$

$$\begin{aligned} \log (7 \times 9) &= 1.7993 \\ \log (8 \times 5) &= \underline{1.6020} \\ \therefore \log \frac{7 \times 9}{8 \times 5} &= 0.1973 \end{aligned}$$

38. Find the logarithm of

$$\frac{0.7 \times 0.9}{0.8 \times 0.5}.$$

$0.7 \times 0.9 = 7 \times 3^2 \times 0.01$;
$0.8 \times 0.5 = 2^3 \times 5 \times 0.01$.

$$\begin{aligned} \log 7 &= 0.8451 \\ \log 3^2 &= 0.9542 \\ \log 0.01 &= \underline{8.0000 - 10} \\ \therefore \log (0.7 \times 0.9) &= 9.7993 - 10 \end{aligned}$$

$$\begin{aligned} \log 2^3 &= 0.9030 \\ \log 5 &= 0.6990 \\ \log 0.01 &= \underline{8.0000 - 10} \\ \therefore \log (0.8 \times 0.5) &= 9.6020 - 10 \end{aligned}$$

$$\begin{aligned} \log (0.7 \times 0.9) &= 9.7993 - 10 \\ \log (0.8 \times 0.5) &= \underline{9.6020 - 10} \\ \therefore \log \frac{0.7 \times 0.9}{0.8 \times 0.5} &= 0.1973 \end{aligned}$$

39. Find the logarithm of

$$\frac{0.07 \times 1.5}{0.002}.$$

$0.07 \times 1.5 = 7 \times 3 \times 5 \times 0.001$.

$$\begin{aligned} \log 7 &= 0.8451 \\ \log 3 &= 0.4771 \\ \log 5 &= 0.6990 \\ \log 0.001 &= \underline{7.0000 - 10} \\ \therefore \log (0.07 \times 1.5) &= 9.0212 - 10 \\ \log 0.002 &= \underline{7.3010 - 10} \\ \therefore \log \frac{0.07 \times 1.5}{0.002} &= 1.7202 \end{aligned}$$

40. Find the logarithm of

$$\frac{0.1 \times 0.7 \times 0.2}{5 \times 0.03}.$$

$$\begin{aligned} \log 0.1 &= 9.0000 - 10 \\ \log 0.7 &= 9.8451 - 10 \\ \log 0.2 &= \underline{9.3010 - 10} \\ \therefore \log (0.1 \times 0.7 \times 0.2) &= 8.1461 - 10 \end{aligned}$$

$$\begin{aligned} \log 5 &= 0.6990 \\ \log 0.03 &= \underline{8.4771 - 10} \\ \therefore \log (5 \times 0.03) &= 9.1761 - 10 \end{aligned}$$

$$\begin{aligned} \log (0.1 \times 0.7 \times 0.2) &= 8.1461 - 10 \\ \log (5 \times 0.03) &= \underline{9.1761 - 10} \\ \therefore \log \frac{0.1 \times 0.7 \times 0.2}{5 \times 0.03} &= 8.9700 - 10 \end{aligned}$$

41. Find the logarithm of

$$\frac{0.02 \times 0.005}{0.07 \times 0.03}.$$

$$\begin{aligned} \log 0.02 &= 8.3010-10 \\ \log 0.005 &= \underline{7.6990-10} \\ \therefore \log (0.02 \times 0.005) &= 6.0000-10 \end{aligned}$$

$$\begin{aligned} \log 0.07 &= 8.8451-10 \\ \log 0.03 &= \underline{8.4771-10} \\ \therefore \log (0.07 \times 0.03) &= 7.3222-10 \end{aligned}$$

$$\begin{aligned} \log (0.02 \times 0.005) &= 6.0000-10 \\ \log (0.07 \times 0.03) &= \underline{7.3222-10} \\ \therefore \log \frac{0.02 \times 0.005}{0.07 \times 0.03} &= 8.6778-10 \end{aligned}$$

42. Find the logarithm of 2^3.

$$\begin{aligned} \log 2 &= 0.3010 \\ &\underline{3} \\ \therefore \log 2^3 &= 0.9030 \end{aligned}$$

43. Find the logarithm of 7^4.

$$\begin{aligned} \log 7 &= 0.8451 \\ &\underline{4} \\ \therefore \log 7^4 &= 3.3804 \end{aligned}$$

44. Find the logarithm of 5^5.

$$\begin{aligned} \log 5 &= 0.6990 \\ &\underline{5} \\ \therefore \log 5^5 &= 3.4950 \end{aligned}$$

45. Find the logarithm of $2^{\frac{1}{4}}$.

$\log 2 = 0.3010.$

$\log 2^{\frac{1}{4}} = \frac{1}{4}$ of $0.3010 = 0.0753.$

46. Find the logarithm of $3^{\frac{1}{3}}$.

$\log 3 = 0.4771.$

$\log 3^{\frac{1}{3}} = \frac{1}{3}$ of $0.4771 = 0.1590.$

47. Find the logarithm of $\sqrt{7}$.

$\sqrt{7} = 7^{\frac{1}{2}}.$

$\log 7 = 0.8451.$

$\log \sqrt{7} = \frac{1}{2}$ of $0.8451 = 0.4226.$

48. Find the logarithm of $3^{\frac{2}{5}}$.

$$\begin{aligned} \log 3 &= 0.4771 \\ &\underline{2} \\ 5&\overline{)0.9542} \\ \therefore \log 3^{\frac{2}{5}} &= 0.1908 \end{aligned}$$

49. Find the logarithm of $3^{\frac{9}{11}}$.

$$\begin{aligned} \log 3 &= 0.4771 \\ &\underline{9} \\ 11&\overline{)4.2939} \\ \therefore \log 3^{\frac{9}{11}} &= 0.3904 \end{aligned}$$

50. Find the logarithm of $\sqrt[3]{35}$.

$\sqrt[3]{35} = 35^{\frac{1}{3}} = (5 \times 7)^{\frac{1}{3}}.$

$$\begin{aligned} \log 5 &= 0.6990 \\ \log 7 &= \underline{0.8451} \\ 3&\overline{)1.5441} \\ \therefore \log \sqrt[3]{35} &= 0.5147 \end{aligned}$$

51. Find the logarithm of $\sqrt{4.2}$.

$\sqrt{4.2} = 4.2^{\frac{1}{2}} = (2 \times 3 \times 7 \times 0.1)^{\frac{1}{2}}.$

$$\begin{aligned} \log 2 &= 0.3010 \\ \log 3 &= 0.4771 \\ \log 7 &= 0.8451 \\ \log 0.1 &= \underline{9.0000 - 10} \\ 2&\overline{)0.6232} \\ \therefore \log \sqrt{4.2} &= 0.3116 \end{aligned}$$

52. Find the logarithm of $2.1^{\frac{1}{3}}$

$2.1^{\frac{1}{3}} = (3 \times 7 \times 0.1)^{\frac{1}{3}}.$

$$\begin{aligned} \log 3 &= 0.4771 \\ \log 7 &= 0.8451 \\ \log 0.1 &= \underline{9.0000 - 10} \\ 3&\overline{)0.3222} \\ \therefore \log 2.1^{\frac{1}{3}} &= 0.1074 \end{aligned}$$

53. Find the logarithm of $7^{\frac{7}{11}}$.

$$\begin{array}{r} \log 7 = 0.8451 \\ 7 \\ 11\,\overline{)5.9157} \\ \therefore\ \log 7^{\frac{7}{11}} = 0.5378 \end{array}$$

54. Find the logarithm of 10.5^3.

$$10.5^3 = (3 \times 5 \times 7 \times 0.1)^3.$$

$$\begin{array}{r} \log 3 = 0.4771 \\ \log 5 = 0.6990 \\ \log 7 = 0.8451 \\ \log 0.1 = \underline{9.0000 - 10} \\ \therefore\ \log 10.5 = 1.0212 \\ \underline{3} \\ \therefore\ \log 10.5^3 = 3.0636 \end{array}$$

55. Find the logarithm of $1.2^{\frac{1}{2}}$.

$$1.2^{\frac{1}{2}} = (2^2 \times 3 \times 0.1)^{\frac{1}{2}}.$$

$$\begin{array}{r} \log 2^2 = 0.6020 \\ \log 3 = 0.4771 \\ \log 0.1 = \underline{9.0000 - 10} \\ 2\,\overline{)0.0791} \\ \therefore\ \log 1.2^{\frac{1}{2}} = 0.0396 \end{array}$$

56. Find the logarithm of $6^3 \times 5^2$.

$$6^3 \times 5^2 = 2^3 \times 3^3 \times 5^2.$$

$$\begin{array}{r} \log 2^3 = 0.9030 \\ \log 3^3 = 1.4313 \\ \log 5^2 = \underline{1.3980} \\ \therefore\ \log (6^3 \times 5^2) = 3.7323 \end{array}$$

57. Find the logarithm of $7^{\frac{3}{5}} \times 3^{\frac{2}{9}}$.

$$\begin{array}{r} \log 7 = 0.8451 \\ 3 \\ 5\,\overline{)2.5353} \\ \therefore\ \log 7^{\frac{3}{5}} = 0.5071 \\ \log 3 = 0.4771 \\ 2 \\ 9\,\overline{)0.9542} \\ \therefore\ \log 3^{\frac{2}{9}} = 0.1060 \\ \log 7^{\frac{3}{5}} = \underline{0.5071} \\ \therefore\ \log (7^{\frac{3}{5}} \times 3^{\frac{2}{9}}) = 0.6131 \end{array}$$

58. Find the logarithm of $1.8^{\frac{1}{3}} \times 2.7^3$.

$$1.8^{\frac{1}{3}} = (2 \times 3^2 \times 0.1)^{\frac{1}{3}};$$
$$2.7^3 = (3^3 \times 0.1)^3.$$

$$\begin{array}{r} \log 2 = 0.3010 \\ \log 3^2 = 0.9542 \\ \log 0.1 = \underline{9.0000 - 10} \\ 3\,\overline{)0.2552} \\ \therefore\ \log 1.8^{\frac{1}{3}} = 0.0851 \\ \log 3^3 = 1.4313 \\ \log 0.1 = \underline{9.0000 - 10} \\ 0.4313 \\ \underline{3} \\ \therefore\ \log 2.7^3 = 1.2939 \\ \log 1.8^{\frac{1}{3}} = 0.0851 \\ \log 2.7^3 = \underline{1.2939} \\ \therefore \log(1.8^{\frac{1}{3}} \times 2.7^3) = 1.3790 \end{array}$$

59. Find the logarithm of $3^{\frac{1}{2}} \times 4^{\frac{1}{3}} \times 5^{\frac{1}{4}}$.

$$4^{\frac{1}{3}} = (2^2)^{\frac{1}{3}} = 2^{\frac{2}{3}}.$$

$$\begin{array}{r} \log 3^{\frac{1}{2}} = \frac{1}{2} \text{ of } 0.4771 = 0.2386 \\ \log 2^{\frac{2}{3}} = \frac{2}{3} \text{ of } 0.3010 = 0.2007 \\ \log 5^{\frac{1}{4}} = \frac{1}{4} \text{ of } 0.6990 = \underline{0.1747} \\ \therefore\ \log (3^{\frac{1}{2}} \times 4^{\frac{1}{3}} \times 5^{\frac{1}{4}}) = 0.6140 \end{array}$$

60. Find the logarithm of $0.02^{\frac{1}{3}}$.

$$\begin{array}{r} \log 0.02 = 8.3010 - 10 \\ 20. \qquad - 20 \\ 3\,\overline{)28.3010 - 30} \\ \therefore\ \log 0.02^{\frac{1}{3}} = 9.4337 - 10 \end{array}$$

61. Find the logarithm of $1.05^{\frac{2}{5}} \times 0.03^{\frac{1}{2}} \times 0.02^{3}$.

$$1.05^{\frac{2}{5}} = (3 \times 5 \times 7 \times 0.01)^{\frac{2}{5}}.$$

$$\begin{aligned}
\log 3 &= 0.4771\\
\log 5 &= 0.6990\\
\log 7 &= 0.8451\\
\log 0.01 &= \underline{8.0000 - 10}\\
\log 1.05 &= 0.0212\\
&\underline{2}\\
5\,&\overline{)0.0424}\\
\therefore \log 1.05^{\frac{2}{5}} &= 0.0085
\end{aligned}$$

$$\begin{aligned}
\log 0.03 &= 8.4771 - 10\\
&\ \,10. \qquad - 10\\
2\,&\overline{)18.4771 - 20}\\
\therefore \log 0.03^{\frac{1}{2}} &= 9.2386 - 10
\end{aligned}$$

$$\begin{aligned}
\log 0.02 &= 8.3010 - 10\\
&\underline{3}\\
\therefore \log 0.02^{3} &= 4.9030 - 10
\end{aligned}$$

$$\begin{aligned}
\log 1.05^{\frac{2}{5}} &= 0.0085\\
\log 0.03^{\frac{1}{2}} &= 9.2386 - 10\\
\log 0.02^{3} &= \underline{4.9030 - 10}\\
\therefore \log (1.05^{\frac{2}{5}} \times 0.03^{\frac{1}{2}} \times 0.02^{3}) &= 4.1501 - 10
\end{aligned}$$

Find from the table the common logarithm of:

62. $\log 57 = 1.7559$.

63. $\log 109 = 2.0374$.

64. $\log 857 = 2.9330$.

65. $\log 9901 = 3.9957$.

66. $\log 5406 = 3.7329$.

67. $\log 4552 = 3.6582$.

68. $\log 5433 = 3.7350$.

69. $\log 90871 = 4.9585$.

70. $\log 10007 = 4.0003$.

71. $\log 10070 = 4.0030$.

72. $\log 56.27 = 1.7503$.

73. $\log 12.16 = 1.0850$.

74. $\log 0.8770 = 9.9430 - 10$.

75. $\log 0.0567 = 8.7536 - 10$.

76. $\log 1.006 = 0.0026$.

77. $\log 8.778 = 0.9434$.

78. $\log 73.896 = 1.8686$.

79. $\log 0.07069 = 8.8493 - 10$.

80. $\log 0.008974 = 7.9530 - 10$.

81. $\log 3.1416 = 0.4971$.

Find the antilogarithms of the following common logarithms:

82. $2.8142 = \log 652$.

83. $3.6064 = \log 4040$.

84. $5.2695 = \log 186{,}000$.

85. $1.7750 = \log 59.57$.

86. $3.8941 = \log 7836$.

87. $2.1562 = \log 143.3$.

88. $1.2982 = \log 19.87$.

89. $\bar{1}.7903 = \log 0.617$.

90. $\bar{3}.6233 = \log 0.004201$.

91. $\bar{4}.1547 = \log 0.0001428$.

92. $0.4382 = \log 2.743$.

93. $4.2488 - 10$
$= \log 0.000001773$.

94. $8.6330 - 10 = \log 0.04295$.

95. $2.5310 - 10$
$= \log 0.00000003396$.

96. $9.9486 - 10 = \log 0.8884$.

97. $1.9730 = \log 93.98$.

98. $9.8800 - 10 = \log 0.7586$.

99. $0.2788 = \log 1.9$.

100. $8.0060 - 10 = \log 0.01014$.

101. $7.0216 - 10 = \log 0.001051$.

102. $8.6582 - 10 = \log 0.04552$.

103. Find the cologarithm corresponding to the logarithm 0.6990.
The required cologarithm is 9.3010 – 10.

104. Find the cologarithm corresponding to the logarithm 2.4843.
The required cologarithm is 7.5157 – 10.

105. Find the cologarithm corresponding to the logarithm $\bar{1}.9274$.
The required cologarithm is 0.0726.

106. Find the cologarithm corresponding to the logarithm 0.3010.
The required cologarithm is 9.6990 – 10.

107. Find the cologarithm corresponding to the logarithm 9.7404 – 10.
The required cologarithm is 0.2596.

108. Find the cologarithm corresponding to the logarithm 2.8779 – 10.
The required cologarithm is 7.1221.

109. Find the cologarithm corresponding to the logarithm 9.6542 – 10.
The required cologarithm is 0.3458.

110. Find the cologarithm corresponding to the logarithm 7.1673 – 10.
The required cologarithm is 2.8327.

111. Find the cologarithm corresponding to the logarithm 3.7559.
The required cologarithm is 6.2441 – 10.

112. Find the cologarithm corresponding to the logarithm 11.4263.
The required cologarithm is 8.5737 – 20.

113. Find the cologarithm corresponding to the logarithm 5.0414 – 10.
The required cologarithm is 4.9586.

114. Find the cologarithm corresponding to the logarithm 3.8260 – 10.
The required cologarithm is 6.1740.

115. Find the cologarithm of 363.

$$\log 363 = 2.5599.$$
$$\therefore \text{colog } 363 = 7.4401 - 10.$$

116. Find the cologarithm of 792.

$$\log 792 = 2.8987.$$
$$\therefore \text{colog } 792 = 7.1013 - 10.$$

117. Find the cologarithm of 8652.

$$\log 8652 = 3.9371.$$
$$\therefore \text{colog } 8652 = 6.0629 - 10.$$

118. Find the cologarithm of 178,000.

$$\log 178000 = 5.2504.$$
$$\therefore \text{colog } 178000 = 4.7496 - 10.$$

119. Find the cologarithm of 4820.

$$\log 4820 = 3.6830.$$
$$\therefore \text{colog } 4820 = 6.3170 - 10.$$

120. Find the cologarithm of 8.104.

$$\log 8.104 = 0.9087.$$
$$\therefore \text{colog } 8.104 = 9.0113 - 10.$$

121. Find the cologarithm of 14.26.

$$\log 14.26 = 1.1541.$$
$$\therefore \text{colog } 14.26 = 8.8459 - 10.$$

122. Find the cologarithm of 0.0243.

$$\log 0.0243 = 8.3856 - 10.$$
$$\therefore \text{colog } 0.0243 = 1.6144.$$

123. Find the cologarithm of 43.256.

$$\log 43.256 = 1.6361.$$
$$\therefore \text{colog } 43.256 = 8.3639 - 10.$$

124. Find the cologarithm of 15.7643.

$$\log 15.7643 = 1.1977.$$
$$\therefore \text{colog } 15.7643 = 8.8023 - 10.$$

125. Find the cologarithm of 0.02634.

$$\log 0.02634 = 8.4206 - 10.$$
$$\therefore \text{colog } 0.02634 = 1.5794.$$

126. Find the cologarithm of 2,716,000.

$$\log 2716000 = 6.4340.$$
$$\therefore \text{colog } 2716000 = 3.5660 - 10.$$

127. Find the cologarithm of 0.0003827.

$$\log 0.0003827 = 6.5829 - 10.$$
$$\therefore \text{colog } 0.0003827 = 3.4171.$$

128. Find the cologarithm of 0.045826.

$$\log 0.045826 = 8.6611 - 10.$$
$$\therefore \text{colog } 0.045826 = 1.3389.$$

129. Find the cologarithm of 0.002835.

$$\log 0.002835 = 7.4526 - 10.$$
$$\therefore \text{colog } 0.002835 = 2.5474.$$

Exercise 138. Page 383

Find by logarithms the value of :

1. 562.3×0.7854.

$$\begin{aligned} \log 562.3 &= 2.7499 \\ \log 0.7854 &= \underline{9.8951 - 10} \\ & \ 2.6450 \\ &= \log 441.6. \end{aligned}$$

Therefore, the required value is 441.6.

2. 22.87×0.05826.

$$\begin{aligned} \log 22.87 &= 1.3592 \\ \log 0.05826 &= \underline{8.7654 - 10} \\ & \ 0.1246 \\ &= \log 1.332. \end{aligned}$$

Therefore, the required value is 1.332.

3. 0.31416×282.4.

$$\begin{aligned} \log 0.31416 &= 9.4971 - 10 \\ \log 282.4 &= \underline{2.4508} \\ & \ 1.9479 \\ &= \log 88.7. \end{aligned}$$

Therefore, the required value is 88.7.

4. 5985×4268.

$$\begin{aligned} \log 5985 &= 3.7771 \\ \log 4268 &= \underline{3.6302} \\ & \ 7.4073 \\ &= \log 25550000. \end{aligned}$$

Therefore, the required value is 25,550,000.

5. 820.25×2.6354.

$$\begin{aligned} \log 820.25 &= 2.9139 \\ \log 2.6354 &= \underline{0.4209} \\ & \ 3.3348 \\ &= \log 2162. \end{aligned}$$

Therefore, the required value is 2162.

6. 0.00857×0.00693.

$$\begin{aligned} \log 0.00857 &= 7.9330 - 10 \\ \log 0.00693 &= \underline{7.8407 - 10} \\ & \ 5.7737 - 10 \\ &= \log 0.00005939. \end{aligned}$$

Therefore, the required value is 0.00005939.

7. -46.82×94.37.

$$\begin{aligned} \log 46.82 &= 1.6704\,n \\ \log 94.37 &= \underline{1.9749} \\ & \ 3.6453\,n \\ &= \log 4419. \end{aligned}$$

Therefore, the required value is -4419.

8. $5280 \times (-63.42)$.

$$\begin{aligned} \log 5280 &= 3.7226 \\ \log 63.42 &= \underline{1.8022\,n} \\ & \ 5.5248\,n \\ &= \log 334800. \end{aligned}$$

Therefore, the required value is $-334{,}800$.

9. $-62{,}762 \times (-0.0046)$.

$$\begin{aligned} \log 62762 &= 4.7977\,n \\ \log 0.0046 &= \underline{7.6628 - 10\,n} \\ &\ 2.4605 \\ &= \log 288.7. \end{aligned}$$

Therefore, the required value is 288.7.

10. $28.46 \div 7.423$.

$$\begin{aligned} \log 28.46 &= 1.4542 \\ \text{colog}\, 7.423 &= \underline{9.1294 - 10} \\ &\ 0.5836 \\ &= \log 3.834. \end{aligned}$$

Therefore, the required value is 3.834.

11. $26{,}250 \div 832.6$.

$$\begin{aligned} \log 26250 &= 4.4192 \\ \text{colog}\, 832.6 &= \underline{7.0796 - 10} \\ &\ 1.4988 \\ &= \log 31.54. \end{aligned}$$

Therefore, the required value is 31.54.

12. $0.04287 \div 0.56875$.

$$\begin{aligned} \log 0.04287 &= 8.6322 - 10 \\ \text{colog}\, 0.56875 &= \underline{0.2451} \\ &\ 8.8773 - 10 \\ &= \log 0.07538. \end{aligned}$$

Therefore, the required value is 0.07538.

13. $0.003625 \div 0.000987$.

$$\begin{aligned} \log 0.003625 &= 7.5593 - 10 \\ \text{colog}\, 0.000987 &= \underline{3.0057} \\ &\ 0.5650 \\ &= \log 3.673. \end{aligned}$$

Therefore, the required value is 3.673.

14. $(38.42 \times 63.95) \div 428.3$.

$$\begin{aligned} \log 38.42 &= 1.5845 \\ \log 63.95 &= 1.8059 \\ \text{colog}\, 428.3 &= \underline{7.3683 - 10} \\ &\ 0.7587 \\ &= \log 5.737. \end{aligned}$$

Therefore, the required value is 5.737.

15. $263 \times (-512) \div 1728$.

$$\begin{aligned} \log 263 &= 2.4200 \\ \log 512 &= 2.7093\,n \\ \text{colog}\, 1728 &= \underline{6.7625 - 10} \\ &\ 1.8918\,n \\ &= \log 77.95. \end{aligned}$$

Therefore, the required value is -77.95.

16. $5.8404 \div 0.003764$.

$$\begin{aligned} \log 5.8404 &= 0.7664 \\ \text{colog}\, 0.003764 &= \underline{2.4244} \\ &\ 3.1908 \\ &= \log 1552. \end{aligned}$$

Therefore, the required value is 1552.

17. $48.792 \div 148.79$.

$$\begin{aligned} \log 48.792 &= 1.6883 \\ \text{colog}\, 148.79 &= \underline{7.8274 - 10} \\ &\ 9.5157 - 10 \\ &= \log 0.3279. \end{aligned}$$

Therefore, the required value is 0.3279.

18. $31.58 \div 0.0007854$.

$$\begin{aligned}
\log 31.58 &= 1.4994\\
\operatorname{colog} 0.0007854 &= \underline{3.1049}\\
&4.6043\\
&= \log 40210.
\end{aligned}$$

Therefore, the required value is 40,210.

19. $\dfrac{862 \times 978 \times 562}{718 \times 1255 \times 243}$.

$$\begin{aligned}
\log 862 &= 2.9355\\
\log 978 &= 2.9903\\
\log 562 &= 2.7497\\
\operatorname{colog} 718 &= 7.1439 - 10\\
\operatorname{colog} 1255 &= 6.9013 - 10\\
\operatorname{colog} 243 &= \underline{7.6144 - 10}\\
&0.3351\\
&= \log 2.163.
\end{aligned}$$

Therefore, the required value is 2.163.

20. $\dfrac{52.634 \times (-36.875)}{-152.6 \times (-87.65)}$.

$$\begin{aligned}
\log 52.634 &= 1.7213\\
\log 36.875 &= 1.5667\,n\\
\operatorname{colog} 152.6 &= 7.8165 - 10\,n\\
\operatorname{colog} 87.65 &= \underline{8.0572 - 10\,n}\\
&9.1617 - 10\,n\\
&= 0.1451.
\end{aligned}$$

Therefore, the required value is -0.1451.

21. $\dfrac{84.296 \times 48.75}{7.862 \times (-6.827)}$.

$$\begin{aligned}
\log 84.296 &= 1.9258\\
\log 48.75 &= 1.6880\\
\operatorname{colog} 7.862 &= 9.1045 - 10\\
\operatorname{colog} 6.827 &= \underline{9.1658 - 10\,n}\\
&1.8841\,n\\
&= \log 76.58.
\end{aligned}$$

Therefore, the required value is -76.58.

22. $\dfrac{87.57 \times 495 \times 0.823}{3.1416 \times 0.045 \times 8662}$.

$$\begin{aligned}
\log 87.57 &= 1.9424\\
\log 495 &= 2.6946\\
\log 0.823 &= 9.9154 - 10\\
\operatorname{colog} 3.1416 &= 9.5029 - 10\\
\operatorname{colog} 0.045 &= 1.3468\\
\operatorname{colog} 8662 &= \underline{6.0624 - 10}\\
&1.4645\\
&= \log 29.14.
\end{aligned}$$

Therefore, the required value is 29.14.

23. $\dfrac{0.007854 \times 0.09863}{363 \times 98.47 \times 106.8}$.

$$\begin{aligned}
\log 0.007854 &= 7.8951 - 10\\
\log 0.09863 &= 8.9940 - 10\\
\operatorname{colog} 363 &= 7.4401 - 10\\
\operatorname{colog} 98.47 &= 8.0067 - 10\\
\operatorname{colog} 106.8 &= \underline{7.9714 - 10}\\
&0.3073 - 10\\
&= \log 0.0000000002029.
\end{aligned}$$

Therefore, the required value is 0.0000000002029.

24. $\dfrac{89.76 \times 98.54 \times 26.63}{0.005862 \times 0.8271}$.

$$\begin{aligned} \log 89.76 &= 1.9531 \\ \log 98.54 &= 1.9936 \\ \log 26.63 &= 1.4254 \\ \operatorname{colog} 0.005862 &= 2.2320 \\ \operatorname{colog} 0.8271 &= \underline{0.0824} \\ &\ 7.6865 \\ &= \log 48590000. \end{aligned}$$

Therefore, the required value is 48,590,000.

25. 3.1416^{12}.

$$\begin{aligned} \log 3.1416 &= 0.4971 \\ &\ \underline{12} \\ &\ 5.9652 \\ &= \log 923000. \end{aligned}$$

Therefore, the required value is 923,000.

26. 862.7^5.

$$\begin{aligned} \log 862.7 &= 2.9359 \\ &\ \underline{5} \\ &\ 14.6795 \\ &= \log 478100000000000. \end{aligned}$$

Therefore, the required value is 478,100,000,000,000.

27. 47.58^3.

$$\begin{aligned} \log 47.58 &= 1.6774 \\ &\ \underline{3} \\ &\ 5.0322 \\ &= \log 107700. \end{aligned}$$

Therefore, the required value is 107,700.

28. $81.48^{\frac{1}{2}}$.

$$\begin{aligned} \log 81.48 &= 1.9111. \\ 2\,)\,&\underline{1.9111} \\ &\ 0.9555 = \log 9.026. \end{aligned}$$

Therefore, the required value is 9.026.

29. $16{,}327^{\frac{1}{3}}$.

$$\begin{aligned} \log 16327 &= 4.2129. \\ 3\,)\,&\underline{4.2129} \\ &\ 1.4043 = \log 25.37. \end{aligned}$$

Therefore, the required value is 25.37.

30. $26.87^{\frac{1}{5}}$.

$$\begin{aligned} \log 26.87 &= 1.4293. \\ 5\,)\,&\underline{1.4293} \\ &\ 0.2859 = \log 1.931. \end{aligned}$$

Therefore, the required value is 1.931.

31. $\sqrt[5]{0.00452}$.

$$\begin{aligned} \log 0.00452 &= 7.6551 - 10 \\ &\ \underline{40.\ - 40} \\ &\ 5\,)\,47.6551 - 50 \\ &\ 9.5310 - 10 \\ &= \log 0.3396. \end{aligned}$$

Therefore, the required value is 0.3396.

32. $\sqrt[4]{0.8751}$.

$$\begin{aligned} \log 0.8751 &= 9.9421 - 10 \\ &\ \underline{30.\ - 30} \\ &\ 4\,)\,39.9421 - 40 \\ &\ 9.9855 - 10 \\ &= \log 0.9672. \end{aligned}$$

Therefore, the required value is 0.9672.

33. $802.53^{\frac{1}{4}}$.

$$\begin{array}{l}\log 802.53 = 2.9045.\\ 4\,\underline{)2.9045}\\ \quad 0.7261 = \log 5.323.\end{array}$$

Therefore, the required value is 5.323.

34. 2.851^{7}.

$$\begin{array}{r}\log 2.851 = 0.4550\\ \underline{7}\\ 3.1850\\ = \log 1531.\end{array}$$

Therefore, the required value is 1531.

35. $(\frac{51}{43})^{\frac{1}{20}}$.

$$\begin{array}{r}\log 51 = 1.7076\\ \operatorname{colog} 43 = \underline{8.3665 - 10}\\ 20\,\underline{)0.0741}\\ 0.0037\\ = \log 1.009.\end{array}$$

Therefore, the required value is 1.009.

36. $0.00893^{-\frac{1}{7}} = \dfrac{1}{0.00893^{\frac{1}{7}}}$.

$$\begin{array}{l}\operatorname{colog} 0.00893 = 2.0491.\\ 7\,\underline{)2.0491}\\ \quad 0.2927 = \log 1.962.\end{array}$$

Therefore, the required value is 1.962.

37. 0.5137^{5}.

$$\begin{array}{r}\log 0.5137 = 9.7107 - 10\\ \underline{5\qquad}\\ 8.5535 - 10\\ = \log 0.03577.\end{array}$$

Therefore, the required value is 0.03577.

38. 0.2675^{7}.

$$\begin{array}{r}\log 0.2675 = 9.4273 - 10\\ \underline{7\qquad}\\ 5.9911 - 10\\ = \log 0.00009798.\end{array}$$

Therefore, the required value is 0.00009798.

39. 0.1818^{9}.

$$\begin{array}{r}\log 0.1818 = 9.2596 - 10\\ \underline{9\qquad}\\ 3.3364 - 10\\ = \log 0.000000217.\end{array}$$

Therefore, the required value is 0.000000217.

40. $0.02852^{\frac{1}{5}}$.

$$\begin{array}{r}\log 0.02852 = \quad 8.4551 - 10\\ \underline{40. \qquad - 40}\\ 5\,\underline{)48.4551 - 50}\\ 9.6910 - 10\\ = \log 0.4909.\end{array}$$

Therefore, the required value is 0.4909.

41. $0.07225^{\frac{1}{6}}$.

$$\begin{array}{r}\log 0.07225 = \quad 8.8588 - 10\\ \underline{50. \qquad - 50}\\ 6\,\underline{)58.8588 - 60}\\ 9.8098 - 10\\ = \log 0.6453.\end{array}$$

Therefore, the required value is 0.6453.

42. $0.03687^{\frac{1}{7}}$.

$$\begin{array}{r}\log 0.03687 = \quad 8.5666 - 10\\ \underline{60. \qquad - 60}\\ 7\,\underline{)68.5666 - 70}\\ 9.7952 - 10\\ = \log 0.6240.\end{array}$$

Therefore, the required value is 0.624.

43. $\dfrac{\sqrt[3]{0.0047}}{\sqrt[4]{0.00872}}$.

$$\begin{array}{rlll}
\frac{1}{3}\log 0.0047 &= \frac{1}{3}(7.6721-10) &= &9.2240-10\\
\frac{1}{4}\operatorname{colog} 0.00872 &= \frac{1}{4}(2.0595) &= &\underline{0.5149}\\
 & & &9.7389-10=\log 0.5481.
\end{array}$$

Therefore, the required value is 0.5481.

44. $\dfrac{4\times(0.5326)^{\frac{3}{5}}}{(-41.752)^{\frac{2}{3}}}$.

$$\begin{array}{rlll}
\log 4 &= 0.6021 &= &0.6021\\
\frac{3}{5}\log 0.5326 &= \frac{3}{5}(9.7264-10) &= &9.8358-10\\
\frac{2}{3}\operatorname{colog} 41.752 &= \frac{2}{3}(8.3793-10) &= &\underline{8.9195-10}\\
 & & &9.3574-10=\log 0.2277.
\end{array}$$

Therefore, the required value is 0.2277.

45. $\dfrac{(-0.04563)^{\frac{4}{5}}}{(-0.6578)^{\frac{4}{7}}}$.

$$\begin{array}{rlll}
\frac{4}{5}\log 0.04563 &= \frac{4}{5}(8.6593-10) &= &8.9274-10\\
\frac{4}{7}\operatorname{colog} 0.6578 &= \frac{4}{7}(0.1819) &= &\underline{0.1039}\\
 & & &9.0313-10=\log 0.1075.
\end{array}$$

Therefore, the required value is 0.1075.

46. $\dfrac{5086\times 0.0008769^3}{9802\times 0.001984^4}$.

$$\begin{array}{rlll}
\log 5086 &= 3.7064 &= &3.7064\\
3\log 0.0008769 &= 3(6.9430-10) &= &0.8290-10\\
\operatorname{colog} 9802 &= 6.0087-10 &= &6.0087-10\\
4\operatorname{colog} 0.001984 &= 4(2.7024) &= &\underline{10.8096}\\
 & & &1.3537=\log 22.58.
\end{array}$$

Therefore, the required value is 22.58.

47. $\dfrac{8094\times\sqrt[5]{0.031}}{5408\times\sqrt[6]{0.017}}$.

$$\begin{array}{rlll}
\log 8094 &= 3.9081 &= &3.9081\\
\frac{1}{5}\log 0.031 &= \frac{1}{5}(8.4914-10) &= &9.6983-10\\
\operatorname{colog} 5408 &= 6.2670-10 &= &6.2670-10\\
\frac{1}{6}\operatorname{colog} 0.017 &= \frac{1}{6}(1.7696) &= &\underline{0.2949}\\
 & & &0.1683=\log 1.473.
\end{array}$$

Therefore, the required value is 1.473.

48. $\dfrac{109}{716}\sqrt[5]{\dfrac{7628}{9317}}$.

$$\begin{array}{rl}
\log 7628 = & 3.8824 \\
\operatorname{colog} 9317 = & \underline{6.0307 - 10} \\
 & 9.9131 - 10 \\
 & \underline{40. \qquad - 40} \\
5\,) & \underline{49.9131 - 50} \\
 & 9.9826 - 10 \\
\log 109 = & 2.0374 \\
\operatorname{colog} 716 = & \underline{7.1451 - 10} \\
 & 9.1651 - 10 = \log 0.1462.
\end{array}$$

Therefore, the required value is 0.1462.

49. $\sqrt[4]{\dfrac{348.7^2 \times 2.685^{\frac{1}{3}} \times 3.082}{2.687^{\frac{1}{5}} \times 0.08216^{\frac{1}{4}} \times 8000}}$.

$$\begin{array}{rll}
2 \log 348.7 & = 2\,(2.5424) & = 5.0848 \\
\frac{1}{3} \log 2.685 & = \frac{1}{3}\,(0.4290) & = 0.1430 \\
\log 3.082 & = 0.4889 & = 0.4889 \\
\frac{1}{5} \operatorname{colog} 2.687 & = \frac{1}{5}\,(9.5707 - 10) & = 9.9141 - 10 \\
\frac{1}{4} \operatorname{colog} 0.08216 & = \frac{1}{4}\,(1.0853) & = 0.2713 \\
\operatorname{colog} 8000 & = 6.0969 - 10 & = \underline{6.0969 - 10} \\
 & & 4\,)\,\underline{1.9990} \\
 & & \quad 0.4998 = \log 3.161.
\end{array}$$

Therefore, the required value is 3.161.

50. $\sqrt[5]{\dfrac{0.002452^{\frac{1}{4}} \times 86.47^3 \times 128.72}{5280 \times 0.07115^2 \times 62.47}}$.

$$\begin{array}{rll}
\frac{1}{4} \log 0.002452 & = \frac{1}{4}\,(7.3895 - 10) & = 9.3474 - 10 \\
3 \log 86.47 & = 3\,(1.9369) & = 5.8107 \\
\log 128.72 & = 2.1096 & = 2.1096 \\
\operatorname{colog} 5280 & = 6.2774 - 10 & = 6.2774 - 10 \\
2 \operatorname{colog} 0.07115 & = 2\,(1.1478) & = 2.2956 \\
\operatorname{colog} 62.47 & = 8.2043 - 10 & = \underline{8.2043 - 10} \\
 & & 5\,)\,\underline{4.0450} \\
 & & \quad 0.8090 = \log 6.441.
\end{array}$$

Therefore, the required value is 6.441.

51. $\sqrt[6]{\dfrac{4.382^{\frac{3}{5}} \times 36.15^4 \times 0.00819^{\frac{2}{3}}}{128600 \times 42.9^{\frac{1}{5}} \times 72.653^{\frac{1}{4}}}}.$

$$\begin{array}{lll}
\frac{3}{5}\log 4.382 & = \frac{3}{5}(0.6417) & = 0.3850 \\
4\log 36.15 & = 4(1.5581) & = 6.2324 \\
\frac{2}{3}\log 0.00819 & = \frac{2}{3}(7.9133 - 10) & = 8.6089 - 10 \\
\text{colog}\, 128600 & = 4.8908 - 10 & = 4.8908 - 10 \\
\frac{1}{5}\text{colog}\, 42.9 & = \frac{1}{5}(8.3675 - 10) & = 9.6735 - 10 \\
\frac{1}{4}\text{colog}\, 72.653 & = \frac{1}{4}(8.1388 - 10) & = \underline{9.5347 - 10} \\
 & & 9.3253 - 10 \\
 & & 50. \qquad - 50 \\
 & & 6\overline{)\,59.3253 - 60} \\
 & & 9.8876 - 10 = \log 0.772.
\end{array}$$

Therefore, the required value is 0.772.

52. $\sqrt[7]{\dfrac{3742^2 \times 0.006543^3 \times 22.63^4}{52.78^{\frac{1}{4}} \times 0.0008157^{\frac{1}{3}} \times 4618^{\frac{2}{3}}}}.$

$$\begin{array}{lll}
2\log 3742 & = 2(3.5731) & = 7.1462 \\
3\log 0.006543 & = 3(7.8158 - 10) & = 3.4474 - 10 \\
4\log 22.63 & = 4(1.3547) & = 5.4188 \\
\frac{1}{4}\text{colog}\, 52.78 & = \frac{1}{4}(8.2776 - 10) & = 9.5694 - 10 \\
\frac{1}{3}\text{colog}\, 0.0008157 & = \frac{1}{3}(3.0884) & = 1.0295 \\
\frac{2}{3}\text{colog}\, 4618 & = \frac{2}{3}(6.3356 - 10) & = \underline{7.5571 - 10} \\
 & & 7\overline{)\,4.1684} \\
 & & 0.5955 = \log 3.94.
\end{array}$$

Therefore, the required value is 3.94.

53. Find x from the equation $8^x = 35$.

$$\begin{aligned}
8^x &= 35. \\
\log(8^x) &= \log 35. \\
x \log 8 &= \log 35. \\
\therefore\ x &= \frac{\log 35}{\log 8} = \frac{1.5441}{0.9031} = 1.7098.
\end{aligned}$$

Therefore, $x = 1.7098$.

54. Find x from the equation $12^x = 7$.

$$\begin{aligned} 12^x &= 7. \\ \log(12^x) &= \log 7. \\ x\log 12 &= \log 7. \\ \therefore\ x &= \frac{\log 7}{\log 12} = \frac{0.8451}{1.0792} = 0.7831. \end{aligned}$$

Therefore, $x = 0.7831$.

55. Find x from the equation $1.5^{x+3} = 32$.

$$\begin{aligned} 1.5^{x+3} &= 32. \\ \log(1.5^{x+3}) &= \log 32. \\ (x+3)\log 1.5 &= \log 32. \\ x+3 &= \frac{\log 32}{\log 1.5} = \frac{1.5051}{0.1761} = 8.5468. \\ \therefore\ x &= 5.5468. \end{aligned}$$

Therefore, $x = 5.5468$.

56. Find x from the equation $2.66^{x-5} = 12$.

$$\begin{aligned} 2.66^{x-5} &= 12. \\ \log(2.66^{x-5}) &= \log 12. \\ (x-5)\log 2.66 &= \log 12. \\ x-5 &= \frac{\log 12}{\log 2.66} = \frac{1.0792}{0.4249} = 2.5399. \\ \therefore\ x &= 7.5399. \end{aligned}$$

Therefore, $x = 7.5399$.

57. Find x from the equation $32.5^{\frac{1}{x}} = 8$.

$$\begin{aligned} 32.5^{\frac{1}{x}} &= 8. \\ 32.5 &= 8^x. \\ \log(8^x) &= \log 32.5. \\ x\log 8 &= \log 32.5. \\ \therefore\ x &= \frac{\log 32.5}{\log 8} = \frac{1.5119}{0.9031} = 1.6741. \end{aligned}$$

Therefore, $x = 1.6741$.

58. Find x from the equation $4.02^{\frac{1}{x-4}} = 2.37$.

$$4.02^{\frac{1}{x-4}} = 2.37.$$

$$4.02 = 2.37^{x-4}.$$

$$\log(2.37^{x-4}) = \log 4.02.$$

$$(x-4)\log 2.37 = \log 4.02.$$

$$x - 4 = \frac{\log 4.02}{\log 2.37} = \frac{0.6042}{0.3747} = 1.6125.$$

$$\therefore\ x = 5.6125.$$

Therefore, $x = 5.6125$.

59. Express in logarithmic form the following:

$$\frac{a^2}{4}\sqrt{3};\ \pi r^2;\ \sqrt{s(s-a)(s-b)(s-c)};\ \frac{c}{2\pi};\ \frac{4}{3}\pi r^3.$$

Let $x = \frac{a^2}{4}\sqrt{3}$.

$$\log x = \log\left(\frac{a^2}{4}\sqrt{3}\right).$$

$$\therefore\ \log x = 2\log a + \operatorname{colog} 4 + \tfrac{1}{2}\log 3.$$

Let $x = \pi r^2$.

$$\log x = \log(\pi r^2).$$

$$\therefore\ \log x = \log\pi + 2\log r.$$

Let $x = \sqrt{s(s-a)(s-b)(s-c)}$.

$$\log x = \log\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore\ \log x = \tfrac{1}{2}[\log s + \log(s-a) + \log(s-b) + \log(s-c)].$$

Let $x = \frac{c}{2\pi}$.

$$\log x = \log\left(\frac{c}{2\pi}\right).$$

$$\therefore\ \log x = \log c + 2\operatorname{colog}\pi.$$

Let $x = \frac{4}{3}\pi r^3$.

$$\log x = \log(\tfrac{4}{3}\pi r^3).$$

$$\therefore\ \log x = \log 4 + \operatorname{colog} 3 + \log\pi + 3\log r.$$

60. The indicated area of a triangle is $\sqrt{153 \times 52 \times 51 \times 50}$. Find the value by means of logarithms.

$$\begin{aligned} \log 153 &= 2.1847 \\ \log 52 &= 1.7160 \\ \log 51 &= 1.7076 \\ \log 50 &= 1.6990 \\ 2\,)\,&\underline{7.3073} \\ &3.6537 = \log 4505. \end{aligned}$$

Therefore, the area of the triangle is 4505.

61. The formula for the number of terms in a geometrical series, in terms of l, a, and r, is $r^{n-1} = \frac{l}{a}$. Put this equation in logarithmic form and solve for n.

$$\begin{aligned} r^{n-1} &= \frac{l}{a}. \\ \log(r^{n-1}) &= \log\left(\frac{l}{a}\right). \\ (n-1)\log r &= \log l + \operatorname{colog} a. \\ n - 1 &= \frac{\log l + \operatorname{colog} a}{\log r}. \\ \therefore\ n &= 1 + \frac{\log l + \operatorname{colog} a}{\log r} \\ &= \frac{\log r + \log l + \operatorname{colog} a}{\log r}. \end{aligned}$$

Therefore, $n = \dfrac{\log r + \log l + \operatorname{colog} a}{\log r}$.

62. Find the logarithm of 3 to the base 2.

Let $x = \log_2 3$.

Then $2^x = 3$.

$$\begin{aligned} x \log 2 &= \log 3. \\ \therefore\ x &= \frac{\log 3}{\log 2} = \frac{0.4771}{0.3010} = 1.5850. \end{aligned}$$

Therefore, $\log_2 3 = 1.5850$.

63. Find the logarithm of 100 to the base 12.

Let $$x = \log_{12} 100.$$

Then $$12^x = 100.$$

$$x \log 12 = \log 100.$$

$$\therefore\ x = \frac{\log 100}{\log 12} = \frac{2}{1.0792} = 1.8532.$$

Therefore, $\log_{12} 100 = 1.8532$.

64. Find the logarithm of 10 to the base 12.

Let $$x = \log_{12} 10.$$

Then $$12^x = 10.$$

$$x \log 12 = \log 10.$$

$$\therefore\ x = \frac{\log 10}{\log 12} = \frac{1}{1.0792} = 0.9266.$$

Therefore, $\log_{12} 10 = 0.9266$.

65. Find the logarithm of 0.1 to the base 12.

Let $$x = \log_{12} 0.1.$$

Then $$12^x = 0.1.$$

$$x \log 12 = \log 0.1.$$

$$\therefore\ x = \frac{\log 0.1}{\log 12} = \frac{-1}{1.0792} = -(0.9266).$$

$$\therefore\ \log_{12} 0.1 = -(0.9266).$$

66. Solve for n the equation $l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.

$$l(s-l)^{n-1} - a(s-a)^{n-1} = 0.$$

$$l(s-l)^{n-1} = a(s-a)^{n-1}.$$

$$\log l + (n-1)\log(s-l) = \log a + (n-1)\log(s-a).$$

$$(n-1)\log(s-l) - (n-1)\log(s-a) = \log a - \log l.$$

$$(n-1)[\log(s-l) - \log(s-a)] = \log a - \log l.$$

$$n-1 = \frac{\log a - \log l}{\log(s-l) - \log(s-a)}.$$

$$\therefore\ n = 1 + \frac{\log a - \log l}{\log(s-l) - \log(s-a)}$$

$$= 1 + \frac{\log a + \operatorname{colog} l}{\log(s-l) + \operatorname{colog}(s-a)}.$$

67. Find $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$, $\sqrt[5]{5}$, $\sqrt[6]{6}$, $\sqrt[7]{7}$, $\sqrt[8]{8}$, $\sqrt[9]{9}$, $\sqrt[10]{10}$.

$$\begin{aligned}
\log 2 &= 0.3010.\\
\log \sqrt{2} &= \tfrac{1}{2} \text{ of } 0.3010 = 0.1505 = \log 1.414.\\
\therefore\ \sqrt{2} &= 1.414.\\
\log 3 &= 0.4771.\\
\log \sqrt[3]{3} &= \tfrac{1}{3} \text{ of } 0.4771 = 0.1590 = \log 1.442.\\
\therefore\ \sqrt[3]{3} &= 1.442.\\
\log 4 &= 0.6021.\\
\log \sqrt[4]{4} &= \tfrac{1}{4} \text{ of } 0.6021 = 0.1505 = \log 1.414.\\
\therefore\ \sqrt[4]{4} &= 1.414.\\
\log 5 &= 0.6990.\\
\log \sqrt[5]{5} &= \tfrac{1}{5} \text{ of } 0.6990 = 0.1398 = \log 1.380.\\
\therefore\ \sqrt[5]{5} &= 1.380.\\
\log 6 &= 0.7782.\\
\log \sqrt[6]{6} &= \tfrac{1}{6} \text{ of } 0.7782 = 0.1297 = \log 1.348.\\
\therefore\ \sqrt[6]{6} &= 1.348.\\
\log 7 &= 0.8451.\\
\log \sqrt[7]{7} &= \tfrac{1}{7} \text{ of } 0.8451 = 0.1207 = \log 1.320.\\
\therefore\ \sqrt[7]{7} &= 1.320.\\
\log 8 &= 0.9031.\\
\log \sqrt[8]{8} &= \tfrac{1}{8} \text{ of } 0.9031 = 0.1129 = \log 1.297.\\
\therefore\ \sqrt[8]{8} &= 1.297.\\
\log 9 &= 0.9542.\\
\log \sqrt[9]{9} &= \tfrac{1}{9} \text{ of } 0.9542 = 0.1060 = \log 1.276.\\
\therefore\ \sqrt[9]{9} &= 1.276.\\
\log 10 &= 1.0000.\\
\log \sqrt[10]{10} &= \tfrac{1}{10} \text{ of } 1.0000 = 0.1000 = \log 1.259.\\
\therefore\ \sqrt[10]{10} &= 1.259.
\end{aligned}$$

Therefore, $\sqrt{2} = 1.414$; $\sqrt[3]{3} = 1.442$; $\sqrt[4]{4} = 1.414$; $\sqrt[5]{5} = 1.380$; $\sqrt[6]{6} = 1.348$; $\sqrt[7]{7} = 1.320$; $\sqrt[8]{8} = 1.297$; $\sqrt[9]{9} = 1.276$; $\sqrt[10]{10} = 1.259$.

68. Show that $\log \sqrt{a^2 - b^2} = \frac{1}{2}[\log(a+b) + \log(a-b)]$.

Now
$$\begin{aligned}
\sqrt{a^2 - b^2} &= \sqrt{(a+b)(a-b)}.\\
\therefore\ \log \sqrt{a^2 - b^2} &= \log \sqrt{(a+b)(a-b)}\\
&= \log(\sqrt{a+b} \times \sqrt{a-b})\\
&= \tfrac{1}{2}\log(a+b) + \tfrac{1}{2}\log(a-b).\\
\therefore\ \log \sqrt{a^2 - b^2} &= \tfrac{1}{2}[\log(a+b) + \log(a-b)].
\end{aligned}$$

Exercise 139. Page 389

1. Find the amount of $6326 in 25 years at 4 per cent compound interest, the interest being compounded annually.

$$A = PR^n.$$

Here $$P = 6326,\ R = 1.04,\ n = 25.$$

$$\therefore\ A = 6326 \times 1.04^{25}.$$

$$\begin{aligned} \log 6326 &= 3.8011 \\ \log 1.04^{25} &= \underline{0.4250} \\ \log A &= 4.2261 \\ \therefore\ A &= 16830. \end{aligned}$$

Therefore, the required amount is $16,830.

2. Find the amount of $6326 in 25 years at 4 per cent compound interest, the interest being compounded semiannually.

$$A = PR^n.$$

Here $$P = 6326,\ R = 1.02,\ n = 50.$$

$$\therefore\ A = 6326 \times 1.02^{50}.$$

$$\begin{aligned} \log 6326 &= 3.8011 \\ \log 1.02^{50} &= \underline{0.4300} \\ \log A &= 4.2311 \\ \therefore\ A &= 17030. \end{aligned}$$

Therefore, the required amount is $17,030.

3. Find the amount of $142.50 in 3 years 8 months at 5 per cent compound interest.

$$A = PR^n.$$

Here $$P = 142.50,\ R = 1.05,\ n = 3\tfrac{2}{3} = \tfrac{11}{3}.$$

$$\therefore\ A = 142.50 \times 1.05^{\frac{11}{3}}.$$

$$\begin{aligned} \log 142.50 &= 2.1538 \\ \log 1.05^{\frac{11}{3}} &= \underline{0.0777} \\ \log A &= 2.2315 \\ \therefore\ A &= 170.4. \end{aligned}$$

Therefore, the required amount is $170.40.

4. What sum will amount to \$5000 in 22 years if put at compound interest at $3\frac{1}{2}$ per cent?

$$A = PR^n.$$

$$\therefore P = \frac{A}{R^n}.$$

Here $$A = 5000,\ R = 1.035,\ n = 22.$$

$$\therefore P = \frac{5000}{1.035^{22}}.$$

$$\begin{aligned} \log 5000 &= 3.6990 \\ \operatorname{colog} 1.035^{22} &= \underline{9.6722 - 10} \\ \log P &= 3.3712 \end{aligned}$$

$$\therefore P = 2351.$$

Therefore, the required sum is \$2351.

5. In how many years will \$2500 amount to \$10,000 at 4 per cent compound interest?

$$A = PR^n.$$

Here $$A = 10000,\ P = 2500,\ R = 1.04.$$

$$\therefore 10000 = 2500 \times 1.04^n.$$

$$\therefore 1.04^n = \tfrac{10000}{2500} = 4.$$

$$\therefore n \times \log 1.04 = \log 4.$$

$$\therefore n = \frac{\log 4}{\log 1.04} = \frac{0.6021}{0.0170} = 35.4177.$$

$$35.4177 \text{ years} = 35 \text{ years } 5 \text{ months}.$$

Therefore, the required time is 35 years 5 months.

6. At what rate per cent will \$1500 amount in 15 years to \$2717.05 at compound interest, compounded semiannually?

$$A = PR^n.$$

$$R^n = \frac{A}{P}.$$

$$\therefore R = \sqrt[n]{\frac{A}{P}}.$$

Here $$A = 2717.05,\ P = 1500,\ n = 30.$$

$$\therefore R = \sqrt[30]{\frac{2717.05}{1500}}.$$

$$\begin{aligned}\log 2717.05 &= 3.4341\\ \operatorname{colog} 1500 &= \underline{6.8239 - 10}\\ 30&\overline{)0.2580}\\ \log R &= 0.0086\\ \therefore\ R &= 1.02.\\ r = R - 1 &= 1.02 - 1 = 0.02.\end{aligned}$$

Therefore, the required rate is 2 per cent semiannually; that is, 4 per cent per annum.

7. If a corporation sets aside each year for 20 years $5000 as a sinking fund and puts it at compound interest at $3\frac{1}{2}$ per cent, what will be the amount of the sinking fund?

$$A = \frac{S(R^n - 1)}{r}.$$

Here $S = 5000,\ R = 1.035,\ n = 20,\ r = 0.035.$

$$\therefore\ A = \frac{5000\,(1.035^{20} - 1)}{0.035}.$$

$$\begin{aligned}\log 1.035 &= 0.0149\\ &\quad\ \ \underline{20}\\ &\quad 0.2980 = \log 1.986.\end{aligned}$$

$$\therefore\ A = \frac{5000 \times 0.986}{0.035}.$$

$$\begin{aligned}\log 5000 &= 3.6990\\ \log 0.986 &= 9.9939 - 10\\ \operatorname{colog} 0.035 &= \underline{1.4559}\\ \log A &= 5.1488\\ \therefore\ A &= 140900.\end{aligned}$$

Therefore, the amount of the sinking fund will be $140,900.

8. A city has a funded debt of $500,000 due in 18 years. What sum must be set apart each year, as a sinking fund, to cancel the debt, provided money is worth 4 per cent?

$$A = \frac{S(R^n - 1)}{r}.$$

$$\therefore\ S = \frac{Ar}{R^n - 1}.$$

Here $A = 500000,\ r = 0.04,\ R = 1.04,\ n = 18.$

$$\therefore\ S = \frac{500000 \times 0.04}{1.04^{18} - 1}.$$

$$\begin{aligned} \log 1.04 &= 0.0170 \\ &\underline{18} \\ &0.3060 = \log 2.023. \end{aligned}$$

$$\therefore\ S = \frac{500000 \times 0.04}{1.023}.$$

$$\begin{aligned} \log 500000 &= 5.6990 \\ \log 0.04 &= 8.6021 - 10 \\ \text{colog } 1.023 &= \underline{9.9901 - 10} \\ \log S &= 4.2912 \end{aligned}$$

$$\therefore\ S = 19550.$$

Therefore, the sum that must be set apart each year is $19,550.

9. Find the present worth of an annuity of $1200 for 10 years at $3\frac{1}{2}$ per cent interest.

$$P = \frac{S}{R^n} \times \frac{R^n - 1}{R - 1}.$$

Here $S = 1200,\ R = 1.035,\ n = 10.$

$$\therefore\ P = \frac{1200}{1.035^{10}} \times \frac{1.035^{10} - 1}{1.035 - 1}.$$

$$\begin{aligned} \log 1.035 &= 0.0149 \\ &\underline{10} \\ &0.1490 = \log 1.409. \end{aligned}$$

$$\therefore\ P = \frac{1200}{1.035^{10}} \times \frac{0.409}{0.035}.$$

$$\begin{aligned} \log 1200 &= 3.0792 \\ \text{colog } 1.035^{10} &= 9.8510 - 10 \\ \log 0.409 &= 9.6117 - 10 \\ \text{colog } 0.035 &= \underline{1.4559} \\ \log P &= 3.9978 \end{aligned}$$

$$\therefore\ P = 9950.$$

Therefore, the present worth of the annuity is $9950.

10. A man buys an annuity of $900 for 15 years. If money is worth $4\frac{1}{2}$ per cent, what should he pay for the annuity?

$$P = \frac{S}{R^n} \times \frac{R^n - 1}{R - 1}.$$

Here $S = 900$, $R = 1.045$, $n = 15$.

$$\therefore P = \frac{900}{1.045^{15}} \times \frac{1.045^{15} - 1}{1.045 - 1}.$$

$$\begin{aligned} \log 1.045 &= 0.0191 \\ &\underline{\quad\ \ 15} \\ &0.2865 = \log 1.934. \end{aligned}$$

$$\therefore P = \frac{900}{1.045^{15}} \times \frac{0.934}{0.045}.$$

$$\begin{aligned} \log 900 &= 2.9542 \\ \text{colog } 1.045^{15} &= 9.7135 - 10 \\ \log 0.934 &= 9.9703 - 10 \\ \text{colog } 0.045 &= \underline{1.3468 \qquad\quad} \\ \log P &= 3.9848 \\ \therefore P &= 9656. \end{aligned}$$

Therefore, the man should pay $9656 for the annuity.

11. An annuity of $500 was unpaid for 10 years. What was the amount due, if interest is reckoned at $3\frac{3}{4}$ per cent?

$$A = \frac{S(R^n - 1)}{r}.$$

Here $S = 500$, $R = 1.0375$, $n = 10$, $r = 0.0375$.

$$\therefore A = \frac{500(1.0375^{10} - 1)}{0.0375}.$$

$$\begin{aligned} \log 1.0375 &= 0.0160 \\ &\underline{\quad\ \ 10} \\ &0.1600 = \log 1.445. \end{aligned}$$

$$\therefore A = \frac{500 \times 0.445}{0.0375}.$$

$$\begin{aligned} \log 500 &= 2.6990 \\ \log 0.445 &= 9.6484 - 10 \\ \text{colog } 0.0375 &= \underline{1.4260 \qquad\quad} \\ \log A &= 3.7734 \\ \therefore A &= 5934. \end{aligned}$$

Therefore, the amount due is $5934.

12. **A man 59 years old is expected to live just 15 years. He has $20,000 with which to buy a life annuity. If money is worth $4\frac{1}{4}$ per cent, how large an annuity may he expect?**

$$S = Pr \times \frac{R^n}{R^n - 1}.$$

Here $P = 20000$, $r = 0.0425$, $R = 1.0425$, $n = 15$.

$$\therefore\ S = 20000 \times 0.0425 \times \frac{1.0425^{15}}{1.0425^{15} - 1}.$$

$$\begin{array}{r l}
\log 1.0425 = & 0.0181 \\
 & \underline{15} \\
 & 0.2715 = \log 1.869.
\end{array}$$

$$\therefore\ S = 20000 \times 0.0425 \times \frac{1.0425^{15}}{0.869}.$$

$$\begin{array}{r l}
\log 20000 = & 4.3010 \\
\log 0.0425 = & 8.6284 - 10 \\
\log 1.0425^{15} = & 0.2715 \\
\text{colog}\, 0.869 = & \underline{0.0610} \\
\log S = & 3.2619 \\
\therefore\ S = & 1827.
\end{array}$$

Therefore, the man may expect an annuity of $1827.

13. A man 35 years old is expected to live just 31 years. He has $18,625 with which to buy a life annuity. If money is worth $3\frac{1}{4}$ per cent, how large an annuity may he expect?

$$S = Pr \times \frac{R^n}{R^n - 1}.$$

Here $P = 18625$, $r = 0.0325$, $R = 1.0325$, $n = 31$.

$$\therefore\ S = 18625 \times 0.0325 \times \frac{1.0325^{31}}{1.0325^{31} - 1}.$$

$$\begin{array}{r l}
\log 1.0325 = & 0.0139 \\
 & \underline{31} \\
 & 0.4309 = \log 2.697.
\end{array}$$

$$\therefore\ S = 18625 \times 0.0325 \times \frac{1.0325^{31}}{1.697}.$$

$$\begin{array}{r l}
\log 18625 = & 4.2701 \\
\log 0.0325 = & 8.5119 - 10 \\
\log 1.0325^{31} = & 0.4309 \\
\text{colog}\, 1.697 = & \underline{9.7703 - 10} \\
\log S = & 2.9832 \\
\therefore\ S = & 962.
\end{array}$$

Therefore, the man may expect an annuity of $962.

14. What interest on his investment will a purchaser receive if he buys at 108 Boston & Maine $4\frac{1}{2}$ per cent bonds due in 30 years with semiannual coupons, if money is worth 4 per cent?

$$1+x=\left[\frac{Sq+Sr(1+q)^n-Sr}{Pq}\right]^{\frac{1}{n}}.$$

Here $S=100,\ P=108,\ q=0.02,\ r=0.0225,\ n=78.$

$$\therefore\ 1+x=\left[\frac{2+2.25\times 1.02^{78}-2.25}{108\times 0.02}\right]^{\frac{1}{78}}.$$

$$\begin{aligned}\log 2.25 &= 0.3522\\ \log 1.02^{78} &= \underline{0.6708}\\ &\ 1.0230=\log 10.54.\end{aligned}$$

$$\therefore\ 1+x=\left[\frac{2+10.54-2.25}{108\times 0.02}\right]^{\frac{1}{78}}$$

$$=\left[\frac{10.29}{108\times 0.02}\right]^{\frac{1}{78}}.$$

$$\begin{aligned}\log 10.29 &= 1.0124\\ \operatorname{colog} 108 &= 7.9666-10\\ \operatorname{colog} 0.02 &= \underline{1.6990}\\ 78\,)&\,0.6780\\ \log(1+x) &= 0.0087\end{aligned}$$

$$\therefore\ 1+x=1.020.$$

$$\therefore\ x=0.020.$$

$$2x=0.040.$$

Therefore, the purchaser will receive 4 per cent on his investment.

15. What interest on his investment will a purchaser receive if he buys at 103 West End 4 per cent bonds due in 12 years with semiannual coupons, if money is worth $3\frac{1}{2}$ per cent?

$$1+x=\left[\frac{Sq+Sr(1+q)^n-Sr}{Pq}\right]^{\frac{1}{n}}.$$

Here $S=100,\ P=103,\ q=0.0175,\ r=0.02,\ n=24.$

$$\therefore\ 1+x=\left[\frac{1.75+2\times 1.0175^{24}-2}{103\times 0.0175}\right]^{\frac{1}{24}}.$$

$$\begin{aligned}\log 2 &= 0.3010\\ \log 1.0175^{24} &= \underline{0.1800}\\ &\ 0.4810=\log 3.027.\end{aligned}$$

$$\therefore\ 1+x=\left[\frac{1.75+3.027-2}{103\times 0.0175}\right]^{\frac{1}{24}}$$
$$=\left[\frac{2.777}{103\times 0.0175}\right]^{\frac{1}{24}}.$$

$$\begin{aligned}\log 2.777 &= 0.4436\\ \operatorname{colog} 103 &= 7.9872-10\\ \operatorname{colog} 0.0175 &= \underline{1.7570}\\ 24\,)&\underline{0.1878}\\ \log 1+x &= 0.0078\end{aligned}$$

$$\therefore\ 1+x=1.018.$$
$$\therefore\ x=0.018.$$
$$2x=0.036.$$

Therefore, the purchaser will receive 3.6 per cent on his investment.

16. At what price must Atchison, Topeka & Santa Fe 4 per cent bonds due in 90 years with semiannual coupons be bought in order to net the purchaser $4\frac{1}{4}$ per cent on his investment, if money is worth $3\frac{3}{4}$ per cent?

$$1+x=\left[\frac{Sq+Sr(1+q)^n-Sr}{Pq}\right]^{\frac{1}{n}}.$$
$$(1+x)^n=\frac{Sq+Sr(1+q)^n-Sr}{Pq}.$$
$$Pq(1+x)^n=Sq+Sr(1+q)^n-Sr.$$
$$\therefore\ P=\frac{Sq+Sr(1+q)^n-Sr}{q(1+x)^n}.$$

Here $S=100,\ q=0.01875,\ r=0.02,\ x=0.02125,\ n=180.$

$$\therefore\ P=\frac{1.875+2\times 1.01875^{180}-2}{0.01875\times 1.02125^{180}}.$$

$$\begin{aligned}\log 2 &= 0.3010\\ \log 1.01875^{180} &= \underline{1.4580}\\ &\ 1.7590=\log 57.41.\end{aligned}$$

$$\therefore\ P=\frac{1.875+57.41-2}{0.01875\times 1.02125^{180}}$$
$$=\frac{57.285}{0.01875\times 1.02125^{180}}.$$

$$\begin{aligned} \log 57.285 &= 1.7581 \\ \text{colog}\,0.01875 &= 1.7270 \\ \text{colog}\,1.02125^{180} &= \underline{8.3620 - 10} \\ \log P &= 1.8471 \end{aligned}$$

$$\therefore P = 70.32.$$

Therefore, the purchase price must be 70.32.

17. At what price must Old Colony 4 per cent bonds due in 33 years with semiannual coupons be bought in order to net the purchaser $3\frac{1}{2}$ per cent on his investment, if money is worth $4\frac{1}{2}$ per cent?

By Example 16, $$P = \frac{Sq + Sr(1+q)^n - Sr}{q(1+x)^n}.$$

Here $S = 100$, $q = 0.0225$, $r = 0.02$, $x = 0.0175$, $n = 66$.

$$\therefore P = \frac{2.25 + 2 \times 1.0225^{66} - 2}{0.0225 \times 1.0175^{66}}.$$

$$\begin{aligned} \log 2 &= 0.3010 \\ \log 1.0225^{66} &= \underline{0.6402} \\ &\ 0.9412 = \log 8.734. \end{aligned}$$

$$\begin{aligned} \therefore P &= \frac{2.25 + 8.734 - 2}{0.0225 \times 1.0175^{66}} \\ &= \frac{8.984}{0.0225 \times 1.0175^{66}}. \end{aligned}$$

$$\begin{aligned} \log 8.984 &= 0.9535 \\ \text{colog}\,0.0225 &= 1.6478 \\ \text{colog}\,1.0175^{66} &= \underline{9.5050 - 10} \\ \log P &= 2.1063 \end{aligned}$$

$$\therefore P = 127.7.$$

Therefore, the purchase price must be 127.7.

Exercise 140. Page 392

1. In how many ways may a girl and a boy be chosen from a school that consists of 18 girls and 24 boys?

One girl may be chosen from 18 girls in 18 ways; and one boy may be chosen from 24 boys in 24 ways.

Therefore, one girl and one boy may be chosen from a school that consists of 18 girls and 24 boys in 18 × 24, or 432, ways.

2. On a shelf are a set of 6 English books, a set of 8 French books, and a set of 10 German books. In how many ways can three books be chosen so that there shall be one from each set?

One English book may be chosen from 6 English books in 6 ways; one French book may be chosen from 8 French books in 8 ways; one German book may be chosen from 10 German books in 10 ways.

Therefore, one English book, one French book, and one German book may be chosen from 6 English books, 8 French books, and 10 German books in 6 × 8 × 10, or 480, ways.

3. A boy has 5 different routes to his school building. In how many ways may he go and return by a different route?

The boy may go to the school in any one of the 5 ways; in returning, he may take any one of the 4 remaining ways.

Therefore, the boy may go to school and return by a different route in 5 × 4, or 20, ways.

4. In how many ways can a vowel and a consonant be chosen from the alphabet of 6 vowels and 20 consonants?

One vowel may be chosen from 6 vowels in 6 ways; and one consonant may be chosen from 20 consonants in 20 ways.

Therefore, one vowel and one consonant can be chosen from 6 vowels and 20 consonants in 6 × 20, or 120, ways.

5. After a particular vowel and a particular consonant have been chosen, in how many ways can a two-lettered word be made? In how many different ways can a two-lettered word be made, if it is to consist of a vowel and a consonant?

After a particular vowel and a particular consonant have been chosen, a two-lettered word can be made in 2 ways. For the vowel can be placed first and the consonant second, or the consonant can be placed first and the vowel second.

A two-lettered word can be made in 120 × 2, or 240, ways. For each of the 120 ways of selecting a vowel and a consonant from the alphabet (Example 4), there are 2 ways of arranging these letters. That is, there are 120 × 2, or 240, ways in all.

Exercise 141. Page 395

1. Find the value of $5!$; of $\frac{10!}{6!}$; of $\frac{12!}{3!} \times \frac{8!}{7!} \times \frac{15!}{16!}$.

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

$$\begin{aligned} \frac{10!}{6!} &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= 10 \times 9 \times 8 \times 7 = 5040. \end{aligned}$$

$$\begin{aligned} \frac{12!}{3!} \times \frac{8!}{7!} \times \frac{15!}{16!} &= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} \\ &\quad \times \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &\quad \times \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 8 \times \tfrac{1}{16} \\ &= 39{,}916{,}800. \end{aligned}$$

2. Find the value of $P_{9,5}$; of $P_{20,1}$; of $P_{6,6}$; of $(P_{4,4})^2$.

$$\begin{aligned} P_{9,5} &= \frac{9!}{4!} = 9 \times 8 \times 7 \times 6 \times 5 = 15{,}120. \\ P_{20,1} &= \frac{20!}{19!} = 20. \\ P_{6,6} &= 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720. \\ (P_{4,4})^2 &= (4!)^2 = (4 \times 3 \times 2 \times 1)^2 = 24^2 = 576. \end{aligned}$$

3. Find the value of each in Example 2 when repetitions are allowed.

When repetitions are allowed,

$$\begin{aligned} P_{9,5} &= 9^5 = 59{,}049; \\ P_{20,1} &= 20^1 = 20; \\ P_{6,6} &= 6^6 = 46{,}656; \\ (P_{4,4})^2 &= (4^4)^2 = 4^8 = 65{,}536. \end{aligned}$$

4. In how many ways can a class of ten pupils be arranged at a blackboard?

The number of ways is $10! = 3{,}628{,}800$.

5. How many changes can be rung with a peal of 6 bells?

The number of changes is $6! = 720$.

6. How many changes can be rung with a peal of 7 bells, a particular one always being first?

The number of changes is $1 \times 6! = 6! = 720$.

7. If $P_{8,r} = 1680$, find the value of r.

$$P_{8,r} = \frac{8!}{(8-r)!}.$$

$$\therefore \frac{8!}{(8-r)!} = 1680.$$

$$(8-r)! = \frac{8!}{1680} = \frac{8 \times 7 \times 6 \times 5 \times 4!}{8 \times 7 \times 6 \times 5} = 4!.$$

$$\therefore 8 - r = 4.$$

$$\therefore r = 4.$$

8. If $P_{n,n} = 362,880$, find the value of n.

$$P_{n,n} = n!.$$

$$\therefore n! = 362880 = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 = 9!.$$

$$\therefore n = 9.$$

9. If $P_{n,2} = 110$, find the value of n.

$$P_{n,2} = \frac{n!}{(n-2)!}.$$

$$\therefore \frac{n!}{(n-2)!} = 110.$$

$$\frac{n(n-1)(n-2)!}{(n-2)!} = 110.$$

$$n(n-1) = 110.$$

$$n^2 - n = 110.$$

$$n^2 - n - 110 = 0.$$

$$(n-11)(n+10) = 0.$$

$$\therefore n = 11, \text{ or } -10.$$

10. In how many ways can the members of a baseball nine be arranged in the field, if the same two must always pitch and catch?

The position of catcher can be filled in only 1 way; the position of pitcher can be filled in only 1 way; and the other positions can be filled in 7! ways.

Therefore, the nine can be arranged in $1 \times 1 \times 7! = 7! = 5040$ ways.

11. How many numbers of four figures each can be made with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, repetitions not being allowed? How many if repetitions are allowed?

The first place can be filled in 9 ways; the second in 8 ways; the third in 7 ways; and the fourth in 6 ways.

Therefore, the whole number of numbers is $9 \times 8 \times 7 \times 6 = 3024$.

If repetitions are allowed, the number of numbers is $9^4 = 6561$.

12. In how many ways can eight guests be seated along one side of a table?

The number of ways is $8! = 40,320$.

13. In how many ways can eight guests be seated at a round table relative to the eight places at the table? relative to each other?

The first place can be filled in 8 ways, then the second place in 7 ways, then the third place in 6 ways, and so on.

Therefore, relative to the eight places at the table, eight guests can be seated in $8! = 40,320$ ways.

The number of ways the eight guests can be seated relative to each other may be determined by placing any one particular guest in a seat and then seating the other seven guests in different ways with respect to him. This can be done in $7! = 5040$ different ways.

Therefore, relative to each other, eight guests can be seated at a round table in 5040 ways.

14. In how many ways can the letters of the word *combine* be arranged, if *b* is always to be the middle letter?

If *b* is always to be the middle letter, the fourth place can be filled in 1 way, then the first place in 6 ways, then the second place in 5 ways, then the third place in 4 ways, then the fifth place in 3 ways, then the sixth place in 2 ways, then the seventh place in 1 way.

Therefore, the whole number of ways is $1 \times 6! = 6! = 720$.

15. How many different signals can be given with 6 differently colored flags, if 3 are displayed on a staff each time? How many, if any number of flags are displayed each time?

If 3 flags are displayed at a time, the number of arrangements is

$$\frac{6!}{3!} = 6 \times 5 \times 4 = 120.$$

If any number of flags are displayed at a time, the number of arrangements is

$$\frac{6!}{1!}+\frac{6!}{2!}+\frac{6!}{3!}+\frac{6!}{4!}+\frac{6!}{5!}+\frac{6!}{6!}$$
$$=6\times5\times4\times3\times2+6\times5\times4\times3+6\times5\times4+6\times5+6+1$$
$$=720+360+120+30+6+1$$
$$=1237.$$

Exercise 142. Page 397

1. Find the value of $C_{9,5}$; of $C_{20,1}$; of $C_{6,6}$.

$$C_{9,5}=\frac{9!}{4!\,5!}=\frac{9\times8\times7\times6}{4\times3\times2\times1}=126.$$
$$C_{20,1}=\frac{20!}{19!\,1!}=20.$$
$$C_{6,6}=\frac{6!}{6!}=1.$$

2. Show that $C_{n,r}=\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$.

By § 487, $P_{n,r}=C_{n,r}\times r!$.

$$\therefore\ C_{n,r}=\frac{P_{n,r}}{r!}=\frac{n!}{(n-r)!}\div r!=\frac{n!}{(n-r)!\,r!}$$
$$=\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

3. Show that the number of combinations of 9 things, taken 5 at a time, is the same as the number of combinations taken 4 at a time.

The number of combinations of 9 things, taken 5 at a time, is $\dfrac{9!}{4!\,5!}$; the number of combinations of 9 things, taken 4 at a time, is $\dfrac{9!}{5!\,4!}$.

Therefore, the number of combinations of 9 things, taken 5 at a time, is the same as the number of combinations taken 4 at a time.

4. What is the number of combinations of 100 things taken 98 at a time?

The number of combinations is $\dfrac{100!}{2!\,98!}=\dfrac{100\times99}{2\times1}=4950.$

5. In an examination paper there are 10 questions, from which 8 are to be selected by the pupil. In how many ways can he do this?

The number of ways required is $\frac{10!}{2!\,8!}=\frac{10\times 9}{2\times 1}=45.$

6. How many ways are there of electing five members of a board of education from eight candidates?

The number of ways required is $\frac{8!}{3!\,5!}=\frac{8\times 7\times 6}{3\times 2\times 1}=56.$

7. If $C_{6,r}=20$, find the value of r.

$$C_{6,r}=\frac{6!}{(6-r)!\,r!}.$$

$$\therefore\ \frac{6!}{(6-r)!\,r!}=20.$$

$$\frac{6\times 5\times 4\times 3\times 2\times 1}{(6-r)!\,r!}=20.$$

$$\frac{6\times 3\times 2\times 1}{(6-r)!\,r!}=1.$$

$$\frac{3!\times 3!}{(6-r)!\,r!}=1.$$

$$\therefore\ r!=3!.$$

$$\therefore\ r=3.$$

8. If $24\,C_{n,r}=P_{n,r}$, find the value of r.

$$C_{n,r}=\frac{n!}{(n-r)!\,r!},\ \text{and}\ P_{n,r}=\frac{n!}{(n-r)!}.$$

$$\therefore\ \frac{24\times n!}{(n-r)!\,r!}=\frac{n!}{(n-r)!}.$$

Divide by $\frac{n!}{(n-r)!}$, $\quad\frac{24}{r!}=1.$

$$\therefore\ r!=24.$$

$$r!=1\times 2\times 3\times 4.$$

$$r!=4!$$

$$\therefore\ r=4.$$

9. There are 9 coins of different denominations on the table. How many different sums of money can be made up by taking 4 coins each time?

The number of different sums of money is

$$\frac{9!}{5!\,4!}=\frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1}=126.$$

10. Expand $(x+y)^5$ and show that it is equal to

$$x^5+C_{5,1}x^4y+C_{5,2}x^3y^2+C_{5,3}x^2y^3+C_{5,4}xy^4+C_{5,5}y^5.$$

By multiplication,

$$(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5.$$

Now $$C_{5,1}=\frac{5!}{4!\,1!}=5;$$

$$C_{5,2}=\frac{5!}{3!\,2!}=\frac{5\times 4}{2\times 1}=10;$$

$$C_{5,3}=\frac{5!}{2!\,3!}=10;$$

$$C_{5,4}=\frac{5!}{1!\,4!}=5;$$

and $$C_{5,5}=\frac{5!}{5!}=1.$$

$$\therefore\ (x+y)^5=x^5+C_{5,1}x^4y+C_{5,2}x^3y^2+C_{5,3}x^2y^3+C_{5,4}xy^4+C_{5,5}y^5.$$

11. In how many ways can a committee of three be chosen from a council of eight members?

The number of ways in which the committee can be chosen is

$$\frac{8!}{5!\,3!}=\frac{8\times 7\times 6}{3\times 2\times 1}=56.$$

12. If one of the committee of Example 11 is to be chairman, one secretary, and one treasurer, in how many ways may the committee be organized?

When any particular committee has been chosen, it can be organized in 3!, or 6, ways.

Therefore, the whole number of ways in which the committee can be organized is 56 × 6 = 336.

13. If $C_{16,r} = C_{16,r+2}$, find the value of r.

By § 488,
$$r + r + 2 = 16.$$
$$2r = 14.$$
$$\therefore r = 7.$$

14. Show that $C_{n,1} = P_{n,1}$, but that $C_{n,r} \neq P_{n,r}$ unless $r = 1$.

$$C_{n,1} = \frac{n!}{(n-1)!\,1!} = n.$$
$$P_{n,1} = \frac{n!}{(n-1)!} = n.$$
$$\therefore C_{n,1} = P_{n,1}.$$

Again,
$$C_{n,r} = \frac{n!}{(n-r)!\,r!},\text{ and } P_{n,r} = \frac{n!}{(n-r)!}.$$

Therefore, $C_{n,r} \neq P_{n,r}$ unless $r! = 1$; that is, unless $r = 1$.

15. Make up a problem of your own in combinations or permutations and then solve it.

Exercise 143. Page 403

1. Expand $(a - 3b)^5$.

$$\begin{aligned}(a-3b)^5 &= a^5 - 5a^4(3b) + 10a^3(3b)^2 - 10a^2(3b)^3 + 5a(3b)^4 - (3b)^5\\ &= a^5 - 15a^4b + 90a^3b^2 - 270a^2b^3 + 405ab^4 - 243b^5.\end{aligned}$$

2. Expand $(2x - 5y)^6$.

$$\begin{aligned}(2x-5y)^6 &= (2x)^6 - 6(2x)^5(5y) + 15(2x)^4(5y)^2 - 20(2x)^3(5y)^3\\ &\quad + 15(2x)^2(5y)^4 - 6(2x)(5y)^5 + (5y)^6\\ &= 64x^6 - 960x^5y + 6000x^4y^2 - 20{,}000x^3y^3 + 37{,}500x^2y^4\\ &\quad - 37{,}500xy^5 + 15{,}625y^6.\end{aligned}$$

3. Expand $\left(\frac{1}{x} - \frac{x^2}{2}\right)^4$.

$$\begin{aligned}\left(\frac{1}{x} - \frac{x^2}{2}\right)^4 &= \left(\frac{1}{x}\right)^4 - 4\left(\frac{1}{x}\right)^3\left(\frac{x^2}{2}\right) + 6\left(\frac{1}{x}\right)^2\left(\frac{x^2}{2}\right)^2\\ &\quad - 4\left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right)^3 + \left(\frac{x^2}{2}\right)^4\\ &= \frac{1}{x^4} - \frac{2}{x} + \frac{3x^2}{2} - \frac{x^5}{2} + \frac{x^8}{16}.\end{aligned}$$

4. Expand $(3-2x^2)^7$.

$$\begin{aligned}(3-2x^2)^7 &= 3^7 - 7(3)^6(2x^2) + 21(3)^5(2x^2)^2\\ &\quad - 35(3)^4(2x^2)^3 + 35(3)^3(2x^2)^4\\ &\quad - 21(3)^2(2x^2)^5 + 7(3)(2x^2)^6 - (2x^2)^7\\ &= 2187 - 10{,}206x^2 + 20{,}412x^4 - 22{,}680x^6\\ &\quad + 15{,}120x^8 - 6048x^{10} + 1344x^{12} - 128x^{14}.\end{aligned}$$

5. Expand $(1+2a)^8$.

$$\begin{aligned}(1+2a)^8 &= 1^8 + 8(1)^7(2a) + 28(1)^6(2a)^2 + 56(1)^5(2a)^3\\ &\quad + 70(1)^4(2a)^4 + 56(1)^3(2a)^5 + 28(1)^2(2a)^6\\ &\quad + 8(1)(2a)^7 + (2a)^8\\ &= 1 + 16a + 112a^2 + 448a^3 + 1120a^4 + 1792a^5\\ &\quad + 1792a^6 + 1024a^7 + 256a^8.\end{aligned}$$

6. Expand $\left(\frac{2x^2}{y} + \frac{y^2}{3x}\right)^5$.

$$\begin{aligned}&\left(\frac{2x^2}{y} + \frac{y^2}{3x}\right)^5\\ &= \left(\frac{2x^2}{y}\right)^5 + 5\left(\frac{2x^2}{y}\right)^4\left(\frac{y^2}{3x}\right)\\ &\quad + 10\left(\frac{2x^2}{y}\right)^3\left(\frac{y^2}{3x}\right)^2 + 10\left(\frac{2x^2}{y}\right)^2\left(\frac{y^2}{3x}\right)^3\\ &\quad + 5\left(\frac{2x^2}{y}\right)\left(\frac{y^2}{3x}\right)^4 + \left(\frac{y^2}{3x}\right)^5\\ &= \frac{32x^{10}}{y^5} + \frac{80x^7}{3y^2} + \frac{80x^4y}{9} + \frac{40xy^4}{27} + \frac{10y^7}{81x^2} + \frac{y^{10}}{243x^5}.\end{aligned}$$

7. Expand $(2x+3y)^9$.

$$\begin{aligned}(2x+3y)^9 &= (2x)^9 + 9(2x)^8(3y) + 36(2x)^7(3y)^2\\ &\quad + 84(2x)^6(3y)^3 + 126(2x)^5(3y)^4\\ &\quad + 126(2x)^4(3y)^5 + 84(2x)^3(3y)^6\\ &\quad + 36(2x)^2(3y)^7 + 9(2x)(3y)^8 + (3y)^9\\ &= 512x^9 + 6912x^8y + 41{,}472x^7y^2 + 145{,}152x^6y^3\\ &\quad + 326{,}592x^5y^4 + 489{,}888x^4y^5 + 489{,}888x^3y^6\\ &\quad + 314{,}928x^2y^7 + 118{,}098xy^8 + 19{,}683y^9.\end{aligned}$$

8. Expand $(4x+y)^{10}$.

$$\begin{aligned}(4x+y)^{10} &= (4x)^{10} + 10(4x)^9 y + 45(4x)^8y^2 + 120(4x)^7y^3 \\ &\quad + 210(4x)^6y^4 + 252(4x)^5y^5 + 210(4x)^4y^6 \\ &\quad + 120(4x)^3y^7 + 45(4x)^2y^8 + 10(4x)y^9 + y^{10} \\ &= 1{,}048{,}576\,x^{10} + 2{,}621{,}440\,x^9y + 2{,}949{,}120\,x^8y^2 \\ &\quad + 1{,}966{,}080\,x^7y^3 + 860{,}160\,x^6y^4 \\ &\quad + 258{,}048\,x^5y^5 + 53{,}760\,x^4y^6 + 7680\,x^3y^7 \\ &\quad + 720\,x^2y^8 + 40\,xy^9 + y^{10}.\end{aligned}$$

9. Expand $\left(\frac{3x}{2a} - \frac{y}{4b}\right)^6$.

$$\begin{aligned}\left(\frac{3x}{2a} - \frac{y}{4b}\right)^6 &= \left(\frac{3x}{2a}\right)^6 - 6\left(\frac{3x}{2a}\right)^5\left(\frac{y}{4b}\right) + 15\left(\frac{3x}{2a}\right)^4\left(\frac{y}{4b}\right)^2 \\ &\quad - 20\left(\frac{3x}{2a}\right)^3\left(\frac{y}{4b}\right)^3 + 15\left(\frac{3x}{2a}\right)^2\left(\frac{y}{4b}\right)^4 \\ &\quad - 6\left(\frac{3x}{2a}\right)\left(\frac{y}{4b}\right)^5 + \left(\frac{y}{4b}\right)^6 \\ &= \frac{729x^6}{64a^6} - \frac{729x^5y}{64a^5b} + \frac{1215x^4y^2}{256a^4b^2} - \frac{135x^3y^3}{128a^3b^3} \\ &\quad + \frac{135x^2y^4}{1024a^2b^4} - \frac{9xy^5}{1024ab^5} + \frac{y^6}{4096b^6}.\end{aligned}$$

10. Expand $\left(\sqrt{\frac{x}{2}} + \sqrt[3]{\frac{y}{3}}\right)^7$.

$$\begin{aligned}\left(\sqrt{\frac{x}{2}} + \sqrt[3]{\frac{y}{3}}\right)^7 &= \left(\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}} + \frac{y^{\frac{1}{3}}}{3^{\frac{1}{3}}}\right)^7 \\ &= \left(\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right)^7 + 7\left(\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right)^6\left(\frac{y^{\frac{1}{3}}}{3^{\frac{1}{3}}}\right) + 21\left(\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right)^5\left(\frac{y^{\frac{1}{3}}}{3^{\frac{1}{3}}}\right)^2 \\ &\quad + 35\left(\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right)^4\left(\frac{y^{\frac{1}{3}}}{3^{\frac{1}{3}}}\right)^3 + 35\left(\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right)^3\left(\frac{y^{\frac{1}{3}}}{3^{\frac{1}{3}}}\right)^4 \\ &\quad + 21\left(\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right)^2\left(\frac{y^{\frac{1}{3}}}{3^{\frac{1}{3}}}\right)^5 + 7\left(\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right)\left(\frac{y^{\frac{1}{3}}}{3^{\frac{1}{3}}}\right)^6 + \left(\frac{y^{\frac{1}{3}}}{3^{\frac{1}{3}}}\right)^7 \\ &= \frac{x^{\frac{7}{2}}}{2^{\frac{7}{2}}} + \frac{7x^3y^{\frac{1}{3}}}{2^3 \times 3^{\frac{1}{3}}} + \frac{21x^{\frac{5}{2}}y^{\frac{2}{3}}}{2^{\frac{5}{2}} \times 3^{\frac{2}{3}}} + \frac{35x^2y}{2^2 \times 3} + \frac{35x^{\frac{3}{2}}y^{\frac{4}{3}}}{2^{\frac{3}{2}} \times 3^{\frac{4}{3}}} \\ &\quad + \frac{21xy^{\frac{5}{3}}}{2 \times 3^{\frac{5}{3}}} + \frac{7x^{\frac{1}{2}}y^2}{2^{\frac{1}{2}} \times 3^2} + \frac{y^{\frac{7}{3}}}{3^{\frac{7}{3}}}.\end{aligned}$$

11. Expand $\left(\sqrt[3]{\frac{x^2}{2}}-\frac{1}{2\sqrt{x}}\right)^8$.

$$\left(\sqrt[3]{\frac{x^2}{2}}-\frac{1}{2\sqrt{x}}\right)^8$$

$$=\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}-\frac{1}{2x^{\frac{1}{2}}}\right)^8$$

$$=\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)^8-8\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)^7\left(\frac{1}{2x^{\frac{1}{2}}}\right)+28\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)^6\left(\frac{1}{2x^{\frac{1}{2}}}\right)^2$$

$$-56\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)^5\left(\frac{1}{2x^{\frac{1}{2}}}\right)^3+70\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)^4\left(\frac{1}{2x^{\frac{1}{2}}}\right)^4$$

$$-56\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)^3\left(\frac{1}{2x^{\frac{1}{2}}}\right)^5+28\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)^2\left(\frac{1}{2x^{\frac{1}{2}}}\right)^6$$

$$-8\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)\left(\frac{1}{2x^{\frac{1}{2}}}\right)^7+\left(\frac{1}{2x^{\frac{1}{2}}}\right)^8$$

$$=\frac{x^{\frac{16}{3}}}{2^{\frac{8}{3}}}-\frac{x^{\frac{25}{6}}}{2^{\frac{1}{3}}}+\frac{7x^3}{4}-\frac{7x^{\frac{11}{6}}}{2^{\frac{5}{3}}}+\frac{35x^{\frac{2}{3}}}{2^{\frac{13}{3}}}-\frac{7}{8x^{\frac{1}{2}}}$$

$$+\frac{7}{2^{\frac{14}{3}}x^{\frac{5}{3}}}-\frac{1}{2^{\frac{13}{3}}x^{\frac{17}{6}}}+\frac{1}{2^8x^4}.$$

12. Expand $(1-2x-3x^2)^4$.

$$(1-2x-3x^2)^4$$

$$=[(1-2x)-3x^2]^4$$

$$=(1-2x)^4-4(1-2x)^3(3x^2)+6(1-2x)^2(3x^2)^2$$

$$-4(1-2x)(3x^2)^3+(3x^2)^4$$

$$=[1^4-4(1)^3(2x)+6(1)^2(2x)^2-4(1)(2x)^3+(2x)^4]$$

$$-4[1^3-3(1)^2(2x)+3(1)(2x)^2-(2x)^3](3x^2)$$

$$+6[1^2-2(1)(2x)+(2x)^2](9x^4)$$

$$-4(1-2x)(27x^6)+81x^8$$

$$=1-8x+24x^2-32x^3+16x^4$$

$$-4(1-6x+12x^2-8x^3)(3x^2)$$

$$+6(1-4x+4x^2)(9x^4)-108x^6+216x^7+81x^8$$

$$=1-8x+24x^2-32x^3+16x^4-12x^2+72x^3$$

$$-144x^4+96x^5+54x^4-216x^5+216x^6$$

$$-108x^6+216x^7+81x^8$$

$$=1-8x+12x^2+40x^3-74x^4-120x^5$$

$$+108x^6+216x^7+81x^8.$$

13. Expand $\left(\sqrt{-1}+\dfrac{\sqrt[3]{x}}{2\sqrt{-1}}\right)^5$.

$$\begin{aligned}
&\left(\sqrt{-1}+\frac{\sqrt[3]{x}}{2\sqrt{-1}}\right)^5\\
&\quad=(\sqrt{-1})^5+5(\sqrt{-1})^4\left(\frac{\sqrt[3]{x}}{2\sqrt{-1}}\right)\\
&\qquad+10(\sqrt{-1})^3\left(\frac{\sqrt[3]{x}}{2\sqrt{-1}}\right)^2+10(\sqrt{-1})^2\left(\frac{\sqrt[3]{x}}{2\sqrt{-1}}\right)^3\\
&\qquad+5(\sqrt{-1})\left(\frac{\sqrt[3]{x}}{2\sqrt{-1}}\right)^4+\left(\frac{\sqrt[3]{x}}{2\sqrt{-1}}\right)^5\\
&\quad=\sqrt{-1}+\frac{5\sqrt[3]{x}}{2\sqrt{-1}}+\frac{5}{2}\sqrt{-1}.\sqrt[3]{x^2}+\frac{5x}{4\sqrt{-1}}\\
&\qquad+\frac{5x\sqrt{-1}\sqrt[3]{x}}{16}+\frac{x\sqrt[3]{x^2}}{32\sqrt{-1}}\\
&\quad=\sqrt{-1}-\tfrac{5}{2}\sqrt[3]{x}\sqrt{-1}+\tfrac{5}{2}\sqrt[3]{x^2}\sqrt{-1}-\tfrac{5}{4}x\sqrt{-1}\\
&\qquad+\tfrac{5}{16}x\sqrt[3]{x}\sqrt{-1}-\tfrac{1}{32}x\sqrt[3]{x^2}\sqrt{-1}\\
&\quad=(1-\tfrac{5}{2}\sqrt[3]{x}+\tfrac{5}{2}\sqrt[3]{x^2}-\tfrac{5}{4}x+\tfrac{5}{16}x\sqrt[3]{x}-\tfrac{1}{32}x\sqrt[3]{x^2})\sqrt{-1}.
\end{aligned}$$

14. Expand $\left(\dfrac{a\sqrt{-1}}{3}+\dfrac{1}{b\sqrt{-1}}\right)^6$.

$$\begin{aligned}
&\left(\frac{a\sqrt{-1}}{3}+\frac{1}{b\sqrt{-1}}\right)^6\\
&\quad=\left(\frac{ai}{3}+\frac{1}{bi}\right)^6\\
&\quad=\left(\frac{ai}{3}\right)^6+6\left(\frac{ai}{3}\right)^5\left(\frac{1}{bi}\right)+15\left(\frac{ai}{3}\right)^4\left(\frac{1}{bi}\right)^2\\
&\qquad+20\left(\frac{ai}{3}\right)^3\left(\frac{1}{bi}\right)^3+15\left(\frac{ai}{3}\right)^2\left(\frac{1}{bi}\right)^4+6\left(\frac{ai}{3}\right)\left(\frac{1}{bi}\right)^5+\left(\frac{1}{bi}\right)^6\\
&\quad=\frac{a^6i^6}{729}+\frac{2a^5i^4}{81b}+\frac{5a^4i^2}{27b^2}+\frac{20a^3}{27b^3}+\frac{5a^2}{3b^4i^2}+\frac{2a}{b^5i^4}+\frac{1}{b^6i^6}\\
&\quad=\frac{a^6(-1)}{729}+\frac{2a^5(+1)}{81b}+\frac{5a^4(-1)}{27b^2}+\frac{20a^3}{27b^3}\\
&\qquad+\frac{5a^2}{3b^4(-1)}+\frac{2a}{b^5(+1)}+\frac{1}{b^6(-1)}\\
&\quad=-\frac{a^6}{729}+\frac{2a^5}{81b}-\frac{5a^4}{27b^2}+\frac{20a^3}{27b^3}-\frac{5a^2}{3b^4}+\frac{2a}{b^5}-\frac{1}{b^6}.
\end{aligned}$$

15. Expand $(a^3 - 2a^2b + 3ab^2 - 4b^3)^3$.

$$\begin{aligned}
(a^3 - 2a^2b &+ 3ab^2 - 4b^3)^3 \\
&= [a^2(a-2b) + b^2(3a-4b)]^3 \\
&= [a^2(a-2b)]^3 + 3[\{a^2(a-2b)\}^2\{b^2(3a-4b)\}] \\
&\quad + 3[\{a^2(a-2b)\}\{b^2(3a-4b)\}^2] + [b^2(3a-4b)]^3 \\
&= a^6(a-2b)^3 + 3a^4b^2(a-2b)^2(3a-4b) \\
&\quad + 3a^2b^4(a-2b)(3a-4b)^2 + b^6(3a-4b)^3 \\
&= a^6[a^3 - 3(a)^2(2b) + 3(a)(2b)^2 - (2b)^3] \\
&\quad + 3a^4b^2[a^2 - 2(a)(2b) + (2b)^2](3a-4b) \\
&\quad + 3a^2b^4(a-2b)[(3a)^2 - 2(3a)(4b) + (4b)^2] \\
&\quad + b^6[(3a)^3 - 3(3a)^2(4b) + 3(3a)(4b)^2 - (4b)^3] \\
&= a^6(a^3 - 6a^2b + 12ab^2 - 8b^3) + 3a^4b^2(a^2 - 4ab + 4b^2)(3a-4b) \\
&\quad + 3a^2b^4(a-2b)(9a^2 - 24ab + 16b^2) \\
&\quad + b^6(27a^3 - 108a^2b + 144ab^2 - 64b^3) \\
&= a^9 - 6a^8b + 12a^7b^2 - 8a^6b^3 + 9a^7b^2 - 48a^6b^3 \\
&\quad + 84a^5b^4 - 48a^4b^5 + 27a^5b^4 - 126a^4b^5 \\
&\quad + 192a^3b^6 - 96a^2b^7 + 27a^3b^6 - 108a^2b^7 + 144ab^8 - 64b^9 \\
&= a^9 - 6a^8b + 21a^7b^2 - 56a^6b^3 + 111a^5b^4 - 174a^4b^5 \\
&\quad + 219a^3b^6 - 204a^2b^7 + 144ab^8 - 64b^9.
\end{aligned}$$

16. Find the third term of $(2x - \frac{1}{2}y)^8$.

Here $n = 8$, and $r = 2$.

The third term of $(2x - \frac{1}{2}y)^8 = \frac{8 \times 7}{1 \times 2}(2x)^6(\frac{1}{2}y)^2 = 448x^6y^2$.

17. Find the sixteenth term of $(3x - 2y)^{18}$.

The coefficient of the sixteenth term, or fourth term from the end, is the same as the coefficient of the fourth term.

The sixteenth term of $(3x - 2y)^{18}$

$$= -\frac{18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3}(3x)^3(2y)^{15} = -721,944,576\,x^3y^{15}.$$

18. Find the thirty-second term of $(3x - \frac{1}{3}y)^{37}$.

The thirty-second term of $(3x - \frac{1}{3}y)^{37}$

$$= -\frac{37 \cdot 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(3x)^6(\tfrac{1}{3}y)^{31} = -\frac{774928}{3^{24}}x^6y^{31}.$$

19. **Find the middle term of $(3a - 2b)^{12}$.**

In this case the middle term is the seventh term.
The middle term of $(3a - 2b)^{12}$

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(3a)^6(2b)^6 = 924 \cdot 3^6 \cdot 2^6 a^6 b^6.$$

20. **Find the fifth term of $\left(x + \frac{1}{2x}\right)^9$.**

The fifth term of $\left(x + \frac{1}{2x}\right)^9 = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}x^5\left(\frac{1}{2x}\right)^4 = \frac{63}{8}x.$

21. **Find the eighth term of $\left(x - \frac{1}{4x^2}\right)^{11}$.**

The eighth term of $\left(x - \frac{1}{4x^2}\right)^{11}$

$$= -\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}x^4\left(\frac{1}{4x^2}\right)^7 = -\frac{165}{2^{13}x^{10}}.$$

22. **Find the $(r + 1)$th term of $\left(\frac{3}{4x} - \sqrt[3]{\frac{x^2}{2}}\right)^{15}$.**

The $(r + 1)$th term of $\left(\frac{3}{4x} - \sqrt[3]{\frac{x^2}{2}}\right)^{15}$

$$= \pm\frac{15 \cdot 14 \cdot 13 \cdot 12 \cdots (16 - r)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots r}\left(\frac{3}{4x}\right)^{15-r}\left(\sqrt[3]{\frac{x^2}{2}}\right)^r$$

$$= \pm\frac{15 \cdot 14 \cdot 13 \cdot 12 \cdots (16 - r)}{r!}\left(\frac{3}{2^2x}\right)^{15-r}\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)^r$$

$$= \pm\frac{15 \cdot 14 \cdot 13 \cdot 12 \cdots (16 - r)}{r!} \cdot \frac{3^{15-r}}{2^{30-2r}x^{15-r}} \cdot \frac{x^{\frac{2r}{3}}}{2^{\frac{r}{3}}}$$

$$= \pm\frac{15 \cdot 14 \cdot 13 \cdot 12 \cdots (16 - r)}{r!} \cdot 3^{15-r} \cdot 2^{\frac{5r}{3}-30}x^{\frac{5r}{3}-15}.$$

The sign of the $(r + 1)$th term is positive or negative according as r is even or odd.

23. Find the two middle terms of $\left(\frac{a}{\sqrt{x}} - \frac{b}{\sqrt[3]{y}}\right)^{21}$.

The two middle terms are the eleventh and the twelfth. The coefficients of these terms are

$$\frac{21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}.$$

Hence the terms required are

$$\frac{21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}\left(\frac{a}{\sqrt{x}}\right)^{11}\left(\frac{b}{\sqrt[3]{y}}\right)^{10}$$

and
$$-\frac{21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}\left(\frac{a}{\sqrt{x}}\right)^{10}\left(\frac{b}{\sqrt[3]{y}}\right)^{11},$$

or
$$352{,}716\,\frac{a^{11}b^{10}}{x^{\frac{11}{2}}y^{\frac{10}{3}}} \text{ and } -352{,}716\,\frac{a^{10}b^{11}}{x^{5}y^{\frac{11}{3}}}.$$

24. Expand $(a^{\frac{1}{2}} + b^{\frac{1}{3}})^5$.

$$\begin{aligned}
&(a^{\frac{1}{2}} + b^{\frac{1}{3}})^5 \\
&\quad = (a^{\frac{1}{2}})^5 + 5(a^{\frac{1}{2}})^4(b^{\frac{1}{3}}) + 10(a^{\frac{1}{2}})^3(b^{\frac{1}{3}})^2 + 10(a^{\frac{1}{2}})^2(b^{\frac{1}{3}})^3 \\
&\qquad + 5(a^{\frac{1}{2}})(b^{\frac{1}{3}})^4 + (b^{\frac{1}{3}})^5 \\
&\quad = a^{\frac{5}{2}} + 5a^2b^{\frac{1}{3}} + 10a^{\frac{3}{2}}b^{\frac{2}{3}} + 10ab + 5a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}}.
\end{aligned}$$

25. Expand $(x^{-2} - y^{-3})^6$.

$$\begin{aligned}
&(x^{-2} - y^{-3})^6 \\
&\quad = (x^{-2})^6 - 6(x^{-2})^5(y^{-3}) + 15(x^{-2})^4(y^{-3})^2 - 20(x^{-2})^3(y^{-3})^3 \\
&\qquad + 15(x^{-2})^2(y^{-3})^4 - 6(x^{-2})(y^{-3})^5 + (y^{-3})^6 \\
&\quad = x^{-12} - 6x^{-10}y^{-3} + 15x^{-8}y^{-6} - 20x^{-6}y^{-9} + 15x^{-4}y^{-12} \\
&\qquad - 6x^{-2}y^{-15} + y^{-18}.
\end{aligned}$$

26. Expand $(m^{-\frac{1}{2}} - n^2)^5$.

$$\begin{aligned}
&(m^{-\frac{1}{2}} - n^2)^5 \\
&\quad = (m^{-\frac{1}{2}})^5 - 5(m^{-\frac{1}{2}})^4(n^2) + 10(m^{-\frac{1}{2}})^3(n^2)^2 - 10(m^{-\frac{1}{2}})^2(n^2)^3 \\
&\qquad + 5(m^{-\frac{1}{2}})(n^2)^4 - (n^2)^5 \\
&\quad = m^{-\frac{5}{2}} - 5m^{-2}n^2 + 10m^{-\frac{3}{2}}n^4 - 10m^{-1}n^6 + 5m^{-\frac{1}{2}}n^8 - n^{10}.
\end{aligned}$$

27. Expand $(a^{-3} + b^{-\frac{2}{3}})^5$.

$$\begin{aligned}(a^{-3} + b^{-\frac{2}{3}})^5 &= (a^{-3})^5 + 5(a^{-3})^4(b^{-\frac{2}{3}}) + 10(a^{-3})^3(b^{-\frac{2}{3}})^2 + 10(a^{-3})^2(b^{-\frac{2}{3}})^3\\ &\quad + 5(a^{-3})(b^{-\frac{2}{3}})^4 + (b^{-\frac{2}{3}})^5\\ &= a^{-15} + 5a^{-12}b^{-\frac{2}{3}} + 10a^{-9}b^{-\frac{4}{3}} + 10a^{-6}b^{-2}\\ &\quad + 5a^{-3}b^{-\frac{8}{3}} + b^{-\frac{10}{3}}.\end{aligned}$$

28. Expand $(3a^{-2} + 2b^{-3})^7$.

$$\begin{aligned}(3a^{-2} + 2b^{-3})^7 &= (3a^{-2})^7 + 7(3a^{-2})^6(2b^{-3}) + 21(3a^{-2})^5(2b^{-3})^2\\ &\quad + 35(3a^{-2})^4(2b^{-3})^3 + 35(3a^{-2})^3(2b^{-3})^4\\ &\quad + 21(3a^{-2})^2(2b^{-3})^5 + 7(3a^{-2})(2b^{-3})^6 + (2b^{-3})^7\\ &= 2187a^{-14} + 10,206a^{-12}b^{-3} + 20,412a^{-10}b^{-6}\\ &\quad + 22,680a^{-8}b^{-9} + 15,120a^{-6}b^{-12} + 6048a^{-4}b^{-15}\\ &\quad + 1344a^{-2}b^{-18} + 128b^{-21}.\end{aligned}$$

29. Expand $(x^{-\frac{3}{4}} - y^{-\frac{2}{3}})^6$.

$$\begin{aligned}(x^{-\frac{3}{4}} - y^{-\frac{2}{3}})^6 &= (x^{-\frac{3}{4}})^6 - 6(x^{-\frac{3}{4}})^5(y^{-\frac{2}{3}}) + 15(x^{-\frac{3}{4}})^4(y^{-\frac{2}{3}})^2 - 20(x^{-\frac{3}{4}})^3(y^{-\frac{2}{3}})^1\\ &\quad + 15(x^{-\frac{3}{4}})^2(y^{-\frac{2}{3}})^4 - 6(x^{-\frac{3}{4}})(y^{-\frac{2}{3}})^5 + (y^{-\frac{2}{3}})^6\\ &= x^{-\frac{9}{2}} - 6x^{-\frac{15}{4}}y^{-\frac{2}{3}} + 15x^{-3}y^{-\frac{4}{3}} - 20x^{-\frac{9}{4}}y^{-2}\\ &\quad + 15x^{-\frac{3}{2}}y^{-\frac{8}{3}} - 6x^{-\frac{3}{4}}y^{-\frac{10}{3}} + y^{-4}.\end{aligned}$$

30. Expand $(2x^{-\frac{1}{2}} + 3y^{-\frac{2}{3}})^8$.

$$\begin{aligned}(2x^{-\frac{1}{2}} + 3y^{-\frac{2}{3}})^8 &= (2x^{-\frac{1}{2}})^8 + 8(2x^{-\frac{1}{2}})^7(3y^{-\frac{2}{3}}) + 28(2x^{-\frac{1}{2}})^6(3y^{-\frac{2}{3}})^2\\ &\quad + 56(2x^{-\frac{1}{2}})^5(3y^{-\frac{2}{3}})^3 + 70(2x^{-\frac{1}{2}})^4(3y^{-\frac{2}{3}})^4\\ &\quad + 56(2x^{-\frac{1}{2}})^3(3y^{-\frac{2}{3}})^5 + 28(2x^{-\frac{1}{2}})^2(3y^{-\frac{2}{3}})^6\\ &\quad + 8(2x^{-\frac{1}{2}})(3y^{-\frac{2}{3}})^7 + (3y^{-\frac{2}{3}})^8\\ &= 256x^{-4} + 3072x^{-\frac{7}{2}}y^{-\frac{2}{3}} + 16,128x^{-3}y^{-\frac{4}{3}} + 48,384x^{-\frac{5}{2}}y^{-2}\\ &\quad + 90,720x^{-2}y^{-\frac{8}{3}} + 108,864x^{-\frac{3}{2}}y^{-\frac{10}{3}} + 81,648x^{-1}y^{-4}\\ &\quad + 34,992x^{-\frac{1}{2}}y^{-\frac{14}{3}} + 6561y^{-\frac{16}{3}}.\end{aligned}$$

31. Expand $(2x^{-2}y^{\frac{1}{2}} - y\sqrt{x})^6$.

$$\begin{aligned}
&(2x^{-2}y^{\frac{1}{2}} - y\sqrt{x})^6\\
&\quad= (2x^{-2}y^{\frac{1}{2}} - x^{\frac{1}{2}}y)^6\\
&\quad= (2x^{-2}y^{\frac{1}{2}})^6 - 6(2x^{-2}y^{\frac{1}{2}})^5(x^{\frac{1}{2}}y) + 15(2x^{-2}y^{\frac{1}{2}})^4(x^{\frac{1}{2}}y)^2\\
&\qquad - 20(2x^{-2}y^{\frac{1}{2}})^3(x^{\frac{1}{2}}y)^3 + 15(2x^{-2}y^{\frac{1}{2}})^2(x^{\frac{1}{2}}y)^4\\
&\qquad - 6(2x^{-2}y^{\frac{1}{2}})(x^{\frac{1}{2}}y)^5 + (x^{\frac{1}{2}}y)^6\\
&\quad= 64x^{-12}y^3 - 192x^{-\frac{19}{2}}y^{\frac{7}{2}} + 240x^{-7}y^4 - 160x^{-\frac{9}{2}}y^{\frac{9}{2}}\\
&\qquad + 60x^{-2}y^5 - 12x^{\frac{1}{2}}y^{\frac{11}{2}} + x^3y^6.
\end{aligned}$$

32. Expand $(3x\sqrt{-1} + 2y\sqrt{-1})^5$.

$$\begin{aligned}
&(3x\sqrt{-1} + 2y\sqrt{-1})^5\\
&\quad= [\sqrt{-1}(3x + 2y)]^5\\
&\quad= (\sqrt{-1})^5(3x + 2y)^5\\
&\quad= \sqrt{-1}[(3x)^5 + 5(3x)^4(2y) + 10(3x)^3(2y)^2 + 10(3x)^2(2y)^3\\
&\qquad + 5(3x)(2y)^4 + (2y)^5]\\
&\quad= \sqrt{-1}(243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4\\
&\qquad + 32y^5).
\end{aligned}$$

33. Expand $(\frac{1}{2}a\sqrt{-1} - \frac{1}{3}b\sqrt{-1})^8$.

$$\begin{aligned}
&(\tfrac{1}{2}a\sqrt{-1} - \tfrac{1}{3}b\sqrt{-1})^8\\
&\quad= [\sqrt{-1}(\tfrac{1}{2}a - \tfrac{1}{3}b)]^8\\
&\quad= (\sqrt{-1})^8(\tfrac{1}{2}a - \tfrac{1}{3}b)^8\\
&\quad= (\tfrac{1}{2}a - \tfrac{1}{3}b)^8\\
&\quad= (\tfrac{1}{2}a)^8 - 8(\tfrac{1}{2}a)^7(\tfrac{1}{3}b) + 28(\tfrac{1}{2}a)^6(\tfrac{1}{3}b)^2 - 56(\tfrac{1}{2}a)^5(\tfrac{1}{3}b)^3\\
&\qquad + 70(\tfrac{1}{2}a)^4(\tfrac{1}{3}b)^4 - 56(\tfrac{1}{2}a)^3(\tfrac{1}{3}b)^5 + 28(\tfrac{1}{2}a)^2(\tfrac{1}{3}b)^6\\
&\qquad - 8(\tfrac{1}{2}a)(\tfrac{1}{3}b)^7 + (\tfrac{1}{3}b)^8\\
&\quad= \tfrac{1}{256}a^8 - \tfrac{1}{48}a^7b + \tfrac{7}{144}a^6b^2 - \tfrac{7}{108}a^5b^3 + \tfrac{35}{648}a^4b^4 - \tfrac{7}{243}a^3b^5\\
&\qquad + \tfrac{7}{729}a^2b^6 - \tfrac{4}{2187}ab^7 + \tfrac{1}{6561}b^8.
\end{aligned}$$

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1. Expand $(1 + x)^{\frac{1}{2}}$ to four terms.

$$\begin{aligned}
(1 + x)^{\frac{1}{2}} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{1 \cdot 2}x^2 + \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{1 \cdot 2 \cdot 3}x^3 + \cdots\\
&= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \cdots
\end{aligned}$$

2. Expand $(1+x)^{-2}$ to four terms.

$$(1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-2-1)}{1 \cdot 2}x^2 + \frac{(-2)(-2-1)(-2-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$
$$= 1 - 2x + 3x^2 - 4x^3 + \cdots$$

3. Expand $(1+x)^{-\frac{1}{2}}$ to four terms.

$$(1+x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)x + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{1 \cdot 2}x^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$
$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots$$

4. Expand $(1-x)^{-3}$ to four terms.

$$(1-x)^{-3} = 1 - (-3)x + \frac{(-3)(-3-1)}{1 \cdot 2}x^2 - \frac{(-3)(-3-1)(-3-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$
$$= 1 + 3x + 6x^2 + 10x^3 + \cdots$$

5. Expand $(1-x)^{-\frac{3}{2}}$ to four terms.

$$(1-x)^{-\frac{3}{2}} = 1 - \left(-\frac{3}{2}\right)x + \frac{(-\frac{3}{2})(-\frac{3}{2}-1)}{1 \cdot 2}x^2 - \frac{(-\frac{3}{2})(-\frac{3}{2}-1)(-\frac{3}{2}-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$
$$= 1 + \tfrac{3}{2}x + \tfrac{15}{8}x^2 + \tfrac{35}{16}x^3 + \cdots$$

6. Expand $(1-3x)^{\frac{1}{3}}$ to four terms.

$$(1-3x)^{\frac{1}{3}} = 1 - \frac{1}{3}(3x) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1 \cdot 2}(3x)^2 - \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1 \cdot 2 \cdot 3}(3x)^3 + \cdots$$
$$= 1 - x - x^2 - \tfrac{5}{3}x^3 - \cdots$$

7. Expand $(a-bx)^{-\frac{2}{3}}$ to four terms.

$$(a-bx)^{-\frac{2}{3}} = (a)^{-\frac{2}{3}} - \left(-\frac{2}{3}\right)(a)^{-\frac{5}{3}}(bx) + \frac{-\frac{2}{3} \cdot -\frac{5}{3}}{1 \cdot 2}(a)^{-\frac{8}{3}}(bx)^2 - \frac{-\frac{2}{3} \cdot -\frac{5}{3} \cdot -\frac{8}{3}}{1 \cdot 2 \cdot 3}(a)^{-\frac{11}{3}}(bx)^3 + \cdots$$
$$= a^{-\frac{2}{3}} + \tfrac{2}{3}a^{-\frac{5}{3}}bx + \tfrac{5}{9}a^{-\frac{8}{3}}b^2x^2 + \tfrac{40}{81}a^{-\frac{11}{3}}b^3x^3 + \cdots$$

8. Expand $(2a-3b)^{\frac{2}{3}}$ to four terms.

$$(2a-3b)^{\frac{2}{3}}=(2a)^{\frac{2}{3}}-\frac{2}{3}(2a)^{-\frac{1}{3}}(3b)+\frac{\frac{2}{3}\cdot-\frac{1}{3}}{1\cdot 2}(2a)^{-\frac{4}{3}}(3b)^2$$
$$-\frac{\frac{2}{3}\cdot-\frac{1}{3}\cdot-\frac{4}{3}}{1\cdot 2\cdot 3}(2a)^{-\frac{7}{3}}(3b)^3+\cdots$$
$$=2^{\frac{2}{3}}a^{\frac{2}{3}}-2^{\frac{2}{3}}a^{-\frac{1}{3}}b-2^{-\frac{4}{3}}a^{-\frac{4}{3}}b^2-\frac{2^{-\frac{1}{3}}}{3}a^{-\frac{7}{3}}b^3-\cdots$$

9. Expand $(3x^2+2y)^{-\frac{3}{2}}$ to four terms.

$$(3x^2+2y)^{-\frac{3}{2}}$$
$$=(3x^2)^{-\frac{3}{2}}+\left(-\frac{3}{2}\right)(3x^2)^{-\frac{5}{2}}(2y)+\frac{-\frac{3}{2}\cdot-\frac{5}{2}}{1\cdot 2}(3x^2)^{-\frac{7}{2}}(2y)^2$$
$$+\frac{-\frac{3}{2}\cdot-\frac{5}{2}\cdot-\frac{7}{2}}{1\cdot 2\cdot 3}(3x^2)^{-\frac{9}{2}}(2y)^3+\cdots$$
$$=3^{-\frac{3}{2}}x^{-3}-3^{-\frac{3}{2}}x^{-5}y+\frac{5\cdot 3^{-\frac{5}{2}}}{2}x^{-7}y^2-\frac{35\cdot 3^{-\frac{9}{2}}}{2}x^{-9}y^3+\cdots$$

10. Expand $(a^2-4x^2)^{\frac{3}{4}}$ to four terms.

$$(a^2-4x^2)^{\frac{3}{4}}=(a^2)^{\frac{3}{4}}-\frac{3}{4}(a^2)^{-\frac{1}{4}}(4x^2)+\frac{\frac{3}{4}\cdot-\frac{1}{4}}{1\cdot 2}(a^2)^{-\frac{5}{4}}(4x^2)^2$$
$$-\frac{\frac{3}{4}\cdot-\frac{1}{4}\cdot-\frac{5}{4}}{1\cdot 2\cdot 3}(a^2)^{-\frac{9}{4}}(4x^2)^3+\cdots$$
$$=a^{\frac{3}{2}}-3a^{-\frac{1}{2}}x^2-\tfrac{3}{2}a^{-\frac{5}{2}}x^4-\tfrac{5}{2}a^{-\frac{9}{2}}x^6-\cdots$$

11. Expand $\sqrt[4]{3-2x}$ to four terms.

$$\sqrt[4]{3-2x}=(3-2x)^{\frac{1}{4}}$$
$$=(3)^{\frac{1}{4}}-\frac{1}{4}(3)^{-\frac{3}{4}}(2x)+\frac{\frac{1}{4}\cdot-\frac{3}{4}}{1\cdot 2}(3)^{-\frac{7}{4}}(2x)^2$$
$$-\frac{\frac{1}{4}\cdot-\frac{3}{4}\cdot-\frac{7}{4}}{1\cdot 2\cdot 3}(3)^{-\frac{11}{4}}(2x)^3+\cdots$$
$$=3^{\frac{1}{4}}-\frac{1}{2\cdot 3^{\frac{3}{4}}}x-\frac{1}{8\cdot 3^{\frac{3}{4}}}x^2-\frac{7}{16\cdot 3^{\frac{11}{4}}}x^3-\cdots$$
$$=3^{\frac{1}{4}}-\frac{3^{\frac{1}{4}}}{6}x-\frac{3^{\frac{1}{4}}}{24}x^2-\frac{7\cdot 3^{\frac{1}{4}}}{432}x^3-\cdots$$

12. Expand $\sqrt[5]{1+4a}$ to four terms.

$$\sqrt[5]{1+4a}=(1+4a)^{\frac{1}{5}}$$
$$=1+\frac{1}{5}(4a)+\frac{\frac{1}{5}\cdot-\frac{4}{5}}{1\cdot 2}(4a)^2+\frac{\frac{1}{5}\cdot-\frac{4}{5}\cdot-\frac{9}{5}}{1\cdot 2\cdot 3}(4a)^3+\cdots$$
$$=1+\tfrac{4}{5}a-\tfrac{32}{25}a^2+\tfrac{384}{125}a^3-\cdots$$

13. Expand $\sqrt[6]{3-5b}$ to four terms.

$$\begin{aligned}
\sqrt[6]{3-5b} &= (3-5b)^{\frac{1}{6}}\\
&= 3^{\frac{1}{6}} - \frac{1}{6}(3)^{-\frac{5}{6}}(5b) + \frac{\frac{1}{6}\cdot -\frac{5}{6}}{1\cdot 2}(3)^{-\frac{11}{6}}(5b)^2\\
&\quad - \frac{\frac{1}{6}\cdot -\frac{5}{6}\cdot -\frac{11}{6}}{1\cdot 2\cdot 3}(3)^{-\frac{17}{6}}(5b)^3 + \cdots\\
&= 3^{\frac{1}{6}} - \frac{5}{6\cdot 3^{\frac{5}{6}}}b - \frac{125}{72\cdot 3^{\frac{11}{6}}}b^2 - \frac{6875}{1296\cdot 3^{\frac{17}{6}}}b^3 - \cdots\\
&= 3^{\frac{1}{6}} - \frac{5\cdot 3^{\frac{1}{6}}}{18}b - \frac{125\cdot 3^{\frac{1}{6}}}{648}b^2 - \frac{6875\cdot 3^{\frac{1}{6}}}{34992}b^3 - \cdots
\end{aligned}$$

14. Expand $\sqrt[4]{5a^2+7b^3}$ to four terms.

$$\begin{aligned}
\sqrt[4]{5a^2+7b^3} &= (5a^2+7b^3)^{\frac{1}{4}}\\
&= (5a^2)^{\frac{1}{4}} + \frac{1}{4}(5a^2)^{-\frac{3}{4}}(7b^3) + \frac{\frac{1}{4}\cdot -\frac{3}{4}}{1\cdot 2}(5a^2)^{-\frac{7}{4}}(7b^3)^2\\
&\quad + \frac{\frac{1}{4}\cdot -\frac{3}{4}\cdot -\frac{7}{4}}{1\cdot 2\cdot 3}(5a^2)^{-\frac{11}{4}}(7b^3)^3 + \cdots\\
&= 5^{\frac{1}{4}}a^{\frac{1}{2}} + \frac{7}{4\cdot 5^{\frac{3}{4}}}a^{-\frac{3}{2}}b^3 - \frac{147}{32\cdot 5^{\frac{7}{4}}}a^{-\frac{7}{2}}b^6 + \frac{2401}{128\cdot 5^{\frac{11}{4}}}a^{-\frac{11}{2}}b^9 - \cdots\\
&= 5^{\frac{1}{4}}a^{\frac{1}{2}} + \frac{7\cdot 5^{\frac{1}{4}}}{20}a^{-\frac{3}{2}}b^3 - \frac{147\cdot 5^{\frac{1}{4}}}{800}a^{-\frac{7}{2}}b^6 + \frac{2401\cdot 5^{\frac{1}{4}}}{16000}a^{-\frac{11}{2}}b^9 - \cdots
\end{aligned}$$

15. Expand $(2a-5b)^{-4}$ to four terms.

$$\begin{aligned}
&(2a-5b)^{-4}\\
&\quad = (2a)^{-4} - (-4)(2a)^{-5}(5b) + \frac{(-4)(-5)}{1\cdot 2}(2a)^{-6}(5b)^2\\
&\qquad - \frac{(-4)(-5)(-6)}{1\cdot 2\cdot 3}(2a)^{-7}(5b)^3 + \cdots\\
&\quad = 2^{-4}a^{-4} + 4\cdot 2^{-5}a^{-5}\cdot 5b + 10\cdot 2^{-6}a^{-6}\cdot 25b^2\\
&\qquad + 20\cdot 2^{-7}a^{-7}\cdot 125b^3 + \cdots\\
&\quad = \tfrac{1}{16}a^{-4} + \tfrac{5}{8}a^{-5}b + \tfrac{125}{32}a^{-6}b^2 + \tfrac{625}{32}a^{-7}b^3 + \cdots
\end{aligned}$$

16. Expand $(3x+4y)^{-3}$ to four terms.

$$\begin{aligned}
&(3x+4y)^{-3}\\
&\quad = (3x)^{-3} + (-3)(3x)^{-4}(4y) + \frac{(-3)(-4)}{1\cdot 2}(3x)^{-5}(4y)^2\\
&\qquad + \frac{(-3)(-4)(-5)}{1\cdot 2\cdot 3}(3x)^{-6}(4y)^3 + \cdots\\
&\quad = 3^{-3}x^{-3} - 3\cdot 3^{-4}x^{-4}\cdot 4y + 6\cdot 3^{-5}x^{-5}\cdot 16y^2\\
&\qquad - 10\cdot 3^{-6}x^{-6}\cdot 64y^3 + \cdots\\
&\quad = \tfrac{1}{27}x^{-3} - \tfrac{4}{27}x^{-4}y + \tfrac{32}{81}x^{-5}y^2 - \tfrac{640}{729}x^{-6}y^3 + \cdots
\end{aligned}$$

17. Expand $(2a^2-3b^3)^{\frac{3}{4}}$ to four terms.

$$\begin{aligned}
&(2a^2-3b^3)^{\frac{3}{4}}\\
&\quad=(2a^2)^{\frac{3}{4}}-(\tfrac{3}{4})(2a^2)^{-\frac{1}{4}}(3b^3)+\frac{\frac{3}{4}\cdot-\frac{1}{4}}{1\cdot2}(2a^2)^{-\frac{5}{4}}(3b^3)^2\\
&\qquad-\frac{\frac{3}{4}\cdot-\frac{1}{4}\cdot-\frac{5}{4}}{1\cdot2\cdot3}(2a^2)^{-\frac{9}{4}}(3b^3)^3+\cdots\\
&\quad=2^{\frac{3}{4}}a^{\frac{3}{2}}-\tfrac{3}{4}\cdot2^{-\frac{1}{4}}a^{-\frac{1}{2}}\cdot3b^3-\tfrac{3}{32}\cdot2^{-\frac{5}{4}}a^{-\frac{5}{2}}\cdot9b^6\\
&\qquad-\tfrac{5}{128}\cdot2^{-\frac{9}{4}}a^{-\frac{9}{2}}\cdot27b^9-\cdots\\
&\quad=2^{\frac{3}{4}}a^{\frac{3}{2}}-\frac{9\cdot2^{\frac{3}{4}}}{8}a^{-\frac{1}{2}}b^3-\frac{27\cdot2^{\frac{3}{4}}}{128}a^{-\frac{5}{2}}b^6-\frac{135\cdot2^{\frac{3}{4}}}{1024}a^{-\frac{9}{2}}b^9-\cdots
\end{aligned}$$

18. Expand $(2a^{-2}-3b^{-3})^{-\frac{3}{4}}$ to four terms.

$$\begin{aligned}
&(2a^{-2}-3b^{-3})^{-\frac{3}{4}}\\
&\quad=(2a^{-2})^{-\frac{3}{4}}-(-\tfrac{3}{4})(2a^{-2})^{-\frac{7}{4}}(3b^{-3})\\
&\qquad+\frac{-\frac{3}{4}\cdot-\frac{7}{4}}{1\cdot2}(2a^{-2})^{-\frac{11}{4}}(3b^{-3})^2\\
&\qquad-\frac{-\frac{3}{4}\cdot-\frac{7}{4}\cdot-\frac{11}{4}}{1\cdot2\cdot3}(2a^{-2})^{-\frac{15}{4}}(3b^{-3})^3+\cdots\\
&\quad=2^{-\frac{3}{4}}a^{\frac{3}{2}}+\tfrac{3}{4}\cdot2^{-\frac{7}{4}}a^{\frac{7}{2}}\cdot3b^{-3}+\tfrac{21}{32}\cdot2^{-\frac{11}{4}}a^{\frac{11}{2}}\cdot9b^{-6}\\
&\qquad+\tfrac{77}{128}\cdot2^{-\frac{15}{4}}a^{\frac{15}{2}}\cdot27b^{-9}+\cdots\\
&\quad=\frac{2^{\frac{1}{4}}}{2}a^{\frac{3}{2}}+\frac{9\cdot2^{\frac{1}{4}}}{16}a^{\frac{7}{2}}b^{-3}+\frac{189\cdot2^{\frac{1}{4}}}{256}a^{\frac{11}{2}}b^{-6}+\frac{2079\cdot2^{\frac{1}{4}}}{2048}a^{\frac{15}{2}}b^{-9}+\cdots
\end{aligned}$$

19. Expand $\left(\dfrac{1}{2a}+\dfrac{3}{b}\right)^{\frac{1}{3}}$ to four terms.

$$\begin{aligned}
&\left(\frac{1}{2a}+\frac{3}{b}\right)^{\frac{1}{3}}\\
&\quad=\left(\frac{1}{2a}\right)^{\frac{1}{3}}+\frac{1}{3}\left(\frac{1}{2a}\right)^{-\frac{2}{3}}\left(\frac{3}{b}\right)+\frac{\frac{1}{3}\cdot-\frac{2}{3}}{1\cdot2}\left(\frac{1}{2a}\right)^{-\frac{5}{3}}\left(\frac{3}{b}\right)^2\\
&\qquad+\frac{\frac{1}{3}\cdot-\frac{2}{3}\cdot-\frac{5}{3}}{1\cdot2\cdot3}\left(\frac{1}{2a}\right)^{-\frac{8}{3}}\left(\frac{3}{b}\right)^3+\cdots\\
&\quad=\frac{1}{2^{\frac{1}{3}}a^{\frac{1}{3}}}+\frac{1}{3}\cdot2^{\frac{2}{3}}a^{\frac{2}{3}}\cdot\frac{3}{b}-\frac{1}{9}\cdot2^{\frac{5}{3}}a^{\frac{5}{3}}\cdot\frac{9}{b^2}+\frac{5}{81}\cdot2^{\frac{8}{3}}a^{\frac{8}{3}}\cdot\frac{27}{b^3}-\cdots\\
&\quad=\frac{2^{\frac{2}{3}}a^{\frac{2}{3}}}{2a}+\frac{2^{\frac{2}{3}}a^{\frac{2}{3}}}{b}-\frac{2^{\frac{5}{3}}a^{\frac{5}{3}}}{b^2}+\frac{5\cdot2^{\frac{8}{3}}a^{\frac{8}{3}}}{3b^3}-\cdots
\end{aligned}$$

20. Expand $\left(\frac{a}{3b}+\frac{2m^{\frac{1}{3}}}{x}\right)^{\frac{1}{4}}$ to four terms.

$$\left(\frac{a}{3b}+\frac{2m^{\frac{1}{3}}}{x}\right)^{\frac{1}{4}}$$

$$=\left(\frac{a}{3b}\right)^{\frac{1}{4}}+\frac{1}{4}\left(\frac{a}{3b}\right)^{-\frac{3}{4}}\left(\frac{2m^{\frac{1}{3}}}{x}\right)+\frac{\frac{1}{4}\cdot-\frac{3}{4}}{1\cdot 2}\left(\frac{a}{3b}\right)^{-\frac{7}{4}}\left(\frac{2m^{\frac{1}{3}}}{x}\right)^{2}$$

$$+\frac{\frac{1}{4}\cdot-\frac{3}{4}\cdot-\frac{7}{4}}{1\cdot 2\cdot 3}\left(\frac{a}{3b}\right)^{-\frac{11}{4}}\left(\frac{2m^{\frac{1}{3}}}{x}\right)^{3}+\cdots$$

$$=\frac{a^{\frac{1}{4}}}{3^{\frac{1}{4}}b^{\frac{1}{4}}}+\frac{1}{4}\cdot\frac{3^{\frac{3}{4}}b^{\frac{3}{4}}}{a^{\frac{3}{4}}}\cdot\frac{2m^{\frac{1}{3}}}{x}-\frac{3}{32}\cdot\frac{3^{\frac{7}{4}}b^{\frac{7}{4}}}{a^{\frac{7}{4}}}\cdot\frac{4m^{\frac{2}{3}}}{x^{2}}$$

$$+\frac{7}{128}\cdot\frac{3^{\frac{11}{4}}b^{\frac{11}{4}}}{a^{\frac{11}{4}}}\cdot\frac{8m}{x^{3}}-\cdots$$

$$=\frac{3^{\frac{3}{4}}a^{\frac{1}{4}}}{3b^{\frac{1}{4}}}+\frac{3^{\frac{3}{4}}b^{\frac{3}{4}}m^{\frac{1}{3}}}{2a^{\frac{3}{4}}x}-\frac{3^{\frac{11}{4}}b^{\frac{7}{4}}m^{\frac{2}{3}}}{8a^{\frac{7}{4}}x^{2}}+\frac{7\cdot 3^{\frac{11}{4}}b^{\frac{11}{4}}m}{16a^{\frac{11}{4}}x^{3}}-\cdots$$

21. Expand $\left(\frac{3a}{b}-\frac{2a^{2}}{x}\right)^{-\frac{2}{5}}$ to four terms.

$$\left(\frac{3a}{b}-\frac{2a^{2}}{x}\right)^{-\frac{2}{5}}$$

$$=\left(\frac{3a}{b}\right)^{-\frac{2}{5}}-\left(-\frac{2}{5}\right)\left(\frac{3a}{b}\right)^{-\frac{7}{5}}\left(\frac{2a^{2}}{x}\right)$$

$$+\frac{-\frac{2}{5}\cdot-\frac{7}{5}}{1\cdot 2}\left(\frac{3a}{b}\right)^{-\frac{12}{5}}\left(\frac{2a^{2}}{x}\right)^{2}$$

$$-\frac{-\frac{2}{5}\cdot-\frac{7}{5}\cdot-\frac{12}{5}}{1\cdot 2\cdot 3}\left(\frac{3a}{b}\right)^{-\frac{17}{5}}\left(\frac{2a^{2}}{x}\right)^{3}+\cdots$$

$$=\frac{b^{\frac{2}{5}}}{3^{\frac{2}{5}}a^{\frac{2}{5}}}+\frac{2}{5}\cdot\frac{b^{\frac{7}{5}}}{3^{\frac{7}{5}}a^{\frac{7}{5}}}\cdot\frac{2a^{2}}{x}+\frac{7}{25}\cdot\frac{b^{\frac{12}{5}}}{3^{\frac{12}{5}}a^{\frac{12}{5}}}\cdot\frac{4a^{4}}{x^{2}}$$

$$+\frac{28}{125}\cdot\frac{b^{\frac{17}{5}}}{3^{\frac{17}{5}}a^{\frac{17}{5}}}\cdot\frac{8a^{6}}{x^{3}}+\cdots$$

$$=\frac{3^{\frac{3}{5}}b^{\frac{2}{5}}}{3a^{\frac{2}{5}}}+\frac{4\cdot 3^{\frac{3}{5}}a^{\frac{3}{5}}b^{\frac{7}{5}}}{45x}+\frac{28\cdot 3^{\frac{3}{5}}a^{\frac{8}{5}}b^{\frac{12}{5}}}{675x^{2}}+\frac{224\cdot 3^{\frac{3}{5}}a^{\frac{13}{5}}b^{\frac{17}{5}}}{10125x^{3}}+\cdots$$

22. Expand $\left(\frac{4x}{3y}-\frac{3a}{4b}\right)^{-\frac{3}{4}}$ to four terms.

$$\left(\frac{4x}{3y}-\frac{3a}{4b}\right)^{-\frac{3}{4}}$$

$$=\left(\frac{4x}{3y}\right)^{-\frac{3}{4}}-\left(-\frac{3}{4}\right)\left(\frac{4x}{3y}\right)^{-\frac{7}{4}}\left(\frac{3a}{4b}\right)+\frac{-\frac{3}{4}\cdot-\frac{7}{4}}{1\cdot2}\left(\frac{4x}{3y}\right)^{-\frac{11}{4}}\left(\frac{3a}{4b}\right)^{2}$$

$$-\frac{-\frac{3}{4}\cdot-\frac{7}{4}\cdot-\frac{11}{4}}{1\cdot2\cdot3}\left(\frac{4x}{3y}\right)^{-\frac{15}{4}}\left(\frac{3a}{4b}\right)^{3}+\cdots$$

$$=\frac{3^{\frac{3}{4}}y^{\frac{3}{4}}}{2^{\frac{3}{2}}x^{\frac{3}{4}}}+\frac{3}{4}\cdot\frac{3^{\frac{7}{4}}y^{\frac{7}{4}}}{2^{\frac{7}{2}}x^{\frac{7}{4}}}\cdot\frac{3a}{4b}+\frac{21}{32}\cdot\frac{3^{\frac{11}{4}}y^{\frac{11}{4}}}{2^{\frac{11}{2}}x^{\frac{11}{4}}}\cdot\frac{9a^{2}}{16b^{2}}$$

$$+\frac{77}{128}\cdot\frac{3^{\frac{15}{4}}y^{\frac{15}{4}}}{2^{\frac{15}{2}}x^{\frac{15}{4}}}\cdot\frac{27a^{3}}{64b^{3}}+\cdots$$

$$=\frac{2^{\frac{1}{2}}\cdot3^{\frac{3}{4}}y^{\frac{3}{4}}}{4x^{\frac{3}{4}}}+\frac{27\cdot2^{\frac{1}{2}}\cdot3^{\frac{3}{4}}ay^{\frac{7}{4}}}{256bx^{\frac{7}{4}}}+\frac{1701\cdot2^{\frac{1}{2}}\cdot3^{\frac{3}{4}}a^{2}y^{\frac{11}{4}}}{32768b^{2}x^{\frac{11}{4}}}$$

$$+\frac{56133\cdot2^{\frac{1}{2}}\cdot3^{\frac{3}{4}}a^{3}y^{\frac{15}{4}}}{2097152b^{3}x^{\frac{15}{4}}}+\cdots$$

23. Expand $\left(\frac{2a^{2}}{3b^{3}}+\frac{4x^{3}}{y^{4}}\right)^{-\frac{4}{7}}$ to four terms.

$$\left(\frac{2a^{2}}{3b^{3}}+\frac{4x^{3}}{y^{4}}\right)^{-\frac{4}{7}}$$

$$=\left(\frac{2a^{2}}{3b^{3}}\right)^{-\frac{4}{7}}+\left(-\frac{4}{7}\right)\left(\frac{2a^{2}}{3b^{3}}\right)^{-\frac{11}{7}}\left(\frac{4x^{3}}{y^{4}}\right)$$

$$+\frac{-\frac{4}{7}\cdot-\frac{11}{7}}{1\cdot2}\left(\frac{2a^{2}}{3b^{3}}\right)^{-\frac{18}{7}}\left(\frac{4x^{3}}{y^{4}}\right)^{2}$$

$$+\frac{-\frac{4}{7}\cdot-\frac{11}{7}\cdot-\frac{18}{7}}{1\cdot2\cdot3}\left(\frac{2a^{2}}{3b^{3}}\right)^{-\frac{25}{7}}\left(\frac{4x^{3}}{y^{4}}\right)^{3}+\cdots$$

$$=\frac{3^{\frac{4}{7}}b^{\frac{12}{7}}}{2^{\frac{4}{7}}a^{\frac{8}{7}}}-\frac{2^{2}}{7}\cdot\frac{3^{\frac{11}{7}}b^{\frac{33}{7}}}{2^{\frac{11}{7}}a^{\frac{22}{7}}}\cdot\frac{2^{2}x^{3}}{y^{4}}+\frac{22}{49}\cdot\frac{3^{\frac{18}{7}}b^{\frac{54}{7}}}{2^{\frac{18}{7}}a^{\frac{36}{7}}}\cdot\frac{2^{4}x^{6}}{y^{8}}$$

$$-\frac{132}{343}\cdot\frac{3^{\frac{25}{7}}b^{\frac{75}{7}}}{2^{\frac{25}{7}}a^{\frac{50}{7}}}\cdot\frac{2^{6}x^{9}}{y^{12}}+\cdots$$

$$=\frac{2^{\frac{3}{7}}\cdot3^{\frac{4}{7}}b^{\frac{12}{7}}}{a^{\frac{8}{7}}}-\frac{12\cdot2^{\frac{3}{7}}\cdot3^{\frac{4}{7}}b^{\frac{33}{7}}x^{3}}{7a^{\frac{22}{7}}y^{4}}+\frac{396\cdot2^{\frac{3}{7}}\cdot3^{\frac{4}{7}}b^{\frac{54}{7}}x^{6}}{49a^{\frac{36}{7}}y^{8}}$$

$$-\frac{14256\cdot2^{\frac{3}{7}}\cdot3^{\frac{4}{7}}b^{\frac{75}{7}}x^{9}}{343a^{\frac{50}{7}}y^{12}}+\cdots$$

24. Find the sixth term of $(1-3x)^{\frac{1}{3}}$.

Here $n=\frac{1}{3}$, and $r=5$.

The sixth term of $(1-3x)^{\frac{1}{3}}$

$$=-\frac{\frac{1}{3}\cdot-\frac{2}{3}\cdot-\frac{5}{3}\cdot-\frac{8}{3}\cdot-\frac{11}{3}}{1\cdot2\cdot3\cdot4\cdot5}(3x)^5=-\frac{22}{3}x^5.$$

25. Find the tenth term of $(2a-3b)^{-\frac{3}{4}}$.

Here $n=-\frac{3}{4}$, and $r=9$.

The tenth term of $(2a-3b)^{-\frac{3}{4}}$

$$=-\frac{-\frac{3}{4}\cdot-\frac{7}{4}\cdot-\frac{11}{4}\cdot-\frac{15}{4}\cdot-\frac{19}{4}\cdot-\frac{23}{4}\cdot-\frac{27}{4}\cdot-\frac{31}{4}\cdot-\frac{35}{4}}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}(2a)^{-\frac{39}{4}}(3b)^9$$

$$=\frac{5\cdot7\cdot11\cdot19\cdot23\cdot31\cdot3^{10}\cdot2^{\frac{1}{4}}}{2^{35}}a^{-\frac{39}{4}}b^9.$$

26. Find the eighth term of $(a^2+4x^{-2})^{-\frac{5}{2}}$.

Here $n=-\frac{5}{2}$, and $r=7$.

The eighth term of $(a^2+4x^{-2})^{-\frac{5}{2}}$

$$=\frac{-\frac{5}{2}\cdot-\frac{7}{2}\cdot-\frac{9}{2}\cdot-\frac{11}{2}\cdot-\frac{13}{2}\cdot-\frac{15}{2}\cdot-\frac{17}{2}}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}(a^2)^{-\frac{19}{2}}(2^2x^{-2})^7$$

$$=-2^8\cdot3\cdot5\cdot11\cdot13\cdot17\,a^{-19}x^{-14}.$$

27. Expand $(3a^2-a-2)^{\frac{2}{5}}$ to four terms.

$$\begin{aligned}
&(3a^2-a-2)^{\frac{2}{5}}\\
&\quad=[3a^2-(a+2)]^{\frac{2}{5}}\\
&\quad=(3a^2)^{\frac{2}{5}}-\frac{2}{5}(3a^2)^{-\frac{3}{5}}(a+2)+\frac{\frac{2}{5}\cdot-\frac{3}{5}}{1\cdot2}(3a^2)^{-\frac{8}{5}}(a+2)^2\\
&\qquad-\frac{\frac{2}{5}\cdot-\frac{3}{5}\cdot-\frac{8}{5}}{1\cdot2\cdot3}(3a^2)^{-\frac{13}{5}}(a+2)^3+\cdots\\
&\quad=3^{\frac{2}{5}}a^{\frac{4}{5}}-\frac{2}{5}\cdot\frac{1}{3^{\frac{3}{5}}}a^{-\frac{6}{5}}(a+2)-\frac{3}{25}\cdot\frac{1}{3^{\frac{8}{5}}}a^{-\frac{16}{5}}(a^2+4a+4)\\
&\qquad-\frac{8}{125}\cdot\frac{1}{3^{\frac{13}{5}}}a^{-\frac{26}{5}}(a^3+6a^2+12a+8)-\cdots\\
&\quad=3^{\frac{2}{5}}a^{\frac{4}{5}}-\frac{2\cdot3^{\frac{2}{5}}}{15}a^{-\frac{1}{5}}-\frac{4\cdot3^{\frac{2}{5}}}{15}a^{-\frac{6}{5}}-\frac{3^{\frac{2}{5}}}{75}a^{-\frac{6}{5}}-\frac{4\cdot3^{\frac{2}{5}}}{75}a^{-\frac{11}{5}}-\cdots\\
&\qquad-\frac{8\cdot3^{\frac{2}{5}}}{3375}a^{-\frac{11}{5}}-\cdots\\
&\quad=3^{\frac{2}{5}}a^{\frac{4}{5}}-\frac{2\cdot3^{\frac{2}{5}}}{15}a^{-\frac{1}{5}}-\frac{7\cdot3^{\frac{2}{5}}}{25}a^{-\frac{6}{5}}-\frac{188\cdot3^{\frac{2}{5}}}{3375}a^{-\frac{11}{5}}-\cdots
\end{aligned}$$

28. **Find the sixth root of 66 to six decimal places.**

$$66 = 64 + 2 = 64\left(1 + \tfrac{2}{64}\right) = 2^6\left(1 + \tfrac{1}{32}\right).$$

$$\therefore \sqrt[6]{66} = 2\left(1 + \tfrac{1}{32}\right)^{\frac{1}{6}}$$

$$= 2\left[1 + \frac{1}{6}\left(\frac{1}{32}\right) + \frac{\frac{1}{6}\cdot -\frac{5}{6}}{1\cdot 2}\left(\frac{1}{32}\right)^2 + \frac{\frac{1}{6}\cdot -\frac{5}{6}\cdot -\frac{11}{6}}{1\cdot 2\cdot 3}\left(\frac{1}{32}\right)^3 + \frac{\frac{1}{6}\cdot -\frac{5}{6}\cdot -\frac{11}{6}\cdot -\frac{17}{6}}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{32}\right)^4 + \cdots\right]$$

$$= 2\left(1 + \tfrac{1}{192} - \tfrac{5}{73728} + \tfrac{55}{42467328} - \tfrac{935}{32614907904} + \cdots\right)$$

$$= 2(1 + 0.00520833 - 0.00006782 + 0.00000130 - 0.00000003 + \cdots)$$

$$= 2(1.00514178)$$

$$= 2.01028356.$$

Therefore, to six decimal places the sixth root of 66 is 2.010284.

29. Find the fifth root of 239 to seven decimal places.

$$239 = 243 - 4 = 243\left(1 - \tfrac{4}{243}\right) = 3^5\left(1 - \tfrac{4}{243}\right).$$

$$\therefore \sqrt[5]{239} = 3\left(1 - \tfrac{4}{243}\right)^{\frac{1}{5}}$$

$$= 3\left[1 - \frac{1}{5}\left(\frac{4}{243}\right) + \frac{\frac{1}{5}\cdot -\frac{4}{5}}{1\cdot 2}\left(\frac{4}{243}\right)^2 - \frac{\frac{1}{5}\cdot -\frac{4}{5}\cdot -\frac{9}{5}}{1\cdot 2\cdot 3}\left(\frac{4}{243}\right)^3 + \frac{\frac{1}{5}\cdot -\frac{4}{5}\cdot -\frac{9}{5}\cdot -\frac{14}{5}}{1\cdot 2\cdot 3\cdot 4}\left(\frac{4}{243}\right)^4 - \cdots\right]$$

$$= 3\left(1 - \tfrac{4}{1215} - \tfrac{32}{1476225} - \tfrac{128}{597871125} - \tfrac{1792}{726413416875} - \cdots\right)$$

$$= 3(1 - 0.003292181 - 0.000021677 - 0.000000214 - 0.000000002 - \cdots)$$

$$= 3(0.996685926)$$

$$= 2.990057778.$$

Therefore, to seven decimal places the fifth root of 239 is 2.9900578.

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1. Reduce $\dfrac{2x^4 + 3x^3 + 5x^2 + 9x - 3}{3x^4 - 2x^3 + 10x^2 - 6x + 3}$ to lowest terms.

$$\begin{array}{r|l|l}
2x^4 + 3x^3 + 5x^2 + 9x - 3 & 3x^4 - 2x^3 + 10x^2 - 6x + 3 & \\
 & 2 & \\
\hline
 & 6x^4 - 4x^3 + 20x^2 - 12x + 6 & 3 \\
 & 6x^4 + 9x^3 + 15x^2 + 27x - 9 & \\
\hline
 & -1)\,-13x^3 + 5x^2 - 39x + 15 & \\
\hline
 & 13x^3 - 5x^2 + 39x - 15 & \\
 & = x^2(13x - 5) + 3(13x - 5) & \\
 & = (x^2 + 3)(13x - 5). &
\end{array}$$

It is evident that $13x - 5$ is not a common factor.

$$2x^4 + 3x^3 + 5x^2 + 9x - 3 = (x^2 + 3)(2x^2 + 3x - 1).$$
$$3x^4 - 2x^3 + 10x^2 - 6x + 3 = (x^2 + 3)(3x^2 - 2x + 1).$$
$$\therefore \frac{2x^4 + 3x^3 + 5x^2 + 9x - 3}{3x^4 - 2x^3 + 10x^2 - 6x + 3} = \frac{(x^2 + 3)(2x^2 + 3x - 1)}{(x^2 + 3)(3x^2 - 2x + 1)} = \frac{2x^2 + 3x - 1}{3x^2 - 2x + 1}.$$

2. Reduce $\dfrac{x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5}{x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5}$ to lowest terms.

$$\frac{x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5}{x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5} = \frac{x^4(x + y) + x^2y^2(x + y) + y^4(x + y)}{x^4(x - y) + x^2y^2(x - y) + y^4(x - y)} = \frac{(x^4 + x^2y^2 + y^4)(x + y)}{(x^4 + x^2y^2 + y^4)(x - y)} = \frac{x + y}{x - y}.$$

3. Reduce $\dfrac{x^4 - x^3y - x^2y^2 + xy^3}{x^5 - x^4y - xy^4 + y^5}$ to lowest terms.

$$\frac{x^4 - x^3y - x^2y^2 + xy^3}{x^5 - x^4y - xy^4 + y^5} = \frac{x(x^3 - x^2y - xy^2 + y^3)}{(x^5 - xy^4) - (x^4y - y^5)} = \frac{x[(x^3 + y^3) - (x^2y + xy^2)]}{x(x^4 - y^4) - y(x^4 - y^4)}$$
$$= \frac{x[(x + y)(x^2 - xy + y^2) - xy(x + y)]}{(x - y)(x^4 - y^4)}$$
$$= \frac{x(x + y)(x^2 - 2xy + y^2)}{(x - y)(x^2 - y^2)(x^2 + y^2)} = \frac{x(x + y)(x - y)^2}{(x + y)(x - y)^2(x^2 + y^2)} = \frac{x}{x^2 + y^2}.$$

4. Reduce $\dfrac{(x + y)^5 - x^5 - y^5}{(x + y)^4 + x^4 + y^4}$ to lowest terms.

$$\frac{(x + y)^5 - x^5 - y^5}{(x + y)^4 + x^4 + y^4} = \frac{5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4}{(x + y)^4 - x^2y^2 + (x^2 + y^2)^2 - x^2y^2}$$
$$= \frac{5xy(x^3 + y^3 + 2x^2y + 2xy^2)}{[(x + y)^2 + xy][(x + y)^2 - xy] + (x^2 + y^2 + xy)(x^2 + y^2 - xy)}$$
$$= \frac{5xy[(x + y)(x^2 - xy + y^2) + 2xy(x + y)]}{(x^2 + 3xy + y^2)(x^2 + xy + y^2) + (x^2 + xy + y^2)(x^2 - xy + y^2)}$$

$$= \frac{5\,xy\,(x+y)\,(x^2 - xy + y^2 + 2\,xy)}{(x^2 + xy + y^2)\,(x^2 + 3\,xy + y^2 + x^2 - xy + y^2)}$$

$$= \frac{5\,xy\,(x+y)\,(x^2 + xy + y^2)}{(x^2 + xy + y^2)\,(2\,x^2 + 2\,xy + 2\,y^2)} = \frac{5\,xy\,(x+y)\,(x^2 + xy + y^2)}{2\,(x^2 + xy + y^2)^2}$$

$$= \frac{5\,xy\,(x+y)}{2\,(x^2 + xy + y^2)}.$$

5. Simplify $\dfrac{1+a}{1-a} + \dfrac{4\,a}{1+a^2} + \dfrac{8\,a}{1+a^4} - \dfrac{1-a}{1+a}$.

$$\frac{1+a}{1-a} - \frac{1-a}{1+a} = \frac{1 + 2\,a + a^2 - 1 + 2\,a - a^2}{1-a^2} = \frac{4\,a}{1-a^2}.$$

$$\frac{4\,a}{1-a^2} + \frac{4\,a}{1+a^2} = \frac{4\,a\,(1 + a^2 + 1 - a^2)}{1-a^4} = \frac{8\,a}{1-a^4}.$$

$$\frac{8\,a}{1-a^4} + \frac{8\,a}{1+a^4} = \frac{8\,a\,(1 + a^4 + 1 - a^4)}{1-a^8} = \frac{16\,a}{1-a^8}.$$

$$\therefore \frac{1+a}{1-a} + \frac{4\,a}{1+a^2} + \frac{8\,a}{1+a^4} - \frac{1-a}{1+a} = \frac{16\,a}{1-a^8}.$$

6. Simplify $\dfrac{a}{a-2} + \dfrac{a-9}{a-7} - \dfrac{a+1}{a-1} - \dfrac{a-8}{a-6}$.

By division,

$$\frac{a}{a-2} + \frac{a-9}{a-7} - \frac{a+1}{a-1} - \frac{a-8}{a-6}$$

$$= 1 + \frac{2}{a-2} + 1 - \frac{2}{a-7} - 1 - \frac{2}{a-1} - 1 + \frac{2}{a-6}$$

$$= \frac{2}{a-2} - \frac{2}{a-7} - \frac{2}{a-1} + \frac{2}{a-6}$$

$$= \frac{2\,(a-6) + 2\,(a-2)}{(a-2)\,(a-6)} - \frac{2\,(a-1) + 2\,(a-7)}{(a-1)\,(a-7)}$$

$$= \frac{2\,(2\,a-8)}{(a-2)\,(a-6)} - \frac{2\,(2\,a-8)}{(a-1)\,(a-7)}$$

$$= \frac{4\,(a-4)\,(a^2 - 8\,a + 7 - a^2 + 8\,a - 12)}{(a-1)\,(a-2)\,(a-6)\,(a-7)}$$

$$= \frac{4\,(a-4)\,(-5)}{(a-1)\,(a-2)\,(a-6)\,(a-7)}$$

$$= \frac{20\,(4-a)}{(a-1)\,(a-2)\,(a-6)\,(a-7)}.$$

7. Simplify $\dfrac{a}{a+b}+\dfrac{b}{a-b}-\dfrac{ab}{ab-b^2}+\dfrac{ab}{a^2+ab}$.

$$\frac{a}{a+b}+\frac{b}{a-b}-\frac{ab}{ab-b^2}+\frac{ab}{a^2+ab}=\frac{a}{a+b}+\frac{b}{a-b}-\frac{a}{a-b}+\frac{b}{a+b}$$

$$=\frac{a+b}{a+b}+\frac{b-a}{a-b}=1-1=0.$$

8. Simplify $\dfrac{1}{2x+2}-\dfrac{4}{x+2}+\dfrac{9}{2(x+3)}-\dfrac{x-1}{(x+2)(x+3)}$.

$$\frac{1}{2x+2}-\frac{4}{x+2}+\frac{9}{2(x+3)}-\frac{x-1}{(x+2)(x+3)}$$

$$=\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}-\frac{x-1}{(x+2)(x+3)}.$$

$$\frac{1}{2(x+1)}-\frac{4}{x+2}=\frac{x+2-8x-8}{2(x+1)(x+2)}=\frac{-7x-6}{2(x+1)(x+2)}.$$

$$\frac{9}{2(x+3)}-\frac{x-1}{(x+2)(x+3)}=\frac{9x+18-2x+2}{2(x+2)(x+3)}=\frac{7x+20}{2(x+2)(x+3)}.$$

$$\frac{-7x-6}{2(x+1)(x+2)}+\frac{7x+20}{2(x+2)(x+3)}=\frac{-7x^2-27x-18+7x^2+27x+20}{2(x+1)(x+2)(x+3)}$$

$$=\frac{2}{2(x+1)(x+2)(x+3)}=\frac{1}{(x+1)(x+2)(x+3)}.$$

9. Simplify $\dfrac{a^3+a^2b+ab^2+b^3}{a^3-a^2b-ab^2+b^3}\times\dfrac{(a+b)^2-3ab}{(a-b)^2+3ab}\times\dfrac{(a-b)^3-a^3+b^3}{(a+b)^3-a^3-b^3}$.

$$\frac{a^3+a^2b+ab^2+b^3}{a^3-a^2b-ab^2+b^3}\times\frac{(a+b)^2-3ab}{(a-b)^2+3ab}\times\frac{(a-b)^3-a^3+b^3}{(a+b)^3-a^3-b^3}$$

$$=\frac{a^2(a+b)+b^2(a+b)}{a^2(a-b)-b^2(a-b)}\times\frac{a^2+2ab+b^2-3ab}{a^2-2ab+b^2+3ab}$$

$$\times\frac{a^3-3a^2b+3ab^2-b^3-a^3+b^3}{a^3+3a^2b+3ab^2+b^3-a^3-b^3}$$

$$=\frac{(a+b)(a^2+b^2)}{(a-b)(a^2-b^2)}\times\frac{a^2-ab+b^2}{a^2+ab+b^2}\times\frac{-3a^2b+3ab^2}{3a^2b+3ab^2}$$

$$=\frac{(a+b)(a^2+b^2)}{(a-b)^2(a+b)}\times\frac{a^2-ab+b^2}{a^2+ab+b^2}\times\frac{-3ab(a-b)}{3ab(a+b)}$$

$$=-\frac{(a^2+b^2)(a^2-ab+b^2)}{(a-b)(a+b)(a^2+ab+b^2)}.$$

10. Simplify $\frac{a^2+ab+ac+bc}{ax-ay-x^2+xy} \times \frac{a^2-ax+ay-xy}{a^2+ac+ax+cx} \div \frac{a^2-a(y-b)-by}{x^2-x(y-a)-ay}$

$$\frac{a^2+ab+ac+bc}{ax-ay-x^2+xy} \times \frac{a^2-ax+ay-xy}{a^2+ac+ax+cx} \div \frac{a^2-a(y-b)-by}{x^2-x(y-a)-ay}$$

$$= \frac{a(a+b)+c(a+b)}{a(x-y)-x(x-y)} \times \frac{a(a-x)+y(a-x)}{a(a+c)+x(a+c)} \div \frac{(a-y)(a+b)}{(x+a)(x-y)}$$

$$= \frac{(a+b)(a+c)}{(a-x)(x-y)} \times \frac{(a-x)(a+y)}{(a+x)(a+c)} \times \frac{(x+a)(x-y)}{(a-y)(a+b)} = \frac{a+y}{a-y}.$$

11. Simplify $\frac{ax-ay+bx-by+a+b}{ax-ay-bx+by+a-b} \times \frac{a^2-b^2-ac+bc}{ab+ac+b^2-c^2}$.

$$\frac{ax-ay+bx-by+a+b}{ax-ay-bx+by+a-b} \times \frac{a^2-b^2-ac+bc}{ab+ac+b^2-c^2}$$

$$= \frac{(ax+bx)-(ay+by)+(a+b)}{(ax-bx)-(ay-by)+(a-b)} \times \frac{(a^2-b^2)-(ac-bc)}{(ab+ac)+(b^2-c^2)}$$

$$= \frac{x(a+b)-y(a+b)+(a+b)}{x(a-b)-y(a-b)+(a-b)} \times \frac{(a+b)(a-b)-c(a-b)}{a(b+c)+(b+c)(b-c)}$$

$$= \frac{(a+b)(x-y+1)}{(a-b)(x-y+1)} \times \frac{(a-b)(a+b-c)}{(b+c)(a+b-c)} = \frac{a+b}{b+c}.$$

12. Simplify $\frac{a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5}{a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5} \div \frac{a^2-2ab+b^2}{a^2+2ab+b^2}$.

$$\frac{a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5}{a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5} \div \frac{a^2-2ab+b^2}{a^2+2ab+b^2}$$

$$= \frac{a^3(a^2-ab+b^2)-b^3(a^2-ab+b^2)}{a^3(a^2+ab+b^2)+b^3(a^2+ab+b^2)} \div \frac{(a-b)^2}{(a+b)^2}$$

$$= \frac{(a^3-b^3)(a^2-ab+b^2)}{(a^3+b^3)(a^2+ab+b^2)} \times \frac{(a+b)^2}{(a-b)^2}$$

$$= \frac{(a-b)(a^2+ab+b^2)(a^2-ab+b^2)}{(a+b)(a^2-ab+b^2)(a^2+ab+b^2)} \times \frac{(a+b)^2}{(a-b)^2} = \frac{a+b}{a-b}.$$

13. Solve $$\frac{8x+5}{14} + \frac{7x-3}{6x+2} = \frac{4x+6}{7}.$$

Transpose, $$\frac{8x+5}{14} - \frac{4x+6}{7} = \frac{3-7x}{6x+2}.$$

Simplify the left number, $$-\frac{1}{2} = \frac{3-7x}{6x+2}.$$

$$-3x-1=3-7x.$$
$$4x=4.$$
$$\therefore x=1.$$

14. Solve $$\frac{132x+1}{3x+1}+\frac{8x+5}{x-1}=52.$$

By division, $$44-\frac{43}{3x+1}+8+\frac{13}{x-1}=52.$$
$$\frac{13}{x-1}-\frac{43}{3x+1}=0.$$
$$39x+13-43x+43=0.$$
$$-4x=-56.$$
$$\therefore x=14.$$

15. Solve $$\frac{3}{x-7}+\frac{1}{x-9}=\frac{4}{x-8}.$$

Clear of fractions,
$$3x^2-51x+216+x^2-15x+56=4x^2-64x+252.$$
$$-2x=-20.$$
$$\therefore x=10.$$

16. Solve $$\frac{ax-2a}{ax-2b}=\frac{ax-2b}{ax+2a}.$$

Clear of fractions, $$a^2x^2-4a^2=a^2x^2-4abx+4b^2.$$
$$4abx=4a^2+4b^2.$$
$$\therefore x=\frac{a^2+b^2}{ab}.$$

17. Solve $$\frac{a-x}{bc}+\frac{b-x}{ac}+\frac{c-x}{ab}=0.$$

Clear of fractions,
$$a^2-ax+b^2-bx+c^2-cx=0.$$
$$ax+bx+cx=a^2+b^2+c^2.$$
$$\therefore x=\frac{a^2+b^2+c^2}{a+b+c}.$$

18. Solve $$\frac{ax^2-bx+c}{mx^2-nx+p}=\frac{a}{m}.$$

Clear of fractions, $$amx^2-bmx+cm=amx^2-anx+ap.$$
$$anx-bmx=ap-cm.$$
$$\therefore x=\frac{ap-cm}{an-bm}.$$

19. Solve
$$\frac{b-x}{a+x}+\frac{c-x}{a-x}=\frac{a(c-2x)}{a^2-x^2}.$$

Clear of fractions,
$$(a-x)(b-x)+(a+x)(c-x)=a(c-2x).$$
$$ab-bx-ax+x^2+ac-ax+cx-x^2=ac-2ax.$$
$$bx-cx=ab.$$
$$\therefore x=\frac{ab}{b-c}.$$

20. Solve
$$\frac{x-a}{b+c}+\frac{x-b}{a+c}+\frac{x-c}{b+a}=3.$$

Transpose,
$$\frac{x-a}{b+c}-1+\frac{x-b}{a+c}-1+\frac{x-c}{b+a}-1=0.$$
$$\frac{x-a-b-c}{b+c}+\frac{x-b-a-c}{a+c}+\frac{x-c-b-a}{b+a}=0.$$
$$\frac{x-(a+b+c)}{b+c}+\frac{x-(a+b+c)}{a+c}+\frac{x-(a+b+c)}{b+a}=0.$$

This equation is satisfied if $x-(a+b+c)=0$.
$$\therefore x=a+b+c.$$

21. Solve
$$\frac{2x+a}{3(x-a)}+\frac{3x-a}{2(x+a)}=2\tfrac{1}{6}.$$
$$\frac{4x^2+6ax+2a^2+9x^2-12ax+3a^2}{6(x-a)(x+a)}=\frac{13}{6}.$$
$$\frac{13x^2-6ax+5a^2}{(x-a)(x+a)}=13.$$
$$13x^2-6ax+5a^2=13x^2-13a^2.$$
$$-6ax=-18a^2.$$
$$\therefore x=3a.$$

22. Solve
$$\frac{a}{c}+\frac{cx}{ax-b}=\frac{c}{a}+\frac{ax}{cx-b}.$$

By division,
$$\frac{a}{c}+\frac{c}{a}+\frac{bc}{a(ax-b)}=\frac{c}{a}+\frac{a}{c}+\frac{ab}{c(cx-b)}.$$
$$\frac{bc}{a(ax-b)}=\frac{ab}{c(cx-b)}.$$
$$\frac{c}{a(ax-b)}=\frac{a}{c(cx-b)}.$$

$$c^3x - bc^2 = a^3x - a^2b.$$
$$c^3x - a^3x = bc^2 - a^2b.$$
$$(c^3 - a^3)x = b(c^2 - a^2).$$
$$(c - a)(c^2 + ca + a^2)x = b(c - a)(c + a).$$
$$\therefore\ x = \frac{b(c + a)}{c^2 + ca + a^2}.$$

23. The sum of \$2.37 is divided among A, B, and C. If B is given 20 per cent more than A and 25 per cent more than C, how much does each receive?

Let $x =$ the number of cents C receives.

Then $\frac{5x}{4} =$ the number of cents B receives,

and $\frac{5}{6}$ of $\frac{5x}{4} = \frac{25x}{24} =$ the number of cents A receives.

$$\therefore\ \frac{25x}{24} + \frac{5x}{4} + x = 237.$$
$$25x + 30x + 24x = 5688.$$
$$79x = 5688.$$
$$\therefore\ x = 72.$$

Then $\frac{5x}{4} = 90,$

and $\frac{25x}{24} = 75.$

Therefore, A receives 75 cents; B, 90 cents; and C, 72 cents.

24. A man bought two pairs of shoes, and one pair cost 50 cents more than the other pair. If he had paid 50 cents less for each pair, he would have paid four fifths of what he did pay. How much did each pair cost?

Let $x =$ the number of cents he paid for the cheaper pair.

Then $x + 50 =$ the number of cents he paid for the dearer pair.

$$\therefore\ (x - 50) + (x + 50 - 50) = \tfrac{4}{5}(x + x + 50).$$
$$x - 50 + x = \tfrac{4}{5}(2x + 50).$$
$$2x - 50 = \tfrac{4}{5}(2x + 50).$$
$$10x - 250 = 8x + 200.$$
$$2x = 450.$$
$$\therefore\ x = 225.$$

Then $x + 50 = 275.$

Therefore, the shoes cost him \$2.25 and \$2.75 per pair.

25. A sum of money is divided between A and B. A receives $72 more than B, and A receives seven twelfths of the whole. What is the sum divided?

Let $x =$ the number of dollars B receives.

Then $x + 72 =$ the number of dollars A receives,

and $2x + 72 =$ the number of dollars divided.

$$\therefore\ x + 72 = \tfrac{7}{12}(2x + 72).$$

$$6x + 432 = 7x + 252.$$

$$\therefore\ x = 180.$$

Then $2x + 72 = 432.$

Therefore, the sum divided is $432.

26. A farmer bought some pigs at $2.50 each. He sold 40 of them at $2.60 each and the rest at $3.00 each. If his total gain was 10 per cent, how many pigs did he buy?

Let $x =$ the number of pigs the farmer bought.

Then $250x =$ the number of cents he paid,

and $275x =$ the number of cents he received.

$$\therefore\ 40 \times 260 + (x - 40)300 = 275x.$$

Divide by 25, $416 + 12x - 480 = 11x.$

$$\therefore\ x = 64.$$

Therefore, the farmer bought 64 pigs.

27. A boat that travels $10\frac{1}{2}$ miles an hour downstream requires three times as long to travel a certain distance up the river as down the river. Find the rate of the current.

Let $x =$ the number of miles per hour the current flows.

Then $10\frac{1}{2} - 2x =$ the number of miles per hour the boat can travel upstream,

$\frac{1}{10\frac{1}{2}} =$ the number of hours required to travel 1 mile down,

and $\frac{1}{10\frac{1}{2} - 2x} =$ the number of hours required to travel 1 mile up.

$$\therefore\ \frac{3}{10\frac{1}{2}} = \frac{1}{10\frac{1}{2} - 2x}.$$

$$\frac{2}{7} = \frac{2}{21 - 4x}.$$

$$\therefore\ 7 = 21 - 4x.$$

$$4x = 14.$$

$$\therefore\ x = 3\tfrac{1}{2}.$$

Therefore, the rate of the current is $3\frac{1}{2}$ miles per hour.

28. From the contents of a keg a man draws 1 gallon less than two fifths of the contents, and from the remainder he draws 1 gallon less than three fourths of what is left. He finds that the keg then contains 1 gallon more than one sixth of the original contents. Find the original contents.

Let $x =$ the number of gallons in the keg at first.

Then $\frac{2x}{5} - 1 =$ the number of gallons drawn the first time,

$x - \left(\frac{2x}{5} - 1\right) = \frac{3x}{5} + 1 =$ the number of gallons left after the first drawing,

$\frac{3}{4}\left(\frac{3x}{5} + 1\right) - 1 =$ the number of gallons drawn the second time,

and $\frac{3x}{5} + 1 - \left[\frac{3}{4}\left(\frac{3x}{5} + 1\right) - 1\right] = \frac{3x}{5} + 1 - \frac{9x}{20} + \frac{1}{4} = \frac{3x}{20} + \frac{5}{4}$

$=$ the number of gallons left in the keg.

$$\therefore \frac{3x}{20} + \frac{5}{4} = \frac{x}{6} + 1.$$

$$9x + 75 = 10x + 60.$$

$$\therefore x = 15.$$

Therefore, the original contents was 15 gallons.

29. Solve
$$\left.\begin{aligned} x + 4y &= 37 \\ 2x + 5y &= 53 \end{aligned}\right\} \quad \begin{matrix}(1)\\(2)\end{matrix}$$

Multiply (1) by 2, $2x + 8y = 74.$ (3)

Subtract (2) from (3), $3y = 21.$

$\therefore y = 7.$

Substitute the value of y in (1), $x + 28 = 37.$

$\therefore x = 9.$

Therefore, $x = 9$, and $y = 7$.

30. Solve
$$\left.\begin{aligned} 8x - 15y &= -30 \\ 2x + 3y &= 15 \end{aligned}\right\} \quad \begin{matrix}(1)\\(2)\end{matrix}$$

Multiply (2) by 4, $8x + 12y = 60.$ (3)

Subtract (1) from (3), $27y = 90.$

$\therefore y = 3\frac{1}{3}.$

Substitute the value of y in (2), $2x + 10 = 15.$

$2x = 5.$

$\therefore x = 2\frac{1}{2}.$

Therefore, $x = 2\frac{1}{2}$, and $y = 3\frac{1}{3}$.

31. Solve

$$\left.\begin{aligned} 5x + 7y &= 17 \\ 7x - 5y &= 9 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (1) by 7, $35x + 49y = 119.$ (3)

Multiply (2) by 5, $35x - 25y = 45.$ (4)

Subtract (4) from (3), $74y = 74.$

$\therefore y = 1.$

Substitute the value of y in (2), $7x - 5 = 9.$

$7x = 14.$

$\therefore x = 2.$

Therefore, $x = 2$, and $y = 1$.

32. Solve

$$\left.\begin{aligned} 10x + 7y &= -4 \\ 6x + 5y &= -2 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (1) by 3, $30x + 21y = -12.$ (3)

Multiply (2) by 5, $30x + 25y = -10.$ (4)

Subtract (3) from (4), $4y = 2.$

$\therefore y = \frac{1}{2}.$

Substitute the value of y in (2), $6x + 2\frac{1}{2} = -2.$

$6x = -4\frac{1}{2}.$

$\therefore x = -\frac{3}{4}.$

Therefore, $x = -\frac{3}{4}$, and $y = \frac{1}{2}$.

33. Solve

$$\left.\begin{aligned} \tfrac{2}{3}x + \tfrac{3}{5}y &= 17 \\ \tfrac{3}{4}x + \tfrac{2}{3}y &= 19 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (1) by 15, $10x + 9y = 255.$ (3)

Multiply (2) by 12, $9x + 8y = 228.$ (4)

Subtract (4) from (3), $x + y = 27.$ (5)

Multiply (5) by 8, $8x + 8y = 216.$ (6)

Subtract (6) from (4), $x = 12.$

Substitute the value of x in (5), $12 + y = 27.$

$\therefore y = 15.$

Therefore, $x = 12$, and $y = 15$.

34. Solve

$$\left.\begin{aligned} ax + by &= c \\ mx + ny &= p \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (1) by m, $amx + bmy = cm.$ (3)

Multiply (2) by a, $amx + any = ap.$ (4)

Subtract (3) from (4), $any + bmy = ap - cm.$

$$\therefore y = \frac{ap - cm}{an - bm}.$$

Multiply (1) by n, $anx + bny = cn.$ (5)

Multiply (2) by b, $bmx + bny = bp.$ (6)

Subtract (6) from (5), $anx - bmx = cn - bp.$

$$\therefore x = \frac{cn - bp}{an - bm}.$$

Therefore, $x = \dfrac{cn - bp}{an - bm}$, and $y = \dfrac{ap - cm}{an - bm}$.

35. Solve

$$\left.\begin{aligned} 3x - y &= 2(a + b)^2 \quad (1)\\ 3y - x &= 2(a - b)^2 \quad (2)\end{aligned}\right\}$$

Add (1) and (2), $2x + 2y = 4a^2 + 4b^2.$

$x + y = 2a^2 + 2b^2.$ (3)

Add (1) and (3), $4x = 4a^2 + 4ab + 4b^2.$

$\therefore x = a^2 + ab + b^2.$

Add (2) and (3), $4y = 4a^2 - 4ab + 4b^2.$

$\therefore y = a^2 - ab + b^2.$

Therefore, $x = a^2 + ab + b^2$, and $y = a^2 - ab + b^2$.

36. Solve

$$\left.\begin{aligned} 5x - 2y &= 3(a + 7c) \quad (1)\\ 5y - 2x &= 3(a + 7b) \quad (2)\end{aligned}\right\}$$

Add (1) and (2), $3x + 3y = 6a + 21b + 21c.$ (3)

Multiply (3) by $\frac{2}{3}$, $2x + 2y = 4a + 14b + 14c.$ (4)

Add (1) and (4), $7x = 7a + 14b + 35c.$

$\therefore x = a + 2b + 5c.$

Add (2) and (4), $7y = 7a + 35b + 14c.$

$\therefore y = a + 5b + 2c.$

Therefore, $x = a + 2b + 5c$, and $y = a + 5b + 2c$.

37. Solve

$$\left.\begin{aligned} ax + bc &= by + ac \quad (1)\\ x + y &= c \quad (2)\end{aligned}\right\}$$

From (1), $ax - by = ac - bc.$ (3)

Multiply (2) by b, $bx + by = bc.$ (4)

Add (3) and (4), $ax + bx = ac.$

$$\therefore x = \frac{ac}{a + b}.$$

Multiply (2) by a, $ax + ay = ac.$ (5)

Subtract (3) from (5), $ay + by = bc.$

$$\therefore y = \frac{bc}{a + b}.$$

Therefore, $x = \dfrac{ac}{a + b}$, and $y = \dfrac{bc}{a + b}$.

38. Solve

$$\left.\begin{aligned}(a+b)x+(a+c)y&=a+b\\(a+c)x+(a+b)y&=a+c\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Add (1) and (2),

$$(2a+b+c)x+(2a+b+c)y=2a+b+c.$$

Divide by $2a+b+c$, $x+y=1.$ (3)

Subtract (2) from (1), $(b-c)x+(c-b)y=b-c.$

Divide by $b-c$, $x-y=1.$ (4)

Add (3) and (4), $2x=2.$

$\therefore x=1.$

Subtract (4) from (3), $2y=0.$

$\therefore y=0.$

Therefore, $x=1$, and $y=0$.

39. A market woman spent $1.30 for apples, some at a cent each, and the remainder at three for two cents. She sold the apples for $2.10, thereby gaining half a cent on each apple. How many at each price did she buy?

Let $x=$ the number of apples bought at a cent each,

and $y=$ the number of apples bought at three for two cents.

$$\therefore x+\frac{2y}{3}=130. \qquad (1)$$

Then $x+\frac{x}{2}=$ the number of cents received for the first kind,

and $\frac{2y}{3}+\frac{y}{2}=$ the number of cents received for the second kind.

$$\therefore x+\frac{x}{2}+\frac{2y}{3}+\frac{y}{2}=210. \qquad (2)$$

From (1), $3x+2y=390.$ (3)

From (2), $6x+3x+4y+3y=1260.$

$9x+7y=1260.$ (4)

Multiply (3) by 3, $9x+6y=1170.$ (5)

Subtract (5) from (4), $y=90.$

Substitute the value of y in (1), $x+60=130.$

$\therefore x=70.$

Therefore, the woman bought 70 apples at a cent each and 90 apples at three for two cents.

40. A power boat can travel 15 miles an hour downstream. It is found that the boat can cover a certain distance in 45 minutes in still water and the same distance in 1 hour against the current. Find the distance, the rate of the boat in still water, and the rate of the current.

Let $x =$ the number of miles per hour the boat can go in still water,
and $y =$ the number of miles per hour the current flows.

$\therefore x + y = 15,$ (1)

and $\frac{3}{4}x = x - y.$ (2)

From (2), $3x = 4x - 4y.$

$x = 4y.$ (3)

Substitute the value of x in (1), $4y + y = 15.$

$5y = 15.$

$\therefore y = 3.$

Substitute the value of y in (3), $x = 12.$

Then $x - y = 9.$

Therefore, the distance is 9 miles, the rate of the boat in still water is 12 miles an hour, and the rate of the current is 3 miles an hour.

41. There are two numbers whose sum is 3; and the quotient of the first by the second is also 3. Find the numbers.

Let $x =$ the larger number,
and $y =$ the smaller number.

$\therefore x + y = 3,$ (1)

and $\frac{x}{y} = 3.$ (2)

From (1), $x = 3 - y.$ (3)

From (2), $x = 3y.$ (4)

Equate the values of x, $3y = 3 - y.$

$4y = 3.$

$\therefore y = \frac{3}{4}.$

Substitute the value of y in (4), $x = 2\frac{1}{4}.$

Therefore, the numbers are $2\frac{1}{4}$ and $\frac{3}{4}$.

42. A man has a certain sum of money invested at a certain per cent. If he had $1000 more invested at a per cent ½ less, the interest would be $35 less; but if he had $500 less invested at a per cent ½ higher, the interest would be $50 more. Find the capital and the rate per cent.

Let $x =$ the number of dollars in the sum invested,
and $y =$ the rate per cent.

$$\therefore \left(x + 1000\right)\left(\frac{y - \frac{1}{2}}{100}\right) = \frac{xy}{100} - 35, \quad (1)$$

and
$$\left(x - 500\right)\left(\frac{y + \frac{1}{2}}{100}\right) = \frac{xy}{100} + 50. \quad (2)$$

Multiply (1) by 200, $(x + 1000)(2y - 1) = 2xy - 7000.$

$$2xy + 2000y - x - 1000 = 2xy - 7000.$$
$$x - 2000y = 6000. \quad (3)$$

Multiply (2) by 200, $(x - 500)(2y + 1) = 2xy + 10000.$

$$2xy - 1000y + x - 500 = 2xy + 10000.$$
$$x - 1000y = 10500. \quad (4)$$

Subtract (3) from (4), $1000y = 4500.$

$$\therefore y = 4\tfrac{1}{2}.$$

Substitute the value of y in (3), $x - 9000 = 6000.$

$$\therefore x = 15000.$$

Therefore, the capital is \$15,000, and the rate per cent is $4\frac{1}{2}$.

43. A and B working together can build a wall in 12 days. If A works 2 days and B works 3 days, they will build a fifth of the wall. How long will it take each alone?

Let. $x =$ the number of days it would take A alone,

and $y =$ the number of days it would take B alone.

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{12}, \quad (1)$$

and
$$\frac{2}{x} + \frac{3}{y} = \frac{1}{5}. \quad (2)$$

Multiply (1) by 2,
$$\frac{2}{x} + \frac{2}{y} = \frac{1}{6}. \quad (3)$$

Subtract (3) from (2),
$$\frac{1}{y} = \frac{1}{30}.$$
$$\therefore y = 30.$$

Substitute the value of y in (1),
$$\frac{1}{x} + \frac{1}{30} = \frac{1}{12}.$$
$$\frac{1}{x} = \frac{1}{20}.$$
$$\therefore x = 20.$$

Therefore, it would take A alone 20 days, and B alone 30 days.

44. Solve $x^2 - 17x + 60 = 0.$

Factor, $(x - 5)(x - 12) = 0.$

$$\therefore x = 5, \text{ or } 12.$$

45. Solve

$$\begin{aligned}\tfrac{6}{5}x^2+10&=7x.\\6x^2+50&=35x.\\6x^2-35x&=-50.\\144x^2-(\ )+1225&=25.\\12x-35&=\pm 5.\\12x&=40,\text{ or }30.\\\therefore\ x&=3\tfrac{1}{3},\text{ or }2\tfrac{1}{2}.\end{aligned}$$

46. Solve

$$\begin{aligned}20x^2+159&=136x.\\20x^2-136x&=-159.\\100x^2-680x+1156&=361.\\10x-34&=\pm 19.\\10x&=53,\text{ or }15.\\\therefore\ x&=5\tfrac{3}{10},\text{ or }1\tfrac{1}{2}.\end{aligned}$$

47. Solve

$$\begin{aligned}\frac{7}{2x-3}+\frac{5}{x-1}&=12.\\7x-7+10x-15&=24x^2-60x+36.\\24x^2-77x&=-58.\\4\times 24^2x^2-(\ )+5929&=361.\\48x-77&=\pm 19.\\48x&=96,\text{ or }58.\\\therefore\ x&=2,\text{ or }1\tfrac{5}{24}.\end{aligned}$$

48. Solve

$$\begin{aligned}x+5\sqrt{37-x}&=43.\\5\sqrt{37-x}&=43-x.\\\text{Square,}\quad 925-25x&=1849-86x+x^2.\\x^2-61x+924&=0.\\(x-28)(x-33)&=0.\\\therefore\ x&=28,\text{ or }33\end{aligned}$$

49. Solve

$$\begin{aligned}\sqrt{x+1}+\sqrt{x-4}&=\sqrt{2x+9}\\\text{Square.}\quad x+1+2\sqrt{x^2-3x-4}+x-4&=2x+9.\\2\sqrt{x^2-3x-4}&=12.\\\sqrt{x^2-3x-4}&=6.\\x^2-3x-4&=36.\\x^2-3x-40&=0.\\(x-8)(x+5)&=0.\\\therefore\ x&=8,\text{ or }-5.\end{aligned}$$

But the value $x=-5$ does not satisfy the given equation.

$$\therefore\ x=8.$$

50. Solve

$$3x - 4\sqrt{x-7} = 2(x+2).$$
$$3x - 4\sqrt{x-7} = 2x + 4.$$
$$x - 4 = 4\sqrt{x-7}.$$
$$x^2 - 8x + 16 = 16x - 112.$$
$$x^2 - 24x + 128 = 0.$$
$$(x-8)(x-16) = 0.$$
$$\therefore\ x = 8, \text{ or } 16.$$

51. Solve

$$a^2 - x^2 = (a-x)(b+c-x).$$
$$(a^2 - x^2) - (a-x)(b+c-x) = 0.$$
$$(a-x)(a+x) - (a-x)(b+c-x) = 0.$$
$$(a-x)(a+x-b-c+x) = 0.$$
$$(a-x)(2x+a-b-c) = 0.$$
$$\therefore\ x = a, \text{ or } \frac{b+c-a}{2}.$$

52. Solve

$$\frac{2a+b}{a+x} - \frac{2a-b}{a-x} = \frac{2a}{b}.$$
$$\frac{(2a+b)(a-x)-(2a-b)(a+x)}{a^2-x^2} = \frac{2a}{b}.$$
$$\frac{2ab-4ax}{a^2-x^2} = \frac{2a}{b}.$$

Divide by $2a$,

$$\frac{b-2x}{a^2-x^2} = \frac{1}{b}.$$
$$b^2 - 2bx = a^2 - x^2.$$
$$x^2 - 2bx + b^2 = a^2.$$
$$x - b = \pm a.$$
$$\therefore\ x = b + a, \text{ or } b - a.$$

53. Solve

$$\sqrt{a-x} + \sqrt{x-b} = \sqrt{a-b}.$$
$$a - x + 2\sqrt{(a-x)(x-b)} + x - b = a - b.$$
$$2\sqrt{(a-x)(x-b)} = 0.$$
$$\sqrt{(a-x)(x-b)} = 0.$$
$$(a-x)(x-b) = 0.$$
$$\therefore\ x = a, \text{ or } b.$$

54. A man drives a certain distance in 6 hours. If he had driven three quarters of it at a rate $1\frac{1}{2}$ miles an hour faster, he could have driven the remainder at a rate 3 miles an hour slower and finished in the same time. Find the distance and the rate of driving.

Let $x =$ the number of miles per hour the man drives.

Then $6x =$ the number of miles he drives.

$$\therefore \frac{\frac{3}{4}(6x)}{x+1\frac{1}{2}} + \frac{\frac{1}{4}(6x)}{x-3} = 6.$$

$$\frac{9x}{2x+3} + \frac{3x}{2x-6} = 6.$$

$$\frac{3x}{2x+3} + \frac{x}{2x-6} = 2.$$

$$6x^2 - 18x + 2x^2 + 3x = 8x^2 - 12x - 36.$$

$$3x = 36.$$

$$\therefore x = 12.$$

Then $$6x = 72.$$

Therefore, the distance is 72 miles, and the rate of driving is 12 miles per hour.

55. Two friends, A and B, are 221 miles apart and set out on bicycles to meet. A travels 10 miles an hour and B travels 6 miles less per hour than the number of hours each travels. How many miles does each travel before they meet?

Let $x =$ the number of hours each travels.

Then $10x =$ the number of miles A travels,

and $x(x-6) =$ the number of miles B travels.

$$\therefore x(x-6) + 10x = 221.$$

$$x^2 - 6x + 10x = 221.$$

$$x^2 + 4x = 221.$$

$$x^2 + 4x + 4 = 225.$$

$$x + 2 = \pm 15.$$

$$\therefore x = 13, \text{ or } -17.$$

Then $$10x = 130,$$

and $$x(x-6) = 91.$$

Therefore, A travels 130 miles, and B, 91 miles.

56. The sum of a certain number and its square root is 72. Find the number.

Let $x^2 =$ the number.

$$\therefore x^2 + x = 72.$$

$$x^2 + x - 72 = 0.$$

$$(x-8)(x+9) = 0.$$

$$\therefore x = 8, \text{ or } -9.$$

Then $$x^2 = 64, \text{ or } 81.$$

Therefore, the required number is 64, or 81.

57. A man divides 120 nuts among 9 children, giving half to the boys and half to the girls. If each girl receives 3 more nuts than each boy, how many boys and how many girls are there?

Let $x =$ the number of boys.

Then $9 - x =$ the number of girls.

$$\therefore \frac{60}{9-x} - \frac{60}{x} = 3.$$

$$\frac{20}{9-x} - \frac{20}{x} = 1.$$

$$20x - 180 + 20x = 9x - x^2.$$

$$x^2 + 31x - 180 = 0.$$

$$(x-5)(x+36) = 0.$$

$$\therefore x = 5, \text{ or } -36.$$

Then $9 - x = 4$, or 45.

Therefore, there are 5 boys and 4 girls.

58. The denominator of a certain fraction is greater than the numerator by 3. If 2 is added to each term, the new fraction is $\frac{2}{21}$ greater than the given fraction. Find the fraction.

Let $x =$ the numerator.

Then $x + 3 =$ the denominator,

and $\frac{x}{x+3} =$ the fraction.

$$\therefore \frac{x+2}{x+5} = \frac{x}{x+3} + \frac{2}{21}.$$

$$\frac{x+2}{x+5} - \frac{x}{x+3} = \frac{2}{21}.$$

$$\frac{x^2 + 5x + 6 - x^2 - 5x}{(x+5)(x+3)} = \frac{2}{21}.$$

$$\frac{6}{x^2 + 8x + 15} = \frac{2}{21}.$$

$$126 = 2x^2 + 16x + 30.$$

$$63 = x^2 + 8x + 15.$$

$$x^2 + 8x - 48 = 0.$$

$$(x-4)(x+12) = 0.$$

$$\therefore x = 4, \text{ or } -12.$$

Then $x + 3 = 7$, or -9,

and $\frac{x}{x+3} = \frac{4}{7}$.

Therefore, the fraction required is $\frac{4}{7}$.

59. A tourist who has finished a journey of 540 miles would have required 2 days more if he had gone each day 3 miles less. How many days was he on his journey, and how many miles did he travel each day?

Let $x =$ the number of miles per day he traveled.

Then $\frac{540}{x} =$ the number of days he traveled.

$$\therefore \frac{540}{x-3} - \frac{540}{x} = 2.$$

$$\frac{270}{x-3} - \frac{270}{x} = 1.$$

$$270x - 270x + 810 = x^2 - 3x.$$

$$x^2 - 3x - 810 = 0.$$

$$(x-30)(x+27) = 0.$$

$$\therefore x = 30, \text{ or } -27.$$

Then $$\frac{540}{x} = 18.$$

Therefore, the tourist was 18 days on his journey, and traveled 30 miles a day.

60. A body of troops was marching in solid column with 14 more men in depth than in front. When the front was increased by 828 men there were 5 full lines. Find the number of men.

Let $x =$ the number of men in front.

Then $x + 14 =$ the number of men in depth,

and $x(x+14) =$ the whole number of men.

$$\therefore x(x+14) = 5(x+828).$$

$$x^2 + 14x = 5x + 4140.$$

$$x^2 + 9x = 4140.$$

$$4x^2 + (\ ) + 81 = 16641.$$

$$2x + 9 = \pm 129.$$

$$2x = 120, \text{ or } -138.$$

$$\therefore x = 60, \text{ or } -69.$$

Then $$x + 14 = 74,$$

and $$x(x+14) = 4440.$$

Therefore, the number of men was 4440.

61. Solve $\left.\begin{array}{r}5x^2+2y^2=22\\3x^2-5y^2=7\end{array}\right\}$ (1) (2)

Multiply (1) by 3, $15x^2+6y^2=66.$ (3)

Multiply (2) by 5, $15x^2-25y^2=35.$ (4)

Subtract (4) from (3), $31y^2=31.$

$$y^2=1.$$

$$\therefore\ y=\pm 1.$$

Substitute the value of y in (2), $3x^2-5=7.$

$$3x^2=12.$$

$$x^2=4.$$

$$\therefore\ x=\pm 2.$$

Therefore, $x=\ \ 2$, and $y=\ \ 1$;
$x=\ \ 2$, and $y=-1$;
$x=-2$, and $y=\ \ 1$;
or $x=-2$, and $y=-1$.

62. Solve $\left.\begin{array}{r}x+xy=45\\y+xy=48\end{array}\right\}$ (1) (2)

Subtract (2) from (1), $x-y=-3.$

$$\therefore\ x=y-3. \quad (3)$$

Substitute the value of x in (2), $y+y^2-3y=48.$

$$y^2-2y-48=0.$$

$$(y-8)(y+6)=0.$$

$$\therefore\ y=8, \text{ or } -6.$$

Substitute the value of y in (3), $x=5, \text{ or } -9.$

Therefore, $x=\ \ 5$, and $y=\ \ 8$;
or $x=-9$, and $y=-6$.

63. Solve $\left.\begin{array}{r}x^2+y^2=130\\\dfrac{x+y}{x-y}=8\end{array}\right\}$ (1) (2)

From (2), $x+y=8x-8y.$

$$9y=7x.$$

$$\therefore\ y=\frac{7x}{9}. \quad (3)$$

Substitute the value of y in (1), $x^2+\dfrac{49x^2}{81}=130.$

$$81x^2+49x^2=130\times 81.$$

$$130x^2 = 130 \times 81.$$
$$x^2 = 81.$$
$$\therefore\ x = \pm 9.$$

Substitute the value of x in (3), $y = \pm 7.$

Therefore, $x = \ \ 9$, and $y = \ \ 7$;
or $x = -9$, and $y = -7$.

64. Solve
$$\left.\begin{aligned} x^2 + y^2 &= axy \\ x + y &= bxy \end{aligned}\right\} \quad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Square (2), $x^2 + 2xy + y^2 = b^2x^2y^2.$ (3)
Subtract (1) from (3), $2xy = b^2x^2y^2 - axy.$

$$b^2x^2y^2 - (a+2)xy = 0.$$
$$xy\,[b^2xy - (a+2)] = 0.$$
$$\therefore\ xy = 0, \text{ or } \frac{a+2}{b^2}. \qquad (4)$$

Substitute the value of xy in (1), $x^2 + y^2 = 0$, or $\dfrac{a(a+2)}{b^2}$. (5)

Substitute the value of xy in (2), $x + y = 0$, or $\dfrac{a+2}{b}$. (6)

Multiply (4) by 2, $2xy = 0$, or $\dfrac{2(a+2)}{b^2}$. (7)

Subtract (7) from (5), $x^2 - 2xy + y^2 = 0$, or $\dfrac{a^2-4}{b^2}$.

Extract the root, $x - y = 0$, or $\pm\dfrac{1}{b}\sqrt{a^2-4}$. (8)

Add (6) and (8), $2x = 0$, or $\dfrac{1}{b}(a + 2 \pm \sqrt{a^2-4})$.

$$\therefore\ x = 0, \text{ or } \frac{1}{2b}(a + 2 \pm \sqrt{a^2-4}).$$

Subtract (8) from (6), $2y = 0$, or $\dfrac{1}{b}(a + 2 \mp \sqrt{a^2-4})$.

$$\therefore\ y = 0, \text{ or } \frac{1}{2b}(a + 2 \mp \sqrt{a^2-4}).$$

Therefore, $x = 0$, and $y = 0$;
$x = \dfrac{1}{2b}(a + 2 + \sqrt{a^2-4})$, and $y = \dfrac{1}{2b}(a + 2 - \sqrt{a^2-4})$;
or $x = \dfrac{1}{2b}(a + 2 - \sqrt{a^2-4})$, and $y = \dfrac{1}{2b}(a + 2 + \sqrt{a^2-4})$.

65. Solve
$$\left.\begin{aligned} x &= a(x^2+y^2) \\ y &= b(x^2+y^2) \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Divide (1) by (2),
$$\frac{x}{y}=\frac{a}{b}.$$
$$bx = ay.$$
$$\therefore x = \frac{ay}{b}. \qquad (3)$$

Substitute the value of x in (2),
$$y = b\left(\frac{a^2y^2}{b^2}+y^2\right).$$
$$y = \frac{a^2y^2}{b}+by^2.$$
$$by = a^2y^2+b^2y^2.$$
$$(a^2+b^2)y^2 - by = 0.$$
$$y[(a^2+b^2)y-b] = 0.$$
$$\therefore y = 0, \text{ or } \frac{b}{a^2+b^2}.$$

Substitute the value of y in (3),
$$x = 0, \text{ or } \frac{a}{a^2+b^2}.$$

Therefore, $x = 0$, and $y = 0$;

or $$x = \frac{a}{a^2+b^2}, \text{ and } y = \frac{b}{a^2+b^2}.$$

66. Solve
$$\left.\begin{aligned} b^2x^2 &= a^2y^2 \\ a-x &= b-y \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

From (1)
$$bx = \pm ay.$$
$$\therefore x = \pm\frac{ay}{b}. \qquad (3)$$

From (2)
$$x = a-b+y. \qquad (4)$$

Equate the values of x,
$$\pm\frac{ay}{b} = a-b+y.$$
$$\pm ay = ab-b^2+by.$$
$$\pm ay - by = ab-b^2.$$
$$(\pm a-b)y = b(a-b).$$
$$\therefore y = b, \text{ or } \frac{b(b-a)}{a+b}.$$

Substitute the value of y in (4), $x = a-b+b$, or $a-b+\dfrac{b(b-a)}{a+b}$.
$$\therefore x = a, \text{ or } \frac{a(a-b)}{a+b}.$$

Therefore, $x = a$, and $y = b$;

or $$x = \frac{a(a-b)}{a+b}, \text{ and } y = \frac{b(b-a)}{a+b}.$$

67. Solve
$$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{1}{3}\\ x^2+y^2&=160\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

From (1), $3y + 3x = xy.$

$$2xy = 6x + 6y. \qquad (3)$$

Add (2) and (3), $x^2 + 2xy + y^2 = 6x + 6y + 160.$

$$(x + y)^2 = 6(x + y) + 160.$$

$$(x + y)^2 - 6(x + y) - 160 = 0.$$

Factor, $(x + y - 16)(x + y + 10) = 0.$

$$\therefore x + y = 16, \text{ or } -10. \qquad (4)$$

Substitute the value of $x + y$ in (3), $2xy = 96, \text{ or } -60.$ (5)

Subtract (5) from (2), $x^2 - 2xy + y^2 = 64, \text{ or } 220.$

$$x - y = \pm 8, \text{ or } \pm 2\sqrt{55}. \qquad (6)$$

Add (4) and (6), $2x = 24, 8, \text{ or } -10 \pm 2\sqrt{55}.$

$$\therefore x = 12,\ 4, \text{ or } -5 \pm \sqrt{55}.$$

Subtract (6) from (4), $2y = 8,\ 24, \text{ or} -10 \mp 2\sqrt{55}.$

$$\therefore y = 4,\ 12, \text{ or } -5 \mp \sqrt{55}.$$

Therefore, $x = 12$, and $y = 4$;
$x = 4$, and $y = 12$;
$x = -5 + \sqrt{55}$, and $y = -5 - \sqrt{55}$;
or $x = -5 - \sqrt{55}$, and $y = -5 + \sqrt{55}$.

68. Solve
$$\left.\begin{aligned}x(y + 1) &= 10(y - 1)\\ 2y(x + 1) &= 9(x - 1)\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

From (1), $xy + x = 10y - 10.$ (3)

From (2), $2xy + 2y = 9x - 9.$ (4)

Multiply (3) by 2, $2xy + 2x = 20y - 20.$ (5)

Subtract (4) from (5), $11x - 22y = -11.$

$$x - 2y = -1.$$

$$\therefore x = 2y - 1. \qquad (6)$$

Substitute the value of x in (3),

$$2y^2 - y + 2y - 1 = 10y - 10.$$

$$2y^2 - 9y = -9.$$

$$16y^2 - (\ ) + 81 = 9.$$

$$4y - 9 = \pm 3.$$

$$4y = 12, \text{ or } 6.$$

$$\therefore y = 3, \text{ or } 1\tfrac{1}{2}.$$

Substitute the value of y in (6), $x = 5, \text{ or } 2.$

Therefore, $x = 5$, and $y = 3$;
or $x = 2$, and $y = 1\tfrac{1}{2}$.

69. If the sum of the squares of two numbers is divided by the first number, the quotient is 14 and the remainder 4; if divided by the second number, the quotient is 10 and the remainder 4. Find the numbers.

Let $x =$ the first number,
and $y =$ the second number.

$$\therefore \frac{x^2+y^2}{x} = 14 + \frac{4}{x}, \qquad (1)$$

and
$$\frac{x^2+y^2}{y} = 10 + \frac{4}{y}. \qquad (2)$$

From (1), $x^2 + y^2 = 14x + 4. \qquad (3)$
From (2), $x^2 + y^2 = 10y + 4. \qquad (4)$
Subtract (4) from (3),

$$0 = 14x - 10y.$$
$$14x = 10y.$$
$$\therefore x = \frac{5y}{7}. \qquad (5)$$

Substitute the value of x in (4), $\frac{25y^2}{49} + y^2 = 10y + 4.$

$$25y^2 + 49y^2 = 490y + 196.$$
$$74y^2 - 490y = 196.$$
$$74^2y^2 - (\ ) + 245^2 = 74529.$$
$$74y - 245 = \pm 273.$$
$$74y = 518, \text{ or } -28.$$
$$\therefore y = 7, \text{ or } -\tfrac{14}{37}.$$

Substitute the value of y in (5), $x = 5, \text{ or } -\tfrac{10}{37}.$

Therefore, the numbers required are 5 and 7.

70. The diagonal of a rectangle is 85 feet. If each dimension is increased by 2 feet, the area is increased by 230 square feet. Find the dimensions of the rectangle.

Let $x =$ the number of feet in one dimension,
and $y =$ the number of feet in the other dimension.

$$\therefore x^2 + y^2 = 85^2, \qquad (1)$$

and
$$(x+2)(y+2) = xy + 230. \qquad (2)$$

From (2), $xy + 2x + 2y + 4 = xy + 230.$

$$2x + 2y = 226.$$
$$x + y = 113. \qquad (3)$$

Square (3), $x^2 + 2xy + y^2 = 12769. \qquad (4)$

Subtract (1) from (4), $2xy = 5544.$ (5)

Subtract (5) from (1), $x^2 - 2xy + y^2 = 1681.$

$x - y = \pm 41.$ (6)

Add (3) and (6), $2x = 154, \text{ or } 72.$

$\therefore x = 77, \text{ or } 36.$

Substitute the value of x in (3), $y = 36, \text{ or } 77.$

Therefore, the dimensions of the rectangle are 77 feet and 36 feet.

71. The sum of two numbers is 24, and the sum of the fourth powers of the numbers is 85,922. Find the numbers.

Let x = one number,

and y = the other number.

$\therefore x + y = 24,$ (1)

and $x^4 + y^4 = 85922.$ (2)

Raise (1) to the 4th power,

$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 331776.$ (3)

Subtract (2) from (3),

$$4x^3y + 6x^2y^2 + 4xy^3 = 245854.$$

$$2x^3y + 3x^2y^2 + 2xy^3 = 122927.$$

$xy(2x^2 + 3xy + 2y^2) = 122927.$ (4)

Square (1), $x^2 + 2xy + y^2 = 576.$ (5)

$$x^2 + y^2 = 576 - 2xy.$$

$$2x^2 + 2y^2 = 1152 - 4xy.$$

Substitute the value of $2x^2 + 2y^2$ in (4),

$$xy(1152 - 4xy + 3xy) = 122927.$$

$$xy(1152 - xy) = 122927.$$

$$1152xy - x^2y^2 = 122927.$$

$$x^2y^2 - 1152xy = -122927.$$

$$x^2y^2 - (\ ) + 576^2 = 208849.$$

$$xy - 576 = \pm 457.$$

$$xy = 1033, \text{ or } 119.$$

$4xy = 4132, \text{ or } 476.$ (6)

Subtract (6) from (5), $x^2 - 2xy + y^2 = -3556, \text{ or } 100.$

$x - y = \pm 2\sqrt{-889}, \text{ or } \pm 10.$ (7)

Add (1) and (7), $2x = 24 \pm 2\sqrt{-889}, 34 \text{ or } 14.$

$\therefore x = 12 \pm \sqrt{-889}, 17, \text{ or } 7.$

Substitute the value of x in (1), $y = 12 \mp \sqrt{-889}, 7, \text{ or } 17.$

Therefore, the numbers required are 7 and 17, or $12 + \sqrt{-889}$ and $12 - \sqrt{-889}$.

72. **A garrison is supplied with bread for 11 days. If there were 200 men more, each man would receive each day $\frac{1}{4}$ of a pound less bread; if the garrison was 300 men smaller, each man would receive each day $\frac{1}{2}$ of a pound more bread. How large was the garrison and how large was the daily ration of bread?**

Let $x =$ the number of men in the garrison,

and $y =$ the number of pounds of bread in the daily ration of each man.

$$\therefore\ (x + 200)(y - \tfrac{1}{4}) = xy, \qquad (1)$$

and

$$(x - 300)(y + \tfrac{1}{2}) = xy. \qquad (2)$$

From (1),

$$xy + 200\,y - \frac{x}{4} - 50 = xy.$$

$$\frac{x}{4} - 200\,y = -\,50.$$

$$x - 800\,y = -\,200. \qquad (3)$$

From (2),

$$xy + \frac{x}{2} - 300\,y - 150 = xy.$$

$$\frac{x}{2} - 300\,y = 150.$$

$$x - 600\,y = 300. \qquad (4)$$

Subtract (3) from (4),

$$200\,y = 500.$$

$$\therefore\ y = 2\tfrac{1}{2}.$$

Substitute the value of y in (4),

$$x - 1500 = 300.$$

$$\therefore\ x = 1800.$$

Therefore, there are 1800 men in the garrison, and the daily ration of bread is $2\frac{1}{2}$ pounds per man.

73. A fore wheel of a carriage turns 132 times more than a hind wheel while the carriage goes 1 mile. If the circumference of each was 2 feet greater, a fore wheel would turn only 88 times more. Find the circumference of each wheel.

Let $x =$ the number of feet in the circumference of a fore wheel,

and $y =$ the number of feet in the circumference of a hind wheel.

1 mile = 5280 feet.

$$\therefore\ \frac{5280}{x} - \frac{5280}{y} = 132, \qquad (1)$$

and

$$\frac{5280}{x+2} - \frac{5280}{y+2} = 88. \qquad (2)$$

Divide (1) by 132,

$$\frac{40}{x} - \frac{40}{y} = 1.$$

$$40\,y - 40\,x = xy. \qquad (3)$$

Divide (2) by 88,

$$\frac{60}{x+2} - \frac{60}{y+2} = 1.$$

$$60y + 120 - 60x - 120 = xy + 2x + 2y + 4.$$

$$58y - 62x = xy + 4. \tag{4}$$

Subtract (4) from (3),

$$22x - 18y = -4.$$

$$11x - 9y = -2.$$

$$11x = 9y - 2.$$

$$\therefore x = \frac{9y-2}{11}. \tag{5}$$

Substitute the value of x in (3),

$$40y - \frac{40(9y-2)}{11} = \frac{(9y-2)y}{11}.$$

$$440y - 360y + 80 = 9y^2 - 2y.$$

$$9y^2 - 82y = 80.$$

$$81y^2 - (\ ) + 1681 = 2401.$$

$$9y - 41 = \pm 49.$$

$$9y = 90, \text{ or } -8.$$

$$\therefore y = 10, \text{ or } -\tfrac{8}{9}.$$

Substitute the value of y in (5), $x = 8$, or $-\frac{10}{11}$.

Therefore, the circumference of a fore wheel is 8 feet, and the circumference of a hind wheel is 10 feet.

74. The third digit of a number is the sum of the other two digits. The product of the first and third digits exceeds the square of the second by 5. If 396 is added to the number, the order of the digits is reversed. Find the number.

Let x = the first digit,
y = the second digit,
and z = the third digit.

Then $100x + 10y + z$ = the number,
and $100z + 10y + x$ = the number with the order of the digits reversed.

$$\therefore z = x + y, \tag{1}$$

$$xz - y^2 = 5, \tag{2}$$

and

$$100x + 10y + z + 396 = 100z + 10y + x. \tag{3}$$

From (3),

$$99x - 99z = -396.$$

$$x - z = -4.$$

$$\therefore x = z - 4. \tag{4}$$

Substitute the value of x in (1),

$$z = z - 4 + y.$$

$$\therefore y = 4.$$

Substitute the values of x and y in (2),

$$z^2 - 4z - 16 = 5.$$
$$z^2 - 4z - 21 = 0.$$
$$(z - 7)(z + 3) = 0.$$
$$\therefore z = 7, \text{ or } -3.$$

Substitute the value of z in (4), $x = 3, \text{ or } -7.$

The negative values do not apply.

Then $100x + 10y + z = 347.$

Therefore, the number required is 347.

75. An express train travels a miles in b hours with one stop of 5 minutes. An accommodation train covers the same distance in c hours, making 15 stops of average length of 2 minutes. How many miles will each train travel in one hour, if each moves at a uniform rate between stations? Compare the rates of the two trains.

Let x = the number of miles per hour in the rate of the express train between stations,

and y = the number of miles per hour in the rate of the accommodation train between stations.

5 minutes = $\frac{1}{12}$ hour;

15×2 minutes = 30 minutes = $\frac{1}{2}$ hour.

Then $$x = \frac{a}{b - \frac{1}{12}} = \frac{12a}{12b - 1},$$

and $$y = \frac{a}{c - \frac{1}{2}} = \frac{2a}{2c - 1}.$$

$$\therefore x : y = \frac{12a}{12b-1} : \frac{2a}{2c-1} = \frac{6}{12b-1} : \frac{1}{2c-1} = 12c - 6 : 12b - 1.$$

Therefore, the rate of the express train is $\frac{12a}{12b-1}$ miles per hour; that of the accommodation train is $\frac{2a}{2c-1}$ miles per hour; and the ratio of the rates is $12c - 6 : 12b - 1$.

76. Find two numbers in the ratio $1\frac{1}{3} : 2\frac{2}{3}$ such that when each is increased by 15 they shall be in the ratio of $1\frac{2}{3} : 2\frac{1}{3}$.

Let x = the first number,

and y = the second number.

$$\therefore x : y = 1\tfrac{1}{3} : 2\tfrac{2}{3}, \qquad (1)$$

and $$x + 15 : y + 15 = 1\tfrac{2}{3} : 2\tfrac{1}{3}. \qquad (2)$$

From (1),

$$x : y = 9 : 16.$$

$$9y = 16x.$$

$$\therefore\ y = \frac{16x}{9}. \tag{3}$$

From (2),

$$x + 15 : y + 15 = 10 : 15.$$

$$x + 15 : y + 15 = 2 : 3.$$

$$3x + 45 = 2y + 30.$$

$$3x = 2y - 15. \tag{4}$$

Substitute in (4) the value of y from (3), $3x = \frac{32x}{9} - 15.$

$$27x = 32x - 135.$$

$$-5x = -135.$$

$$\therefore\ x = 27.$$

Substitute the value of x in (3), $y = 48.$

Therefore, the numbers required are 27 and 48.

77. Two companies of a regiment went into a battle in strength as 9 to 11. After the battle the relative strength was as 5 to 8. Of the men in the two companies 35 per cent were disabled, and 30 of these belonged to the second company. Find the strength of each company at first.

Let $x =$ the number of men in the first company,

and $y =$ the number of men in the second company.

$$\therefore\ x : y = 9 : 11. \tag{1}$$

Now $\tfrac{35}{100}(x + y) =$ the number of men missing,

$30 =$ the number missing from the second company,

and $\tfrac{35}{100}(x + y) - 30 =$ the number missing from the first company.

$$\therefore\ x - \left[\tfrac{35}{100}(x + y) - 30\right] : y - 30 = 5 : 8. \tag{2}$$

From (2),

$$x - \frac{35x}{100} - \frac{35y}{100} + 30 : y - 30 = 5 : 8.$$

$$\frac{13x}{20} - \frac{7y}{20} + 30 : y - 30 = 5 : 8.$$

$$\frac{26x}{5} - \frac{14y}{5} + 240 = 5y - 150.$$

$$\frac{26x}{5} - \frac{39y}{5} = -390.$$

$$\frac{2x}{5} - \frac{3y}{5} = -30.$$

$$2x - 3y = -150. \tag{3}$$

From (1), $11x = 9y.$

$$\therefore x = \frac{9y}{11}. \qquad (4)$$

Substitute the value of x in (3), $\frac{18y}{11} - 3y = -150.$

$$18y - 33y = -1650.$$
$$-15y = -1650.$$
$$\therefore y = 110.$$

Substitute the value of y in (4), $x = 90.$

Therefore, the strength of the companies at first was 90 men and 110 men.

78. Show that the product of the least and the greatest of any four consecutive numbers is less by 2 than the product of the two intermediate numbers.

Let $n =$ any positive integer.

Then $n + 1$, $n + 2$, $n + 3 =$ the next three larger positive integers.

To prove $n(n + 3) + 2 = (n + 1)(n + 2).$

Now $n^2 + 3n + 2 = n^2 + 3n + 2.$

$$\therefore n(n + 3) + 2 = (n + 1)(n + 2).$$

79. Show that if 1 is added to the product of any four consecutive numbers, the sum is a perfect square.

Let n, $n + 1$, $n + 2$, $n + 3$ be any four consecutive numbers.

Now
$$\begin{aligned} n(n+1)(n+2)(n+3) + 1 &= (n^2 + n)(n^2 + 5n + 6) + 1 \\ &= n^4 + 6n^3 + 11n^2 + 6n + 1 \\ &= (n^2 + 3n + 1)^2. \end{aligned}$$

80. If $\frac{x^3 + ax^2 - bx + c}{x^3 - ax^2 + bx + c} = \frac{x^2 + ax - b}{x^2 - ax + b}$, show that $x = \frac{b}{a}$.

$$\frac{x^3 + ax^2 - bx + c}{x^3 - ax^2 + bx + c} = \frac{x^2 + ax - b}{x^2 - ax + b}.$$

By alternation, $\frac{x^3 + ax^2 - bx + c}{x^2 + ax - b} = \frac{x^3 - ax^2 + bx + c}{x^2 - ax + b}.$

Divide each numerator by its denominator,

$$x + \frac{c}{x^2 + ax - b} = x + \frac{c}{x^2 - ax + b}.$$
$$\therefore x^2 + ax - b = x^2 - ax + b.$$
$$2ax = 2b.$$
$$\therefore x = \frac{b}{a}.$$

81. $x^4 + x^3 - 16x^2 - 4x + 48$ may be resolved into two factors of the form $x^2 + mx + 6$ and $x^2 + nx + 8$. Find the factors.

$$\begin{aligned} &x^4 + x^3 - 16x^2 - 4x + 48 \\ &\quad = (x^2 + mx + 6)(x^2 + nx + 8) \\ &\quad = x^4 + (m + n)x^3 + (14 + mn)x^2 + (8m + 6n)x + 48. \end{aligned}$$

$$\therefore m + n = 1,\ 14 + mn = -16,\ 8m + 6n = -4;$$

whence $m = -5$, and $n = 6$.

$$\therefore x^4 + x^3 - 16x^2 - 4x + 48 = (x^2 - 5x + 6)(x^2 + 6x + 8).$$

82. A man who has deposited $500 at compound interest each year at 5 per cent finds that at the age of 54 he owns $33,250. At what age did he begin to deposit his money?

By § 475, $$A = \frac{S(R^n - 1)}{r}.$$

Here $$A = 33250,\ S = 500,\ R = 1.05,\ r = 0.05.$$

$$\begin{aligned} \therefore 33250 &= \frac{500(1.05^n - 1)}{0.05}. \\ 33250 &= 10000(1.05^n - 1). \\ 1.05^n - 1 &= \frac{33250}{10000}. \\ 1.05^n &= 3.3250 + 1. \\ 1.05^n &= 4.325. \\ \log(1.05^n) &= \log 4.325. \\ n \log 1.05 &= \log 4.325. \\ \therefore n &= \frac{\log 4.325}{\log 1.05} = \frac{0.6360}{0.0212} = 30. \end{aligned}$$

Hence, the man began to deposit his money 30 years before he was 54 years old.

Therefore, the man began to deposit his money at the age of 24 years.

83. December 31, 1895, a man deposited $1200 in a savings bank that pays 3 per cent compound interest. If he deposited $300 on the last day of December of each year, what will be the amount of his deposit December 31, 1905?

The amount of the deposit, December 31, 1905, is the compound amount of $1200 for 10 years at 3 per cent added to the amount of $300 put at interest each year for 10 years at 3 per cent.

(i) By § 471, $$A = PR^n.$$

Here $$P = 1200,\ R = 1.03,\ n = 10.$$

$$\therefore A = 1200 \times 1.03^{10} = 1611.$$

$$\begin{aligned}\log 1200 &= 3.0792\\ \log 1.03^{10} &= \underline{0.1280}\\ &\ 3.2072 = \log 1611.\end{aligned}$$

(ii) By §475, $A = \frac{S(R^n - 1)}{r}.$

Here $S = 300,\ R = 1.03,\ n = 10,\ r = 0.03.$

$$\therefore A = \frac{300(1.03^{10} - 1)}{0.03} = 10000(1.03^{10} - 1).$$

$$\log 1.03^{10} = 0.1280 = \log 1.343.$$

$$\therefore A = 10000(1.343 - 1) = 10000 \times 0.343 = 3430.$$

$$\$1611 + \$3430 = \$5041.$$

Therefore, the amount of the deposit December 31, 1905, is \$5041.

84. A man borrows \$10,000 at 4 per cent interest, and pays each year $12\frac{1}{2}$ per cent of that sum. In how many years will the debt be paid?

The amount due at the end of the nth year is equal to the compound amount of the principal for n years less the sum of the compound amounts of the payments each for its own proper number of years.

$$\therefore A = PR^n - (SR^{n-1} + SR^{n-2} + SR^{n-3} + \cdots + SR + S).$$

The parenthesis is a geometrical series in which $a = S,\ r = R,\ n = n.$

By § 415, $s = \frac{a(r^n - 1)}{r - 1} = \frac{S(R^n - 1)}{R - 1} = \frac{SR^n - S}{R - 1}.$

$$\therefore A = PR^n - \frac{SR^n - S}{R - 1}.$$

Now $A = 0, \text{ and } P = 8S.$

$$\therefore 0 = 8SR^n - \frac{SR^n - S}{R - 1}.$$

$$0 = 8R^n - \frac{R^n - 1}{R - 1}.$$

$$8R^n = \frac{R^n - 1}{R - 1}.$$

$$8R^{n+1} - 8R^n = R^n - 1.$$

$$9R^n - 8R^{n+1} = 1.$$

$$R^n(9 - 8R) = 1.$$

Now $R = 1.04.$

$$\therefore 1.04^n(9 - 8.32) = 1.$$

$$1.04^n \times 0.68 = 1.$$

$$1.04^n = \frac{1}{0.68}.$$

$$\log(1.04^n) = \log\left(\frac{1}{0.68}\right).$$

$$n \log 1.04 = \text{colog}\ 0.68.$$

$$\therefore\ n = \frac{\text{colog}\ 0.68}{\log 1.04} = \frac{0.1675}{0.0170} = 9.85.$$

Hence, it will require 9.85 years.

Since the last payment is not made until the end of the year, it will take 10 years for the debt to be paid, and the last payment will be somewhat less than $12\frac{1}{2}$ per cent of $10,000.

Exercise 146. Page 413

1. Forces of 5 pounds and 12 pounds act on a point. Find their resultant if the forces act in the same direction; if they act in opposite directions; if they act at right angles.

(i) The resultant = 5 pounds + 12 pounds
= 17 pounds.

(ii) The resultant = 12 pounds − 5 pounds
= 7 pounds, acting in the direction of the force of 12 pounds.

(iii) The resultant $= \sqrt{12^2 + 5^2}$ pounds
$= \sqrt{144 + 25}$ pounds
$= \sqrt{169}$ pounds
= 13 pounds.

2. Two forces of 25 pounds act on a point at right angles. Find the resultant, and the angle it forms with each force.

The resultant $= \sqrt{25^2 + 25^2}$ pounds
$= \sqrt{1250}$ pounds
= 35.36 pounds.

Since the forces are equal and act at right angles, the resultant makes an angle of 45° with each force.

3. The resultant of two concurrent forces acting at right angles is 10 pounds. One of the forces is 8 pounds. Find the other force.

Let x = the number of pounds in the required force.

$$\therefore\ x = \sqrt{10^2 - 8^2} = \sqrt{100 - 64} = \sqrt{36} = 6.$$

Therefore, the required force is 6 pounds.

4. Resolve a force of 75 pounds into two perpendicular components that make equal angles with the given force.

Since the two perpendicular components make equal angles with the given force, the two components are equal.

Let $x =$ the number of pounds in each component.

$$\therefore \sqrt{x^2 + x^2} = 75.$$
$$2x^2 = 5625.$$
$$x^2 = 2812.5.$$
$$\therefore x = 53.03.$$

Therefore, each component is 53.03 pounds.

5. Resolve a force of 40 pounds into two perpendicular components one of which is three times as great as the other.

Let $x =$ the number of pounds in the smaller component.

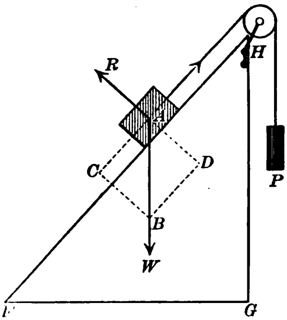
Then $3x =$ the number of pounds in the larger component.

$$\therefore \sqrt{x^2 + (3x)^2} = 40.$$
$$\sqrt{10x^2} = 40.$$
$$10x^2 = 1600.$$
$$x^2 = 160.$$
$$\therefore x = 12.65.$$

Then $$3x = 37.95.$$

Therefore, the components required are 12.65 pounds and 37.95 pounds.

6. What force parallel to a smooth inclined plane 18 feet long and 12 feet high will support on the plane a body weighing 75 pounds?



A weight W is supported on a smooth inclined plane by another weight P, which is connected with W by means of a cord and fixed pulley, as shown in the figure.

Represent W by a line AB drawn from A, where the entire weight of W may be considered to act.

Construct the rectangle $ACBD$ by drawing AC and DB parallel to the length HF of the inclined plane, and AD and CB perpendicular to HF.

The components of W are represented by AC and AD. The component AC pulls the body down the plane and is balanced by P, and therefore $P = AC$. AD pushes the body directly against the plane, and is balanced by the reaction R of the plane.

The triangles ABC and HFG are mutually equiangular, and are therefore similar.

$$\therefore \frac{AC}{AB} = \frac{GH}{FH}, \text{ and } \frac{BC}{AB} = \frac{AD}{AB} = \frac{GF}{FH}.$$

Therefore,
$$\frac{P}{W} = \frac{\text{height of plane}}{\text{length of plane}},$$

and
$$\frac{R}{W} = \frac{\text{base of plane}}{\text{length of plane}}.$$

Here
$$\frac{P}{75} = \frac{12}{18}.$$
$$\frac{P}{75} = \frac{2}{3}.$$
$$3P = 150.$$
$$\therefore P = 50.$$

Therefore, the required force is 50 pounds.

7. The base of a smooth inclined plane is 80 inches and the height is 39 inches. What force acting parallel to the plane will support on it a weight of 178 pounds?

The length of the inclined plane $= \sqrt{80^2 + 39^2}$ inches
$= \sqrt{6400 + 1521}$ inches
$= \sqrt{7921}$ inches
$= 89$ inches.

By Example 6,
$$\frac{P}{178} = \frac{39}{89}.$$
$$89P = 39 \times 178.$$
$$\therefore P = 78.$$

Therefore, the required force is 78 pounds.

8. Resolve a force of 250 pounds acting northeast into two forces, one acting north and the other east.

The two components, since one acts north and one east, make right angles with each other.

The force acts northeast, and therefore makes an angle of 45° with each component.

Therefore, the components are equal.

Let $x =$ the number of pounds in each component.

$$\therefore \sqrt{x^2 + x^2} = 250.$$
$$2x^2 = 62500.$$
$$x^2 = 31250.$$
$$\therefore x = 176.78.$$

Therefore, each required force is 176.78 pounds.

9. Two forces in the ratio of 3 to 5 act at right angles upon a point and produce a resultant force of 136 pounds. Find the value of the forces.

Let $3x =$ the number of pounds in the smaller force.

Then $5x =$ the number of pounds in the larger force.

$$\therefore \sqrt{(3x)^2 + (5x)^2} = 136.$$
$$9x^2 + 25x^2 = 136 \times 136.$$
$$34x^2 = 136 \times 136.$$
$$x^2 = 4 \times 136.$$
$$x^2 = 544.$$
$$\therefore x = 23.324.$$

Then $3x = 69.97,$

and $5x = 116.62.$

Therefore, the forces required are 69.97 pounds and 116.62 pounds.

10. A body moves east with a constant velocity of 25 feet per second, and south with a constant velocity of 30 feet per second. Find the actual rate of motion in a straight line.

Let $x =$ the number of feet the body moves per second in a straight line.

$$\therefore x = \sqrt{25^2 + 30^2} = \sqrt{625 + 900} = \sqrt{1525} = 39.05.$$

Therefore, the body moves 39.05 feet per second in a straight line.

11. A man rows a boat at right angles to the course of a river three times as fast as the river flows, and reaches the opposite bank half a mile below the starting point. What is the width of the river?

$$1 : 3 = \tfrac{1}{2} \text{ mile} : \text{ width of river.}$$
$$\therefore \text{width of river} = 3 \times \tfrac{1}{2} \text{ mile} = 1\tfrac{1}{2} \text{ miles.}$$

Therefore, the width of the river is $1\frac{1}{2}$ miles.

12. A boy sits on his sled on the side of a hill that rises 1 foot in every 6 feet. If the weight of the boy and sled is 100 pounds, what force acting parallel to the ground will keep the sled from sliding down the hill?

By Example 6,

$$\frac{P}{100} = \frac{1}{6}.$$
$$6P = 100.$$
$$\therefore P = 16\tfrac{2}{3}.$$

Therefore, the required force is $16\frac{2}{3}$ pounds.

13. A stone weighing 150 pounds rests on an inclined plane 8 feet long and 3 feet high. What must be the force of friction that will keep the stone from sliding down the plane?

By Example 6,

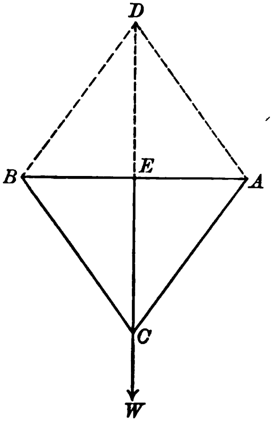
$$\frac{P}{150} = \frac{3}{8}.$$
$$8P = 450.$$
$$\therefore P = 56\tfrac{1}{4}.$$

Therefore, the required force is $56\frac{1}{4}$ pounds.

14. Find the tension on the rope if a mass of 150 pounds is suspended from the middle of a rope 16 feet long fastened to each end of a horizontal beam 12 feet long.

Let AB represent the beam, ACB the rope, and W the weight.

Complete the parallelogram $ACBD$.



$$AC = 8, \text{ and } AE = 6.$$
$$CE = \sqrt{8^2 - 6^2} = \sqrt{64 - 36}$$
$$= \sqrt{28} = 5.29.$$
$$CD = 2 \times 5.29 = 10.58.$$

Then

$$\frac{CD}{AC} = \frac{W}{\text{tension}}.$$
$$\frac{10.58}{8} = \frac{150}{\text{tension}}.$$
$$10.58 \text{ tension} = 8 \times 150.$$
$$\therefore \text{tension} = \frac{8 \times 150}{10.58} = 113.42.$$

Therefore, the tension required is 113.42 pounds.

15. Three concurrent balanced forces act, one toward the north, one toward the east, and one toward the southwest. If the third force is 25 pounds, find the other two forces.

The force that acts toward the southwest is equal to and opposite in direction to the resultant of the forces that act toward the north and east. The opposite of southwest is northeast, and northeast makes equal angles with north and with east. Therefore, the forces that act toward the north and toward the east are equal, and their resultant is 25 pounds.

Let $x =$ the number of pounds in either the north or east force.

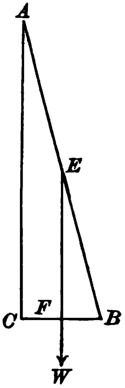
$$\therefore \sqrt{x^2 + x^2} = 25.$$
$$2x^2 = 625.$$
$$x^2 = 312.5.$$
$$\therefore x = 17.68.$$

Therefore, the force toward the north is 17.68 pounds, and the force toward the east is 17.68 pounds.

16. A ladder 24 feet long weighs 36 pounds and rests with one end against a wall and the other end on the ground 6 feet from the wall. The centre of gravity of the ladder is 11½ feet from the base. Find the horizontal pressure against the wall.

Let AB represent the ladder, AC the wall, and BC the ground. Let E be the centre of gravity of the ladder.

$AB = 24$ feet, $EB = 11\frac{1}{2}$ feet, $AE = 12\frac{1}{2}$ feet,
$BC = 6$ feet, and $W = 36$ pounds.
$AC = \sqrt{24^2 - 6^2} = \sqrt{576 - 36} = \sqrt{540} = 23.24$ feet.



In the similar triangles ACB and EFB,

$$AB : CB = EB : FB,$$

or

$$24 : 6 = 11\tfrac{1}{2} : FB.$$

$$\therefore FB = \frac{6 \times 11\frac{1}{2}}{24} = \frac{23}{8} = 2\tfrac{7}{8} \text{ feet.}$$

Let $x =$ the horizontal pressure against the wall.

$$\frac{x}{W} = \frac{FB}{AC}.$$

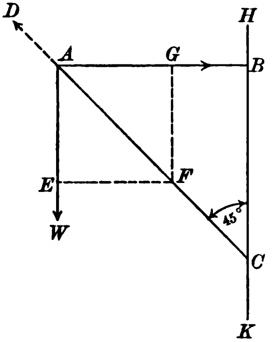
$$\frac{x}{36} = \frac{2\frac{7}{8}}{23.24}.$$

$$\therefore x = \frac{36 \times 2\frac{7}{8}}{23.24} = 4.45.$$

Therefore, the horizontal pressure against the wall is 4.45 pounds.

17. If 200 pounds is supported in front of a wall by a horizontal rod and a brace making an angle of 45° with the rod at its end, find the pull of the rod and the push of the brace.

Let HK represent the wall, AB the horizontal rod, and AC the brace making the angle $ACB = 45°$ with HK.



The forces acting at A are 200 pounds downward, the pull of the rod in the direction AB, and the push of the brace in the direction AD. Take AE to represent 200 pounds, and complete the parallelogram $AEFG$ (here a square).

The push of the brace is just equal to and opposite to AF, the resultant of the weight, 200 pounds, and the tension in the horizontal rod AB.

These two forces make equal angles with this resultant.

Hence, these two forces are equal; that is, the tension in the horizontal rod is equal to 200 pounds.

$\therefore\ AF = \sqrt{200^2 + 200^2}$ pounds $= 200\sqrt{2}$ pounds $= 282.8$ pounds.

Therefore, the pull of the rod is 200 pounds, and the push of the brace is 282.8 pounds.

Exercise 147. Page 415

1. Find the volume of 1 pound of cast iron, specific gravity 7.2. What does the iron weigh under water?

1 cubic foot of water weighs 62.4 pounds.

1 cubic foot of cast iron weighs 7.2 × 62.4 pounds.

Therefore, 7.2 × 62.4 pounds of cast iron occupy 1 cubic foot, or 1728 cubic inches.

Hence, 1 pound of cast iron occupies $\frac{1728}{7.2 \times 62.4}$ cubic inches, or 3.846 cubic inches of space.

Now 3.846 cubic inches of water weigh $\frac{3.846}{1728}$ of 62.4 pounds $= 0.139$ pound.

Hence, the iron weighs under water 1 pound $-$ 0.139 pound $= 0.861$ pound; or the iron weighs under water $\frac{7.2 - 1}{7.2}$ pound $= \frac{6.2}{7.2}$ pound $= 0.861$ pound.

Therefore, the volume of 1 pound of cast iron is 3.846 cubic inches; and the iron weighs 0.861 pound under water.

2. A block of wood, placed in a vessel full of water, floats 0.6 submerged. Sixty cubic centimeters of water runs out. Find the weight, volume, and specific gravity of the wood.

Since 60 cubic centimeters of water runs out, the weight of the block is 60 grams.

Since the block is 0.6 submerged, the volume is $\frac{60}{0.6}$, or 100, cubic centimeters.

Since the block floats 0.6 submerged, the specific gravity of the wood is 0.6.

Therefore, the weight of the wood is 60 grams, the volume is 100 cubic centimeters, and the specific gravity is 0.6.

3. A piece of wood weighs 150 grams in air, and a piece of lead weighs 40 grams in water. They together weigh 30 grams in water. Find the specific gravity of the wood.

The buoyant force of the water on the wood supports the wood and takes away 40 − 30, or 10, grams from the weight of the lead.

Hence, the buoyant force upon the wood, or the weight of an equal bulk of water, is 150 + 10, or 160, grams.

Therefore, the specific gravity of the wood is $\frac{150}{160}$, or 0.9375.

4. A stone weighs 20.4 grams in air, 14.4 grams in water, and 14.94 grams in linseed oil. Find the specific gravity of the stone and of the oil, and the volume of the stone.

The stone loses 20.4 − 14.4, or 6, grams when weighed in water.

Hence, the volume of the water displaced by the stone, that is, the volume of the stone, is 6 cubic centimeters.

The weight of the stone is 20.4 grams, and the weight of an equal bulk of water is 6 grams.

Hence, the specific gravity of the stone is $\frac{20.4}{6} = 3.4$.

The stone loses 20.4 − 14.94, or 5.46, grams when weighed in linseed oil. Therefore, 5.46 grams and 6 grams are the weights of equal bulks of linseed oil and water.

Hence, the specific gravity of linseed oil is $\frac{5.46}{6} = 0.91$.

Therefore, the specific gravity of the stone is 3.4; the specific gravity of linseed oil is 0.91; and the volume of the stone is 6 cubic centimeters.

5. A block of wood weighing 42 grams in air floats in water with 0.8 of its volume submerged. How many grams must be placed upon the block of wood to submerge it?

Since the block of wood floats with 0.8 of its volume submerged, the specific gravity of the wood is 0.8, and the volume is $\frac{42}{0.8}$, or 52.5, cubic centimeters.

When the wood is submerged, it displaces 52.5 cubic centimeters, or 52.5 grams, of water.

Therefore, there must be placed upon the wood to submerge it, 52.5 grams − 42 grams, or 10.5 grams.

6. A block of wood floats in kerosene, specific gravity 0.79, with three fourths of its volume submerged. How much of the wood will be submerged in sea water, specific gravity 1.026?

Let the volume of the wood be a cubic centimeters. Then $\frac{3}{4}a$, or $0.75a$, cubic centimeters of wood are submerged in the kerosene. Therefore, a cubic centimeters of the wood weigh the same as $0.75a$ cubic centimeters of kerosene. Since 1 cubic centimeter of kerosene weighs 0.79 gram, a cubic centimeters of the wood weigh $0.75a \times 0.79$ grams. Since 1 cubic centimeter of sea water weighs 1.026 grams, it will take $\frac{0.75a \times 0.79}{1.026}$, or $0.5775a$, cubic centimeters of sea water to weigh $0.75a \times 0.79$ grams.

Therefore, $\frac{0.5775a}{a}$, or 0.5775, of the wood will be submerged in sea water.

7. How many grams of lead, specific gravity 11.4, must be fastened to a cubic centimeter of cork, specific gravity 0.24, that it may just float in sulphuric acid, specific gravity 1.84?

1 cubic centimeter of cork weighs 0.24 gram, and 1 cubic centimeter of sulphuric acid weighs 1.84 grams.

When 1 cubic centimeter of cork is wholly immersed in sulphuric acid, the buoyant force of the sulphuric acid on the cork is $(1.84 - 0.24)$ grams.

1 cubic centimeter of lead weighs 11.4 grams, and when immersed in sulphuric acid weighs $(11.4 - 1.84)$ grams.

Therefore, the volume of lead required to overcome the buoyant force of the sulphuric acid on the cork is $\frac{1.84-0.24}{11.4-1.84}$, or $\frac{40}{239}$, cubic centimeter.

$\frac{40}{239}$ cubic centimeter of lead weighs $\frac{40}{239}\times 11.4$ grams $= 1.908$ grams.

Therefore, 1.908 grams of lead will be required.

8. A block of lead weighs 236.5 grams. Find the specific gravity of a liquid in which the lead weighs 206.3 grams.

When the block of lead is weighed in the given liquid, the loss in weight is 236.5 − 206.3, or 30.2, grams.

236.5 grams of lead occupy $\frac{236.5}{11.4}$ cubic centimeters of space.

Hence, the weight of $\frac{236.5}{11.4}$ cubic centimeters of the given liquid is 30.2 grams.

Therefore, the specific gravity of the liquid is $30.2 \div \frac{236.5}{11.4} = 1.456$.

9. The specific gravity of pure milk is 1.03. A sample has been adulterated with water and has a specific gravity of 1.0275. What per cent of the sample is water?

Let $x =$ the number of cubic centimeters of water in 1 cubic centimeter of the sample.

Then $x =$ the number of grams of water in 1 cubic centimeter of the sample,

$1 - x =$ the number of cubic centimeters of milk in 1 cubic centimeter of the sample,

$1.03(1-x) =$ the number of grams of milk in 1 cubic centimeter of the sample,

and $1.0275 =$ the number of grams 1 cubic centimeter of the sample weighs.

$$\therefore\ x + 1.03(1-x) = 1.0275.$$
$$x + 1.03 - 1.03x = 1.0275.$$
$$0.03x = 0.0025.$$
$$\therefore\ x = 0.08\tfrac{1}{3}.$$

Therefore, $8\frac{1}{3}$ per cent of the sample is water.

10. A bar of gold contains some copper, and weighs 352.6 grams in air and 332.3 grams in water. Find the number of cubic centimeters of gold and of copper in the bar. The specific gravity of gold is 19.3, that of copper 8.9.

Let x = the number of cubic centimeters of gold in the bar.

Then $352.6 - 332.3 = 20.3$ = the number of grams the bar loses when weighed in water,

20.3 = the number of cubic centimeters in the bar,

$20.3 - x$ = the number of cubic centimeters of copper in the bar,

$19.3x$ = the number of grams of gold in the bar,

and $8.9(20.3 - x)$ = the number of grams of copper in the bar.

$$\therefore\ 19.3x + 8.9(20.3 - x) = 352.6.$$

$$19.3x + 180.67 - 8.9x = 352.6.$$

$$10.4x = 171.93.$$

$$\therefore\ x = 16.532.$$

Then $20.3 - x = 3.768.$

Therefore, the bar contains 16.532 cubic centimeters of gold, and 3.768 cubic centimeters of copper.

11. A piece of lead weighing 45 grams and a piece of copper are fastened to the ends of a string passing over a pulley, and are in equilibrium when immersed in water. Find the weight of the copper if the specific gravity of lead is 11.4, and that of copper is 8.9.

Let x = the number of grams in the weight of the copper.

Then $\frac{10.4}{11.4} \times 45$ = the number of grams the lead weighs in water,

and $\frac{7.9}{8.9}x$ = the number of grams the copper weighs in water.

$$\therefore\ \frac{7.9}{8.9}x = \frac{10.4}{11.4} \times 45.$$

$$\therefore\ x = \frac{10.4}{11.4} \times 45 \times \frac{8.9}{7.9}$$

$$= 46.25.$$

Therefore, the weight of the copper is 46.25 grams.

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